Algebraic Topology
Michaelmas 2013
Question Sheet 1

$\triangle$-complexes, homology of $\triangle$-complexes with field and integer coefficients, Euler characteristic, exact sequences.

1. Find $\triangle$-complexes homeomorphic to S^1, S^2, $\mathbb{R}P^2$ and the torus. For each $\triangle$-complex, compute its homology.

2. Let X be a $\triangle$-complex, F be a field and $C_n^{\triangle}(X;F)$ denote the vector space over F with basis the n-simplices of X. Prove that $(C_\bullet^{\triangle}(X;F), \partial_\bullet)$ is a chain complex where $\partial_\bullet$ is defined as in the definition of simplicial homology. Denote its homology by $H_*(X;F)$.

   Compute $H_*(K;F)$ for a $\triangle$-complex homeomorphic to the Klein bottle and $F = \mathbb{R}, \mathbb{Q}$, or $\mathbb{F}_p$ where p is a prime number and $\mathbb{F}_p$ denotes the field with p elements. [*Watch out for $p = 2$!*]

   What is the relation between $H_*(X;\mathbb{R})$ and $H_*(X;\mathbb{Q})$? Can you say anything about the relation between $H_*(X;\mathbb{R})$ and $H_*(X;\mathbb{Z})$?

3. The Euler characteristic $\chi(X)$ of a finite $\triangle$-complex X is $\Sigma_{n\geq 0}(-1)^n\beta_n$ where β_n is the number of n simplices. [*Note, this generalizes the definition of the Euler characteristic for surfaces given by $V-E+F$.*] For a simplicial complex X define the number $E_{\mathbb{R}}(X) := \Sigma_{n\geq 0}(-1)^n \dim H_n(X;\mathbb{R})$. Prove that

$$E_{\mathbb{R}}(X) = \chi(X).$$

   Is the same true if $\mathbb{R}$ is replaced by any other field?

4. For each of the following exact sequences of abelian groups and homomorphisms, say as much as you can about the unknown group G and/or the unknown homomorphism α:

   (i) $0 \longrightarrow \mathbb{Z}_3 \longrightarrow G \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \longrightarrow 0.$

   (ii) $0 \longrightarrow G \longrightarrow \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \longrightarrow \mathbb{Z}_2 \longrightarrow 0.$