Artin’s map in stable homology

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Abstract: Using a recent theorem of Galatius [G] we identify the map on stable homology induced by Artin’s injection of the braid group $\beta_n$ into the automorphism group of the free group $\text{Aut} F_n$.

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1. Definitions and results.

Let $\beta_n$ be the braid group on $n$ strings and $\text{Aut} F_n$ the automorphism group of the free group $F_n$ on $n$ generators $x_1, \ldots, x_n$. Artin [A] identified $\beta_n$ as a subgroup of $\text{Aut} F_n$ as follows. Let $\sigma_i \in \beta_n$ denote a standard generator, the braid which crosses the $i$-th over the $(i+1)$-st string. Artin’s map
\[ \phi : \beta_n \longrightarrow \text{Aut} F_n \]
is defined by taking $\sigma_i$ to the automorphism
\[ \phi(\sigma_i) : x_j \mapsto \begin{cases} x_j & \text{if } j \neq i, i + 1 \\ x_{i+1} & \text{if } j = i \\ x_{i+1}^{-1} x_i x_{i+1} & \text{if } j = i + 1. \end{cases} \]
$\phi$ extends to a map from $\beta_\infty := \lim_{n \to \infty} \beta_n$ to $\text{Aut} F_\infty := \lim_{n \to \infty} \text{Aut} F_n$. We will describe this map of stable group on homology.

Theorem 1. $\phi_* : H_*(\beta_\infty ; k) \longrightarrow H_*(\text{Aut} F_\infty ; k)$ is trivial when $k = \mathbb{Q}$ or $k = \mathbb{Z}/p\mathbb{Z}$ for any odd prime $p$. It induces an injection on
\[ H_*(\beta_\infty ; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[x_i] \text{ deg } (x_i) = 2^i - 1. \]
Theorem 1 is a corollary of the stronger, homotopy theoretic Theorem 3 below. Another way to state our results in more algebraic terms is in comparison to another homomorphism defined as follows. An element of the symmetric group $\Sigma_n$ acts naturally by permutations of the generators on $F_n$. This defines an embedding $\Sigma_n \subset \text{Aut} F_n$. Precomposition with the natural surjection from the braid group to the symmetric group defines the homomorphism

$$\pi : \beta_n \longrightarrow \Sigma_n \subset \text{Aut} F_n.$$ 

As $\phi$, $\pi$ also commutes with limits and extends to a map of stable groups. Though these maps are very different, they induce the same map on homology.

**Theorem 2.** $\phi_* = \pi_* : H_*(\beta_\infty; \mathbb{Z}) \longrightarrow H_*(\text{Aut} F_\infty; \mathbb{Z})$.

**Remark 1:** Unlike for (co)homology with trivial coefficients, as considered in this note, for (co)homology with twisted coefficients $\phi_*$ can be non-trivial even rationally, cf. for example the recent work of Kawazumi [K].

**Remark 2:** There are many other homomorphisms from $\beta_n$ to $\text{Aut} F_n$. Those that factor through the mapping class group are likely to be trivial in homology. Indeed, in [ST] we show that many algebraically and geometrically defined homomorphisms from the braid group to the mapping class group are homologically trivial, and hence so will the composition to $\text{Aut} F_n$.

### 2. Translation into homotopy.

Juxtaposition of braids and disjoint union of sets respectively induce natural monoidal structures on the disjoint union of the classifying spaces $\coprod_{n \geq 0} B\beta_n$ and $\coprod_{n \geq 0} B\Sigma_n$. Their group completions can be identified respectively as

$$\mathbb{Z} \times B\beta_\infty^+ \cong \Omega^2 S^2$$

$$\mathbb{Z} \times B\Sigma_\infty^+ \cong \Omega^\infty S^\infty.$$
Here “+” denotes Quillen’s plus construction with respect to the maximal perfect subgroup of the fundamental group; \(\Omega^n S^n\) is the space of based maps from the \(n\)-sphere to itself and \(\Omega^\infty S^\infty := \lim_{n \to \infty} \Omega^n S^n\). Recently, Galatius [G] was able to prove an analogue of these results for the automorphism groups of free group:

\[
\mathbb{Z} \times \text{BAut}_n^+ \simeq \Omega^\infty S^\infty.
\]

Furthermore, the inclusion \(\Sigma_n \to \text{Aut}_n\) induces up to homotopy the identity map of \(\Omega^\infty S^\infty\), cf. also [H]. It is also well-known, cf. [CLM], [S], that the surjection \(\beta_n \to \Sigma_n\) induces on group completions up to homotopy the inclusion map \(\Omega^2 S^2 \to \Omega^\infty S^\infty\). As the plus construction does not change the homology of the space Theorem 2 is therefore equivalent to the following.

**Theorem 3.** On group completions \(\phi\) induces up to homotopy the natural inclusion map \(\Omega^2 S^2 \to \Omega^\infty S^\infty\).

**Proof of Theorem 1.** Rationally the homology of \(\Sigma_n\) and \(\Omega^\infty S^\infty\) is trivial, and hence \(\phi_*\) is trivial on rational homology. Recall, F. Cohen in [CLM] describes the homology of the braid group with \(\mathbb{Z}/p\mathbb{Z}\) coefficients for every prime \(p\) in terms of a one-dimensional generator \(x_1 \in H_1(\beta_\infty; \mathbb{Z}/p\mathbb{Z})\) and powers of homology operation applied to \(x_1\). Maps of double loop spaces commute with these homology operations. Hence, by Theorem 3, \(\phi_*\) commutes with them so that its image is determined by its value on \(x_1\). But

\[
H_1(\text{Aut}_\infty; \mathbb{Z}) = H_1(\Sigma_\infty; \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z},
\]

and hence it follows that \(\phi_*\) is zero in all positive dimensions for all odd \(p\). The class \(x_1 \in H_1(\beta_\infty; \mathbb{Z}) = \pi_1 \Omega^2 S^2 = \mathbb{Z}\) corresponds to the Hopf map \(S^3 \to S^2\) which maps under the inclusion map \(\Omega^2 S^2 \to \Omega^\infty S^\infty\) to the non-zero element in the first homology. It is also well-known that the homology operations act freely on the homology of \(\Omega^\infty S^\infty\), so that for \(p = 2\) the map \(\phi_*\) is an injection.
3. Proof of Theorem 3.

From the definition of $\phi(\sigma_i)$ it is clear that $\phi$ acts on the abelianisation of $F_n$ by permutation of the generators. Hence, we have the following commutative diagram of groups:

$$
\begin{array}{ccc}
\beta_n & \xrightarrow{\phi} & \text{Aut} F_n \\
\downarrow{\pi} & & \downarrow{L} \\
\Sigma_n & \longrightarrow & \text{GL}_n \mathbb{Z}.
\end{array}
$$

For a based topological space $X$, let $\mathcal{HE}(X)$ denote the topological monoid of its based homotopy equivalences. Consider the following commutative diagram:

$$
\begin{array}{ccc}
\text{Aut} F_n & \xleftarrow{\pi_0} & \mathcal{HE}(\bigvee_n S^1) \\
\downarrow{L} & & \downarrow \\
\text{GL}_n \mathbb{Z} & \xleftarrow{\pi_0} & \lim_k \mathcal{HE}(\bigvee_n S^k).
\end{array}
$$

The horizontal arrows $\pi_0$ are defined by taking connected components, and the top one is well-known to be a homotopy equivalence. Furthermore, $\Sigma_n$ acts naturally on $\bigvee_n S^k$ by permutation of the summands in the wedge product. Hence the map $\Sigma_n \to \text{GL}_n(\mathbb{Z})$ lifts to $\mathcal{HE}(\bigvee_n S^k)$. Thus on classifying spaces we yield the commutative diagram:

$$
\begin{array}{ccc}
B\beta_n & \xrightarrow{\phi} & B\text{Aut} F_n \\
\downarrow{\pi} & & \downarrow \\
B\Sigma_n & \longrightarrow & B\lim_k \mathcal{HE}(\bigvee_n S^k)
\end{array}
$$

The union over all $n \geq 0$ for each of the four spaces in the above diagram is a monoid, and all maps commute with the monoidal product. After group completion, we thus have:

$$
\begin{array}{ccc}
\mathbb{Z} \times B\beta_\infty^+ & \xrightarrow{\phi} & \mathbb{Z} \times B\text{Aut} F_\infty^+ \\
\downarrow{\pi} & & \downarrow \\
\mathbb{Z} \times B\Sigma_\infty^+ & \longrightarrow & \mathbb{Z} \times B\lim_k \mathcal{HE}(\bigvee_\infty S^k)^+.
\end{array}
$$

The space in the bottom right corner is Waldhausen’s K-theory of a point, $A(*)$, and the bottom horizontal map is split by his trace map $tr : A(*) \to \Omega^\infty S^\infty$, cf. [W]. By Galatius’ result [G] quoted above,

$$
\mathbb{Z} \times B\text{Aut} F_\infty^+ \longrightarrow A(*) \xrightarrow{tr} \Omega^\infty S^\infty
$$
is a homotopy equivalence. Our final commutative diagram implies Theorem 3 (and Theorem 1) immediately:

$$
\begin{array}{ccc}
Z \times B\beta^+ & \xrightarrow{\phi} & Z \times \text{Aut}_F^+ \\
\pi & \downarrow & \simeq \\
Z \times B\Sigma^+ & \rightarrow & \Omega^\infty S^\infty.
\end{array}
$$

Remark 3: The injection $\phi$ defines a braided monoidal structure on the monoid $\coprod_{n \geq 0} B\text{Aut}_F$ in the sense of [F], cf. also [SW]. Hence it induces a map of double loop spaces on group completions. Any double loop map from $\Omega^2 S^2$ is determined by its image on $S^0$. However, apriori, it is not clear whether the induced double loop space structure on the group completion of $\coprod_{n \geq 0} B\text{Aut}_F$ is homotopic to the usual one on $\Omega^\infty S^\infty$. This does therefore not lead to an alternative proof of Theorem 3. On the other hand, Theorem 3 implies that the two double loop space structures on $\Omega^\infty S^\infty$ are indeed homotopic.

References.


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