Algebraic Topology
Hilary 2007
Question Sheet 7

Künneth Formula, manifolds, Poincaré duality.

1. Let $T^n = S^1 \times \ldots \times S^1$ be the n-dimensional torus. Show that $H_k T^n \simeq \mathbb{Z}^{b_k}$ where $b_k = C^n_k = n!/k!(n-k)!$.

2. Let M be a closed, connected, orientable manifold of dimension n.

   (i) Show that $H_{n-1}M$ is torsion free. [*Use Poincaré duality and the Universal Coefficient Theorem.*]

   (ii) Show that the rank of $H_{n-k}M$ is equal to that of $H_k M$.

   (iii) Show that the Euler characteristic of any odd dimensional manifold is zero.

3. Let M and N be two closed manifolds. Show that M and N are orientable if and only if $M \times N$ is orientable.

4. For a map $f : M \to N$ between connected, orientable, closed manifolds of dimension n with fundamental classes $[M]$ and $[N]$, the degree of f is defined to be the integer $deg(f)$ such that $f_*([M]) = deg(f)[N]$. (Note the sign of the degree depends on the choice of fundamental classes.) Show that for any closed, connected, orientable manifold M of dimension n there is a degree 1 map $M \to S^n$.

5. Check you answers to question 2, problem sheet 6 using the Künneth Theorem.