SPACES OF GRAPHS AND SURFACES
- ON THE WORK OF SØREN GALATIUS

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1. Introduction

Galatius' most striking result is easy enough to state. Let Σ_n be the symmetric group on n letters and F_n be the free (non-abelian) group with n generators. The symmetric group Σ_n acts naturally by permutation on the n generators of F_n, and every permutation gives thus rise to an automorphism of the free group. Galatius proves that the map $\Sigma_n \hookrightarrow \mathrm{Aut}F_n$ in homology induces an isomorphism in degrees less than $(n-1)/2$. The homology of the symmetric groups in these ranges is well understood. In particular, in common with all finite groups, it has no non-trivial rational homology. By Galatius' theorem, in low degrees this then is also true for $\mathrm{Aut}F_n$:

$$H_*(AutF_n) \otimes \mathbb{Q} = 0 \quad \text{for} \quad 0 < * < 2n/3,$$

as had been conjectured by Hatcher and Vogtmann.

In this lecture I will put this result in context and explain the connection with previous work on the mapping class group of surfaces and the homotopy theoretic approach to a conjecture by Mumford on its rational, stable cohomology. Galatius' proof in [G] is inspired by this and at the same time improves the methods significantly. This in turn has led to further deep insights into the topology of moduli spaces of manifolds also in higher dimensions.

2. Groups and their (co)homology

I will first step back and say a bit more about the groups mentioned above and discuss their (co)homology in essentially algebraic terms. There are many parallels between mapping class groups and automorphisms of free groups. Indeed, much of the work on $\mathrm{Aut}F_n$ has been inspired by the work on the mapping class group as these groups show very similar behavior.

2.1. Groups of primary interest. I will first introduce the discrete groups that we will mainly be interested in.

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2.1.1. The symmetric group Σ_n on n letters, is finite of size $n!$ and hardly needs further introduction. It has a presentation with generators the transpositions $\sigma_1, \dots, \sigma_{n-1}$ that swop two adjacent letters, and relations $\sigma_i^2 = 1$ and the braid relations $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ and $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i-j| > 1$.

2.1.2. The automorphism group $\mathrm{Aut}F_n$ is the group of invertible homomorphisms of the (non-abelian) free group F_n on n generators to itself. Closely related is the *outer automorphism group* $\mathrm{Out}F_n$ which is a quotient of $\mathrm{Aut}F_n$ by the normal subgroup $\mathrm{Inn}F_n$ of inner automorphisms given by conjugation by a fixed element of F_n. Both groups are infinite (for $n > 1$) and contain the symmetric group Σ_n as a subgroup.

The canonical map from the free group F_n to the free abelian group $\mathbb{Z}^n$ induces a surjective homomorphism

$$L : \mathrm{Aut}F_n \longrightarrow \mathrm{GL}(n, \mathbb{Z}),$$

to the general linear group. The inverse of the special linear group defines a subgroup $S\mathrm{Aut}F_n$ of index 2. A set of generators for this subgroup are the Nielsen transformations λ_{ij} and ρ_{ij} which multiply the ith generator of F_n by the jth on the left and right respectively, and leave all other generators fixed. A nice presentation of this subgroup is given by [Ge84]. To get a full set of generators, one needs to add an automorphism of determinant -1 such as the map that sends the first generator to its inverse and leaves all other generators fixed.

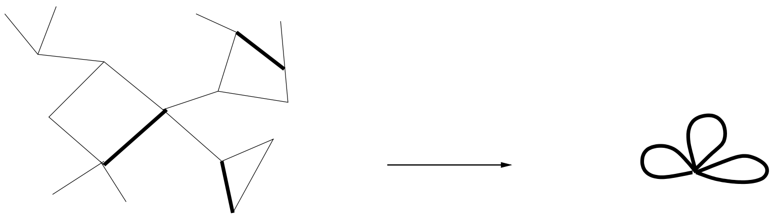

Figure 1: *Collapsing a maximal tree defines a homotopy equivalence.*

F_n is the fundamental group of a bouquet of n circles, or any graph G_n with Euler characteristic $1-n$ more generally. Let $\mathrm{HtEq}(G_n)$ denote the space of homotopy equivalences of G_n and $\mathrm{HtEq}(G_n; *)$ the subspace of homotopy equivalences that fix a basepoint. Their groups of components are $\mathrm{Out}F_n$ and $\mathrm{Aut}F_n$ respectively. Furthermore, each connected component is contractible. We thus have homotopy equivalences

$$\mathrm{HtEq}(G_n; *) \simeq \mathrm{Aut}F_n \quad \text{and} \quad \mathrm{HtEq}(G_n) \simeq \mathrm{Out}F_n.$$

2.1.3. The mapping class group $\Gamma_{g,1}$ *of an oriented surface* $S_{g,1}$ of genus g with one boundary component is the group $\pi_0(\mathrm{Diff}^+(S_{g,1}; \partial))$ of connected components of the group of orientation preserving diffeomorphisms of $S_{g,1}$ that fix the boundary point wise. Closely related is *the mapping class group* Γ_g *of an oriented, closed* *surface* S_g of genus g. They are generated by Dehn twists around simple closed curves defined by the following procedure: cut the surface along the given curve, twist one side by a full turn, and glue it back. A useful presentation was found by Wajnryb [W83]. It is not difficult to see that when two curves intersect once the associated Dehn twists satisfy the braid relation relation $aba = bab$; when two curves don't intersect their associated Dehn twist commute.

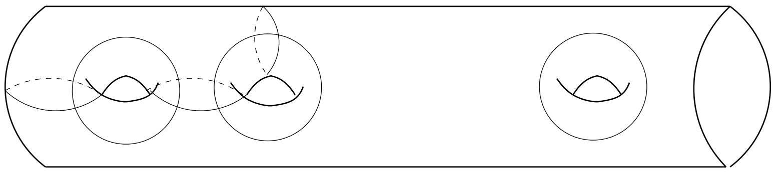

Figure 2: *Dehn twist around the* $2g + 1$ *indicated curves generate* $\Gamma_{g,1}$.

Diffeomorphisms act on the first homology $H_1(S_g) = \mathbb{Z}^{2g}$ of the underlying surface, and when they are orientation preserving they preserve the intersection form. This defines a surjective representation

$$\Gamma_g \longrightarrow \mathrm{SP}(2g, \mathbb{Z}).$$

When the Euler characteristic of the underlying surfaces is negative, Earle and Eells [EE69] showed that the connected components of the diffeomorphism groups are contractible. We thus also have in this case homotopy equivalences

$$\mathrm{Diff}^+(S_{g,1}) \simeq \Gamma_{g,1} \quad \text{and} \quad \mathrm{Diff}^+(S_g) \simeq \Gamma_g.$$

2.1.4. Natural homomorphisms between these groups. For the first three groups we have $\Sigma_n \to \mathrm{Aut}F_n \to \mathrm{Out}F_n$ where the first map is induced by the permutation action on a set of generators for F_n and the second map is the quotient map. By gluing a disc to the boundary of the surface $S_{g,1}$ and extending diffeomorphisms by the identity, we get a natural map $\Gamma_{g,1} \to \Gamma_g$. Finally, every diffeomorphism of $S_{g,1}$ induces an automorphism of the fundamental group $\pi_1 S_{g,1} = F_{2g}$. This defines a map

$$\rho^+ : \Gamma_{g,1} \longrightarrow \mathrm{Aut}F_{2g}.$$

2.2. Group (co)homology. One way to study a discrete group G is through its homology $H_n(G)$ and cohomology $H^n(G)$ groups. These groups can be defined purely algebraically or as the homology and cohomology of a space BG. The space BG is determined (up to homotopy) by the fact that its fundamental group is G and its universal cover is contractible. In practice one constructs such spaces by finding a contractible space EG with a free G action. BG is then the orbit space EG/G. For an easy example, consider the integers acting by translations on the real line. A model for the space $B\mathbb{Z}$ is then given by the circle $S^1 = \mathbb{R}/\mathbb{Z}$.

The first homology group $H_1(G)$ is always the abelianisation $G/[G,G]$ of G. Hopf also found a purely algebraic formula for the second homology group $H_2(G)$. Both these groups can generally be computed when one has a presentation of G (indeed, often a subset of the relations suffices). Presentations for all the groups mentioned above are known by now.[1] For $n > 4$ and $g > 4$, the first two homology groups are as follows:

$$H_1(\Sigma_n) = H_2(\Sigma_n) = \mathbb{Z}/2\mathbb{Z}$$
$$H_1(\mathrm{Aut}F_n) = H_2(\mathrm{Aut}F_n) = \mathbb{Z}/2\mathbb{Z}$$
$$H_1(\Gamma_{g,1}) = 0; \qquad H_2(\Gamma_{g,1}) = \mathbb{Z}.$$

By work of Culler-Vogtmann [CV86] and Harer [H86] we know that both $\mathrm{Out}F_n$ and Γ_g have finite virtual cohomological dimensions:

$$vcd(\mathrm{Out}F_n) = 2n - 3 \quad \text{and} \quad vcd(\Gamma_g) = 4g - 5.$$

In particular this implies that the (co)homology in degrees above these dimensions is all torsion for both groups. Note that the virtual cohomological dimensions depend on n and g. In contrast, in what follows we will only be interested in the (co)homology that is independent of n and g.

2.3. Stable (co)homology and limit groups. The groups that we introduced in Section 2.1 come in families $\{G_n\}_{n\geq 0}$ indexed by the natural numbers. For the symmetric groups Σ_n, the automorphisms of free groups $\mathrm{Aut}F_n$ and the mapping class groups $\Gamma_{g,1}$ there are canonical inclusions $G_n \hookrightarrow G_{n+1}$. Indeed, $\Sigma_n \hookrightarrow \Sigma_{n+1}$ identifies the smaller group with those permutations that leave the $(n+1)$st letter fixed; $\mathrm{Aut}F_n \hookrightarrow \mathrm{Aut}F_{n+1}$ with those automorphisms of F_{n+1} that leave the $(n+1)$st generator fixed and send the first n generators to words not involving the $(n+1)$st; and $\Gamma_{g,1} \hookrightarrow \Gamma_{g+1,1}$ with those mapping classes that come from diffeomorphisms that restrict to the identity on $S_{g+1,1} \setminus S_{g,1}$, a torus with two boundary components. In each case we define the limit groups as $G_\infty := \lim_{n\to\infty} G_n$.

It is natural to ask how the homology of G_n is related to that of G_{n+1} or G_∞. In each of the three cases, the groups satisfy homology stability which means that for a fixed degree the homology does not change once n is large enough. This is the *stable* homology, or equivalently the homology of the group G_∞. For the symmetric groups this was first studied by Nakaoka [Na60], for the mapping class groups by Harer [H85] with improved ranges given by [I89] [B] [RW], and for the automorphism group of free groups by Hatcher and Vogtmann [HV98.C] (see also [HV04] [HVW], and [HW]). Homology stability theorems are generally quite tricky and difficult theorems to prove with the main techniques go back to Quillen, who studied the question for general linear groups. The following holds:

$H_*(\Sigma_n) \to H_*(\Sigma_{n+1})$ is an isomorphism in degrees $* < (n+1)/2$.

[1] Nielsen and McCool had given a presentations of $\mathrm{Aut}F_n$. Simplifications allowed Gersten [G84] to compute the second homology group. For the mapping class group the first presentation was given by Thurston and Hatcher. Building on this Harer determined the second homology group. Further simplifications led to Wajnryb's convenient presentation [W83][BW94].

$H_*(\mathrm{Aut}F_n) \to H_*(\mathrm{Aut}F_{n+1})$ is an isomorphism in degrees $* < (n-1)/2$.
$H_*(\mathrm{Aut}F_n) \to H_*(\mathrm{Out}F_n)$ is an isomorphism in degrees $* < (n-3)/2$.

$H_*(\Gamma_{g,1}) \to H_*(\Gamma_{g+1,1})$ is an isomorphism in degrees $* < 2g/3$.
$H_*(\Gamma_{g,1}) \to H_*(\Gamma_g)$ is an isomorphism in degrees $* < (2g+1)/3$.

These results are crucial for us because it is the homology of the limit group G_∞ that can be computed in each of the cases. Through homology stability also some information on the homology of each G_n can be obtained. For rational homology these stability ranges can often be improved. Indeed, for $\mathrm{Aut}F_n$ this is $2n/3$ as quoted in the introduction, see [HV98.C].

2.4. Products and group completion. At first it is counter-intuitive that the homology of the larger group G_∞ should be more easily determined than that of G_n. But note that we have natural product maps

$$\begin{aligned} \Sigma_n \times \Sigma_m &\longrightarrow \Sigma_{n+m}, \\ \mathrm{Aut}F_n \times \mathrm{Aut}F_m &\longrightarrow \mathrm{Aut}F_{n+m}, \\ \Gamma_{g,1} \times \Gamma_{h,1} &\longrightarrow \Gamma_{g+h,1}. \end{aligned}$$

The first two are given by having the first factor act on the first n points or generators and the second factor on the last m. In case of the mapping class group the product map is induced by gluing $S_{g,1}$ and $S_{h,1}$ to the legs of a pair of pants surface and extending the diffeomorphisms via the identity. Furthermore, the products are commutative up to conjugation by an element in the target group G_{n+m}.[2] It is a standard fact that conjugation induces the identity on group homology. So we see that the products on homology are graded commutative.

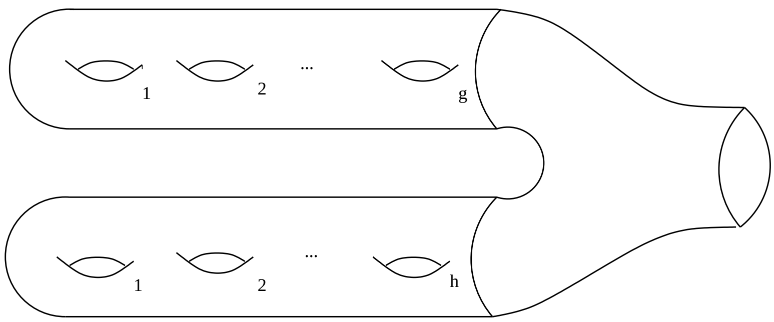

Figure 3: *Pair of pants product for surfaces.*

On the space level the above maps of groups induce a product on the disjoint union $M = \bigsqcup_n BG_n$ making it into a topological monoid. We will need to consider

[2]For the first two groups this is an element of the symmetric group Σ_{n+m} and its square is the identity; but for the mapping class group this corresponds to a twist of the glued on pair of pants surface, a braiding which is of infinite order.

its group completion. Group completion is a powerful tool but also one of the more mysterious constructions.

Just as discrete monoids have a group completion, so do topological monoids. It is a homotopy theoretic construction which associates to a topological monoid M the loop space ΩBM. (Here ΩX denotes the space of maps from a circle to X that send a base point in S^1 to a base point in X.) When $M = G$ is a discrete group, BG is the space mentioned above and the group completion is homotopic to G as it ought to be: $\Omega BG \simeq G$. In general, however, the homotopy of the monoid can be very different from that of its group completion. In our examples, the connected components of M have non-trivial and non-commutative fundamental groups but no higher homotopy while the group completion ΩBM has a small abelian fundamental group but highly non-trivial higher homotopy groups. Nevertheless, the homology groups are related very nicely. The group completion theorem (first instances of which are proved in [BP72] and [Q]) says that in general the homology of a connected component $\Omega_0 BM$ is the limit of the homology groups of the components of M.

To come back to our examples, $M = \bigsqcup_n BG_n$ has product that is commutative on homology. The group completion theorem can therefore be applied, and gives

$$H_*(G_\infty) = H_*(\Omega_0 B(\bigsqcup_n BG_n)).$$

In particular, the limit group G_∞ has the homology of a loop space. Indeed, a much stronger statement is true for our examples. We will see that G_∞ has actually the homology of an infinite loop space. While for the first two groups this has long been known, for the mapping class group it came as a surprise[3] [T97]. Below we will see how to identify these infinite loop spaces. Indeed for the symmetric group, a classical theorem in homotopy theory, the Barratt-Priddy-Quillen Theorem, asserts that the following two spaces are homotopic

$$\Omega B(\bigsqcup_n B\Sigma_n) \simeq \Omega^\infty S^\infty,$$

where $\Omega^\infty S^\infty = \lim_{N\to\infty} \Omega^N S^N$ is the limit space of maps from S^N to itself that fix a chosen basepoint. Its homotopy groups are the notoriously hard to compute stable homotopy groups of spheres. However, for every prime p, the homology with $\mathbb{Z}/p\mathbb{Z}$ coefficients has been computed for $\Omega^\infty S^\infty$, and hence for Σ_∞, see [AK56] and [DL62].

3. Moduli spaces and their (co)homology

We now switch from the algebraic to a more geometric point of view. The groups generally have now a natural topology and we need to distinguish between the homology of the group G as a topological space and that of BG. So the group (co)homology of G will always be thought of and written as the (co)homology of

[3] This is because the twisting for the mapping class group, as explained in the previous footnote, does not square to the identity; it is only a braiding.

the space BG which, as for discrete groups, is the quotient EG/G of a contractible space EG with a continuous, free G action.

3.1. Moduli spaces and characteristic classes. Our interest in groups and their (co)homology comes most often from an interpretation of the group G as the automorphism group $\mathrm{Aut}(W)$ of some geometric object W. We like the model for $B\mathrm{Aut}(W)$ to be a (topological) moduli space in the sense that

(i) the points in $B\mathrm{Aut}(W)$ are representing objects isomorphic to W, and

(ii) any family of objects isomorphic to W and indexed by a space X corresponds to a continuous map $f : X \to B\mathrm{Aut}(W)$.

Such a family E_f is called a *W-bundle* over X. The simplest example of such a bundle is the Cartesian product $W \times X$ which corresponds to the trivial map that sends every point in X to the point in $B\mathrm{Aut}(W)$ that represents W. If f is the trivial map, then so is the map f^* in cohomology. More generally, however, the elements in $f^*(H^*(BAut(W)))$, which we call the *characteristic classes* of E_f, will give information about how twisted the family E_f. It may be helpful to recall the well-known theory of characteristic classes for vector bundles.

Example: To be concrete, take $W = \mathbb{C}^n$. Then $\mathrm{Aut}(W) = \mathrm{GL}(n, \mathbb{C})$ is the group of invertible, linear maps. A good model for $B\mathrm{GL}(n, \mathbb{C})$ is the complex Grassmannian manifold $\mathrm{Gr}^{\mathbb{C}}(n, \infty)$ of n dimensional $\mathbb{C}$- linear subspaces of $\mathbb{C}^\infty := \bigcup_n \mathbb{C}^n$. This is a moduli space in the above sense. Its cohomology is well-known to be

$$H^*(B\mathrm{GL}(n, \mathbb{C})) = \mathbb{Z}[c_1, c_2, \ldots, c_n]$$

where the c_i are the universal Chern classes of degree $2i$. For real vector spaces we have (the historically earlier) Pontryagin classes and Stiefel-Whitney classes.

These characteristic classes for vector bundles, which were discovered in the first half of the last century, have played a central role in the development of topology and geometry ever since. It is natural to ask,

What are the characteristic classes of bundles for more general W?

We might like to take W to be a compact manifold and its group of diffeomorphisms, or a finite simplicial complex and its group (up to homotopy) of homotopy equivalences. When W is a circle, every orientation preserving diffeomorphism is homotopic to a rotation which in turn is homotopic to $\mathrm{GL}(1, \mathbb{C})$. Thus the ring of characteristic classes for S^1-bundles is $\mathbb{Z}[c_1]$. With the proof of Mumford's conjecture and Galatius' theorem, we now also understand the characteristic classes, at least in the stable range, for W an oriented surface and W a simplicial complex of dimension one, as we will now explain.

3.2. Characteristic classes for manifold bundles. We take W to be an oriented, compact, smooth surface and its automorphism group to be the topological group $\mathrm{Diff}^+(W; \partial)$ of orientation preserving diffeomorphisms (which fix the boundary pointwise, if not empty). To construct a topological moduli space consider the space $\mathrm{Emb}(W, \mathbb{R}^\infty)$ of smooth embeddings of W in infinite dimensional Euclidean space. One may think of this as the space of embedded and parameterized surfaces of type W in $\mathbb{R}^\infty$. It is a consequence of Whitney's embedding theorem that this space is contractible. The diffeomorphism group $\mathrm{Diff}^+(W;\partial)$ acts freely on it by precomposing an embedding by a diffeomorphism. The quotient space is the (topological) moduli space for W

$$\mathcal{M}^{top}(W) := \mathrm{Emb}(W, \mathbb{R}^\infty)/\mathrm{Diff}^+(W;\partial) = B\mathrm{Diff}^+(W;\partial).$$

A point in it is a surface in $\mathbb{R}^\infty$. As we already mentioned, if $W = S_{g,1}$ or S_g and $g > 1$ then the diffeomorphism group is homotopic to the mapping class group and hence we have homotopy equivalences

$$\mathcal{M}^{top}(S_{g,1}) \simeq B\Gamma_{g,1} \quad \text{and} \quad \mathcal{M}^{top}(S_g) \simeq B\Gamma_g.$$

3.2.1. Relation to moduli spaces of Riemann surfaces. The homology of the spaces above are also of particular interest to algebraic geometers. The moduli space of Riemann surfaces $\mathcal{M}_g$ of genus g is the quotient of Teichmüller space, which is well-known to be homeomorphic to $\mathbb{R}^{6g-6}$, by the action of the mapping class group Γ_g. Mumford showed that it is a projective variety and moduli space for complex curves. The points of Teichmüller space are Riemann surfaces (with a homotopy class of a homeomorphism to a fixed surface S_g). As a Riemann surface can have at most finitely many automorphisms, the action of the mapping class group has at most finite stabilizers. It follows that rationally the (co)homology of $\mathcal{M}_g$ is the same as $B\Gamma_g$, and hence

$$H_*(\mathcal{M}_g) \otimes \mathbb{Q} \simeq H_*(\mathcal{M}_g^{top}) \otimes \mathbb{Q}.$$

In the early 1980's Mumford [Mu83] constructed characteristic classes κ_i for the $\mathcal{M}_g$ and initiated the systematic study of its cohomology ring. These classes were later also studied by Miller (and independently Morita) in the topological setting who showed that

$$H^*(B\Gamma_\infty) \otimes \mathbb{Q} \supset \mathbb{Q}[\kappa_1, \kappa_2, \dots].$$

The proof in [Mi86] uses Harer's homology stability as well as the commutativity of the product structure on the homology as describe above in Section 2.4. In the light of this, Mumford conjectured that the inclusion above is indeed an isomorphism. This is now a theorem by Madsen and Weiss [MW07]. Indeed, they prove a much stronger statement which was first conjectured in [MT01].

Theorem [MW07]. $\Omega B(\coprod_g B\mathit{Diff}^+(S_{g,1};\partial)) \simeq \Omega^\infty \mathbf{MTSO}(2)$.

Stringing several homotopy equivalences together, we see that the space on the left hand side has the homology of $\mathbb{Z} \times B\Gamma_\infty$ by the group completion theorem. We will now define the space on the right hand side and determine its rational cohomology.

3.2.2. Thom spaces and their rational cohomology. Let $\mathrm{Gr}^+(d,n)$ be the Grassmannian manifold of oriented $\mathbb{R}$-linear d planes in $\mathbb{R}^{d+n}$. There are two canonical vector bundles over $\mathrm{Gr}^+(d,n)$: the canonical d-bundle $\gamma_{d,n}$ with fibers over a plane $P \in \mathrm{Gr}^+(d,n)$ the vectors in P and its orthogonal complement $\gamma_{d,n}^\perp$. We will only use $\gamma_{d,n}^\perp$ and its one-point compactification $(\gamma_{d,n}^\perp)^c$, also known as the Thom space of $\gamma_{d,n}^\perp$. Using the embeddings $\mathrm{Gr}^+(d,n) \to \mathrm{Gr}^+(d,n+1)$ we can form a limit space

$$\Omega^\infty \mathbf{MTSO}(d) := \lim_{n\to\infty} \Omega^{d+n}(\gamma_{d,n}^\perp)^c;$$

here $\Omega^k X$ denotes the space of maps from S^k to X that take the point at infinity of $S^k = (\mathbb{R}^k)^c$ to the base point in X. The rational (co)homology of these spaces are well-understood and can be computed by standard methods in algebraic topology. For a connected component (they are all homotopic), we have

$$H^*(\Omega_0^\infty \mathbf{MTSO}(d)) \otimes \mathbb{Q} = \Lambda(H^{>d}(BSO(d))[-d] \otimes \mathbb{Q});$$

here $\Lambda(V^*)$ for a graded vector space V^* denotes the free graded commutative algebra on V^*. The V^* in question here is given by $V^n = H^{d+n}(BSO(d)) \otimes \mathbb{Q}$. As the rational classes of $H^*(BSO(d))$ are all even, this is just a polynomial algebra. Mumford's conjecture thus follows immediately.

Corollary. $H^*(B\Gamma_\infty) \otimes \mathbb{Q} = \mathbb{Q}[\kappa_1, \kappa_2, \ldots]$.

Madsen and Weiss' theorem above however gives much more information. Thus one is able to determine the divisibility of the κ_i in the integral lattice in $H^*(B\Gamma_\infty)$. In [GMT06] we show that the maximal divisor of κ_{2i} is 2 and that of κ_{2i-1} is the denominator of the $B_i/2i$ where B_i is the ith Bernoulli number.

$\Omega^\infty \mathbf{MTSO}(2)$ has also a vast number of torsion classes, see [MT97], [G04]. Indeed, for every even integer there is an infinite family of torsion homology classes (each essentially a copy of $H_*(B\Sigma_\infty)$). These had not been detected before except for the first family, see [CL84].

3.3. Characteristic classes for graphs. In analogy to the above, Galatius [G] considers a moduli space $\mathcal{G}_n(\mathbb{R}^\infty)$ of embedded finite graphs in $\mathbb{R}^\infty$ that have fundamental group F_n for a fixed n. Its topology is such that the collapse of a (non-loop) edge can be achieved by a continuous path. Thus $\mathcal{G}_n(\mathbb{R}^\infty)$ is connected.[4] Similarly, one can define a based version $\mathcal{G}_n(\mathbb{R}^\infty;*)$ where each graph has a vertex at the origin. Using ideas from Igusa [I02] and Culler-Vogtmann's Outer space [CV86], Galatius proves

$$\mathcal{G}_n(\mathbb{R}^\infty;*) \simeq B\mathrm{Aut}F_n \quad \text{and} \quad \mathcal{G}_n(\mathbb{R}^\infty) \simeq B\mathrm{Out}F_n.$$

We will state now Galatius' theorem in analogue to Madsen and Weiss' theorem on the space level.

[4]The topology is somewhat delicate. In particular one wants the graph to be imbedded in such a way that every point in $\mathbb{R}^\infty$ has a neighborhood which either does not intersect the graph, or intersects it in a small interval (part of an edge), or contains a neighborhood of a vertex (and no more). Collapsing an edge has to be done in such a way that this is always satisfied.

Theorem [G]. $\Omega B(\coprod_{n\geq 0} BAutF_n) \simeq \Omega^\infty S^\infty$.

Previous evidence for this came in the form of a remark by Hatcher in [H95] who proved that $\Omega^\infty S^\infty$ is a direct factor of the left hand side. Hatcher already raises the question whether it could be an equivalence and in particular whether the rational homology of $\mathrm{Aut}F_\infty$ is trivial. Hatcher and Vogtmann in [HV98] prove that $H_*(\mathrm{Aut}F_n)\otimes\mathbb{Q} = 0$ for $0 < * < 7$ with the exception of $H_4(\mathrm{Aut}F_4)\otimes\mathbb{Q} = \mathbb{Q}$ providing strong evidence for the cohomological conjecture. Further evidence for the conjecture on the space level is given by a theorem by Igusa [I02]. It says that the linearisation map $L : \mathrm{Aut}F_n \to \mathrm{GL}(n,\mathbb{Z})$ on classifying spaces and after group completion (i.e. application of ΩB) factors through $\Omega^\infty S^\infty$.

With the Barratt-Priddy-Quillen theorem recalled in 2.4, we see that $\Sigma_\infty \hookrightarrow \mathrm{Aut}F_\infty$ induces an isomorphism in homology and in particular

Corollary. $H_*(BAutF_\infty)\otimes\mathbb{Q} = \mathbb{Q}$ *concentrated in degree 0.*

4. Towards a proof

Our very rough sketch here treats simultaneously the Barratt-Priddy-Quillen theorem, the Madsen-Weiss theorem as well as Galatius' theorem. We mainly follow [G] (and in parts the conceptually closely related [GMTW09]). In particular, we emphasize the role of the scanning map.

4.1. Scanning. In abstract terms, the scanning map can be applied to topological moduli spaces where the points are representing an object W embedded in $\mathbb{R}^N$, for N large. The idea is that the scanning map at the point $W \subset \mathbb{R}^N$ records the local, microscopic picture as a magnifying glass sweeps through $\mathbb{R}^N$.

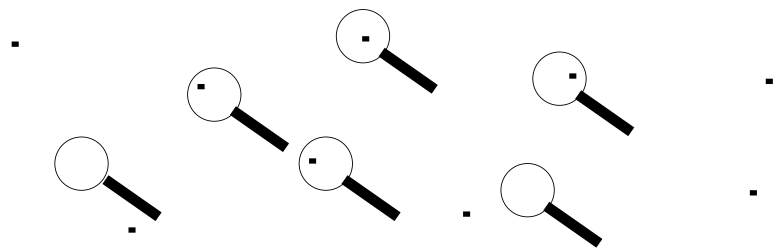

Figure 4: *A configuration of points in* $\mathbb{R}^N$ *sampled by a magnifying glass.*

We first consider the simplest and well known case when W is a set of n points. The ideas here go back to Segal and McDuff. The associated moduli space is the configuration space $\mathcal{C}_n(\mathbb{R}^N)$ of n distinct, unordered points in $\mathbb{R}^N$. When scanning, the lens of the magnifying glass can be taken small enough such that it only sees at most one point. Identifying the lens with a ball B^N in $\mathbb{R}^N$ we see that the space of all local pictures is just the sphere $S^N = (B^N)^c$ where the point at infinity corresponds to the lens giving us the view of the empty set. As the lens moves across $\mathbb{R}^N$ for a given configuration, we thus get a map from $\mathbb{R}^N$ to S^N. But away from a compact set containing the n points the lens sees nothing and the map is constant. We may therefore extend the map to $S^N = (\mathbb{R}^N)^c$ by sending the point at infinity to the empty set. Thus scanning defines a map

$$\coprod_n \mathcal{C}_n(\mathbb{R}^N) \longrightarrow \Omega^N S^N.$$

Similarly, scanning can be applied to the moduli space $\mathcal{M}^{top}(S_g)^N$ of surfaces of type S_g embedded in $\mathbb{R}^N$ This time, unless the lens sees nothing, it will see an oriented 2-plane intersecting B^N, which is the tangent plane T_x of the nearest point x on the surface to the center of the lens. Identifying B^N with $\mathbb{R}^N$, this defines a 2-dimensional subspace $T_x - x$ of $\mathbb{R}^N$ and a vector x perpendicular to it. Thus we see that the Thom space $(\gamma^{\perp}_{2,N-2})^c$ is the space of all local data with the point at infinity corresponding again to the empty lens. Thus for every point in $\mathbb{R}^N$, we get a point in $(\gamma^{\perp}_{2,N-2})^c$, and again we can extend this map continuously to the compactification $S^N = (\mathbb{R}^N)^c$. Thus scanning defines a map

$$\coprod_g \mathcal{M}^{top}(S_g)^N \longrightarrow \Omega^N (\gamma^{\perp}_{2,N-2})^c.$$

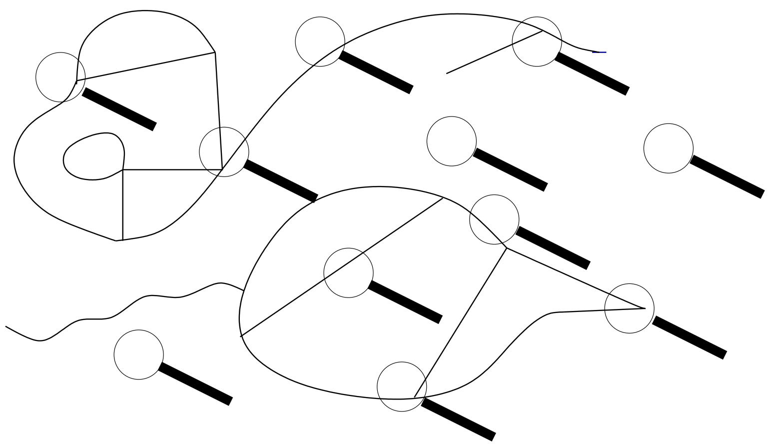

Figure 5: *A graph in $\mathbb{R}^N$ sampled by a magnifying glass.*

The case of graphs is similar, only that the space of all local data is much harder to identify. Indeed, Galatius spends considerable effort to show that a map from the N-sphere S^N to the space of local data in dimension N induces an isomorphism on homotopy groups in degrees $2N - c$ for some constant c. Thus in the limit as

$N \to \infty$ scanning defines (up to homotopy[5]) a map

$$\coprod_n \mathcal{G}_n(\mathbb{R}^\infty) \to \Omega^\infty S^\infty.$$

4.2. Spaces of manifolds and graphs. The above maps can of course not be homotopy equivalences. Not all path-components in the target are hit and the maps are not injective on fundamental groups. We need to enlarge our moduli spaces. Instead of considering only compact objects in $\mathbb{R}^N$ consider manifolds and graphs in $\mathbb{R}^N$ that may also be non-compact. So let $\Phi^{N,N}$ be the space of all manifolds W of a given dimension d or graphs G in $\mathbb{R}^N$. Each W respectively G has to be a closed subset of $\mathbb{R}^N$ but may extend to infinity. It also need not be connected. $\Phi^{N,N}$ is topologized in such a way that manifolds or graphs can be pushed continuously to infinity. The empty set is the basepoint. We have a filtration

$$\Phi^{N,N} \supset \cdots \supset \Phi^{N,1} \supset \Phi^{N,0} \simeq \coprod_n \mathcal{G}_n(\mathbb{R}^N),$$

where $\Phi^{N,i}$ contains only those W or G that are subsets of $\mathbb{R}^i \times (0,1)^{N-i}$. In particular, they are compact in the last $N-i$ coordinate directions.

To prove the theorems by Barratt-Priddy-Quillen, Madsen-Weiss or Galatius one would like to complete three steps. For configuration spaces this is an argument that essentially goes back to Segal [S73].

Step 1. $\Phi^{N,N}$ *is homotopic to the space of local data of the scanning map.*

This is relatively easy. The key here is that the topology of $\Phi^{N,N}$ allows us to push radially away from the origin. At the end of the homotopy what is left is the local data (at the origin).

Step 2. $\Phi^{N,k} \to \Omega\Phi^{N,k+1}$ *is a homotopy equivalence for* $k > 0$.[6]

We can construct a map as follows. For each t one can define a map $\Phi^{N,k} \to \Phi^{N,k+1}$ by sending W to its translate $W - te_{k+1}$ by t in the $(k+1)$st direction. So as t goes to infinity, W gets pushed out of sight and we can extend the map to $S^1 = \mathbb{R}^c$ by sending the point at infinity to the empty set. The case for graphs is just the same.

In the case of configuration spaces, it is straight forward to prove that this is a homotopy equivalence. But some extra argument is required in the case of higher dimensional manifolds and graphs.

Step 3. $\lim_{N\to\infty} \Phi^{N,1}$ *is homotopic to the classifying space of* $\coprod_n BAutW_n$.

Just as Step 2, this is straight forward for configuration spaces. More generally it is not difficult to see that in the manifold case, $\lim_{N\to\infty} \Phi^{N,1}$ is the classifying space

[5]More precisely, the target of the map is really something weakly homotopic to $\Omega^\infty S^\infty$.

[6]This statement is equivalent to saying that $\Phi^{N,k+1}$ is homotopic to $B\Phi^{N,k}$ and the connected components of $\Phi^{N,k}$ form a group. The product in $\Phi^{N,k}$ can be defined as follows: given two graphs W_1 and W_2, move W_2 to its translate $W_2 + e_{k+1}$. The resulting manifolds are disjoint and one can take the disjoint union. Using a homotopy $[0,2] \simeq [0,1]$ one can move the manifold back into $\Phi^{N,k}$. To show that $\Phi^{N,k+1} \simeq B\Phi^{N,k}$ is not too difficult. To show that the connected components from a group requres some argument. Note that it is here that we require the condition on k as the connected components certainly do not form a group when $k = 0$.

of the d-dimensional cobordism category as studied in topological field theory (see for example [GMTW09]; indeed this reproves the Main Theorem in that paper). Similarly, in the graph case, one sees that $\lim_{N\to\infty}\Phi^{N,1}$ is the classifying space of a cobordism category of graphs.

Though interesting in themselves, these identifications of $\lim_{N\to\infty}\Phi^{N,1}$ do not yet allow one to make deductions for $B\mathrm{Aut}W_\infty$ or $B\mathrm{Aut}W_n$. In order to do so, one wants to apply a group completion theorem following the arguments in [T97]. Two things are needed in order to apply it. First one needs to show that the classifying space of the cobordism category is homotopic to that of its subcategory in which the cobordisms are such that each component has non-empty out-going boundary. This is done in [GMTW09] for manifolds (of all dimensions ≥ 2) and in [G] for graphs. Secondly, one needs homology stability – which of course we have for graphs and surfaces.

We note here, that in [GRW] a proof of Madsen and Weiss' theorem in the form stated above is given that no longer uses homology stability. Indeed, Galatius and Randal-Williams show that the inclusion of the monoid $\coprod_n B\mathrm{Aut}W_n$ to the whole category induces a homotopy equivalence on classifying spaces and after group completion (i.e. applying ΩB).

5. Survey of further results

Madsen-Weiss' theorem as well as Galatius' theorem have been generalized in several directions. It is convenient to summarize some of these results in a table.

Σ_n	$(n+1)/2$	$\mathrm{Diff}(n \text{ pts})$	$\Omega^\infty S^\infty$
$\Gamma_{g,1}$	$(2g)/3$	$\mathrm{Diff}^+(S_{g,1};\partial)$	$\Omega^\infty\mathbf{MTSO}(2)$
$\mathcal{N}_{g,1}$	$(n-3)/3$	$\mathrm{Diff}(N_{g,1};\partial)$	$\Omega^\infty\mathbf{MTO}(2)$
.	.	$\mathrm{Diff}^+(\#_n S^k\times S^k\setminus \mathring{B}^{2k};\partial)$	$\Omega^\infty\mathbf{MTSO}(2k)^{\langle k\rangle}$
$\mathrm{Aut}F_n$	$(n-1)/2$	$\mathrm{HtEq}(G_n;*)$	$\Omega^\infty S^\infty$
$\mathcal{H}_{g,1}$	$(g-1)/2$	$\mathrm{Diff}^+(\#_g S^1\times D^2; D^2\subset\partial)$	$\Omega^\infty S^\infty BSO(3)_+$
.	.	$\mathrm{Diff}^+(\#_n S^1\times S^2\setminus \mathring{B}^3;\partial)$	$\Omega^\infty S^\infty BSO(4)_+$

The first column of the table gives the discrete group G_n to be considered; the second column lists the integer k so that the map $G_n\to G_{n+1}$ induces a homology isomorphisms in degrees less than k; the third column gives the automorphism group $\mathrm{Aut}W_n$ of the underlying geometric object; and finally, the fourth column contains a space homotopic to the group completion $\Omega B(\coprod_{n\geq 0} B\mathrm{Aut}W_n)$ and with the same homology as $\mathbb{Z}\times B\mathrm{Aut}W_\infty$.

We discuss now briefly the new entries .

5.1. Non-orientable surfaces. Let $N_{g,1}$ be an non-orientable surface of genus g (i.e. a connected sum of g copies of $\mathbb{R}P^2$s) with a boundary component. The group $\mathrm{Diff}(N_{g,1};\partial)$ has the same homotopy type as the associated mapping class group $\mathcal{N}_g$ by [EE69]. As for oriented surfaces, extending diffeomorphisms by the identity map over a glued on pair of pants surface induces a product $\mathcal{N}_{g,1}\times\mathcal{N}_{h,1}\to\mathcal{N}_{g+h,1}$. Wahl [W08] showed that $\mathcal{N}_{g,1}$ satisfies homology stability. Randal-Williams [RW] improved her range to $(n-3)/3$. (In this paper he also proves homology stability for surfaces with more exotic tangential structures (such as framed, spin and pin) which we have not listed in the above table.) The space $\Omega^\infty\mathbf{MTO}(2)$ is constructed just as in the oriented case only that the Grassmannian of non-oriented planes is considered. This gives the stable homology of the non-oriented mapping class group as

$$H_*(\mathcal{N}_\infty)\otimes\mathbb{Q} = \mathbb{Q}[\xi_i] \quad \text{with } \deg\xi_i = 4i.$$

5.2. A $(k-1)$-connected $2k$-manifold. The connected sum $\#_n S^k \times S^k$ of n copies of $S^k \times S^k$ is a higher dimensional analogue of a surface $S_g = \# S^1 \times S^1$. We cut an open ball $\overset{\circ}{B}{}^{2k}$ out of the manifold and demand that diffeomorphisms fix the boundary. Then a product can be constructed by gluing as for surfaces. At the moment we do not know whether the classifying spaces of the diffeomorphism groups satisfy homology stability. Nor do we know whether the connected components are contractible. Nevertheless, for $k>2$ Galatius and Randal-Williams have identified the group completion of the associated monoid with a slight modification of the space $\Omega^\infty\mathbf{MTSO}(2k)$. To define this space, consider the k-connected cover $\pi : \mathrm{Gr}^+(2k,n)\langle k\rangle \to \mathrm{Gr}^+(2k,n)$, and instead of the universal bundle $\gamma^\perp_{2k,n}$ use its pulled back. (For a space X, its 1-connected cover $X\langle 1\rangle \to X$ is just its universal cover. The k-connected cover $X\langle k\rangle \to X$ is just a generalization of this in that $X\langle k\rangle$ has trivial homotopy groups for $*\leq k$ and the same homotopy groups as X for $*>k$.) The space we are looking for is $\Omega^\infty\mathbf{MTSO}(2k)^{\langle k\rangle} := \lim_{n\to\infty}\Omega^{2k+n}(\pi^*(\gamma^\perp_{2k,n}))^c$.

5.3. Handlebody in dimension 3. Diffeomorphism of the 3-dimensional handlebody $\#_g S^1\times D^2$ of genus g restrict to diffeomorphisms of the boundary surface. Furthermore, its connected components are contractible. This is still the case when we fix a disk $D^2\subset\partial$ on the boundary. Thus its mapping class group $\mathcal{H}_{g,1}$ may be identified with a subgroup of $\Gamma_{g,1}$. Hatcher and Wahl [HW] proved that these groups satisfy homology stability. Very recently, Hatcher showed that the group completion in this case is

$$\Omega^\infty S^\infty BSO(3)_+ = \lim_{k\to\infty}\Omega^k S^k(BSO(3)_+)$$

by thinking of the handlebody as a thickened graph and adopting Galatius' proof. The proof uses another ingredient, the Smale conjecture (see [H83]) which states that $\mathrm{Diff}(D^3)\simeq \mathrm{O}(3)$. Here X_+ denotes X with a disjoint base point. In particular, this gives the stable homology of the handlebody mapping class group as

$$H_*(\mathcal{H}_\infty)\otimes\mathbb{Q} = \mathbb{Q}[\kappa_{2i}] \quad \text{with } \deg\kappa_{2i} = 4i.$$

5.4. A simple 3-dimensional manifold. Finally, the bottom line is also work by Hatcher, recently announced. It concerns the connected sum of n copies of $S^1\times S^2$ with a ball removed so that a product can be defined. Again Hatcher uses a modification of Galatius' argument for graphs and the Smale conjecture. Note however, that in this case the connected components of the diffeomorphism groups are not contractible and we do not know whether the classifying spaces of these groups satisfy homology stability.

6. Conclusion and future directions

We have seen that the method of scanning can be applied to topological moduli spaces $\mathcal{M}^{top}(W)$ of objects isomorphic to W embedded in $\mathbb{R}^\infty$. The target T of the scanning map is a highly structured space, an infinite loop space. In the case of zero dimensional manifolds, graphs and two dimensional manifolds the target of the scanning map T, in the presence of homology stability gives a better and better approximation to the homology of the moduli space $\mathcal{M}^{top}(W)$ as the complexity of W grows. In the previous section we encountered manifolds W of higher dimensions for which the scanning map induces a homotopy equivalence from the group completion of the associated monoid to the target T but for which we do not (yet) have homology stability.

One is naturally led to ask the following questions. Can the table above be completed and homology stability results be found for certain types of manifolds? Are there any other families of manifolds that can be added to the table? Indeed, are there other geometric objects, to which the scanning method can be applied. Galatius considered finite 1-dimensional complexes. Can the methods be pushed to higher dimensional finite complexes?

We emphasized the point of view of moduli spaces and characteristic classes for manifold bundles. Indeed, for every oriented, closed, d-dimensional manifolds W, scanning gives a map $\alpha : \mathcal{M}^{top}(W) \to \Omega^\infty \mathbf{MTSO}(d)$. So the cohomology of $\Omega^\infty \mathbf{MTSO}(d)$ provides characteristic classes for all oriented d-manifolds simultaneously. Ebert has shown that for d even every rational cohomology class c is detected by some manifold, i.e. $\alpha^*(c)$ is non-zero for some W. But this fails for d odd, see [E1] and [E2]. This suggests that the scanning map for d odd is not optimal and should factor through a space $X(d)$. For $d = 1$, $X(1) = \Omega^\infty S^\infty BSO(2)_+$ is the optimal space. Hatcher's last example suggests a similar solution for $d = 3$.

We have tried to give here a glimpse into an active area of research that uses new techniques to study basic questions in geometry and topology. Some of the questions above are already pursued, and we look forward to seeing the theory develop.

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