Algebraic Topology
Michaelmas 2013
Question Sheet 2

Singular homology, naturallity and homotopy invariance; extensions of abelian groups; relation between $\pi_1(X;b)$ and $H_1(X;\mathbb{Z})$ (optional).

1. Using simplicial and/or singular homology (which are indeed isomorphic) compute the homology of (i) $S^1 \vee S^1$; (ii) $\mathbb{R}^2 - \{0\}$; (iii) the Möbius band; and (iv) $\mathbb{R}^2 - \{0,1\}$.

2. Let $x_1, x_2, \ldots, x_k$ be points of a path-connected space X, and $n_1, n_2, \ldots, n_k$ be integers, so that $a = \Sigma_{i=1}^k n_i x_i \in C_0(X)$ is a singular 0-chain. Show that there exists a 1-chain b with $\partial b = a$ if and only if $\Sigma_{i=1}^k n_i = 0$. Deduce that $H_0(X)$ is isomorphic to $\mathbb{Z}$. Describe $H_0(Y)$ for arbitrary spaces Y.

3. An exact sequence of abelian groups $0 \longrightarrow G \xrightarrow{\alpha} H \xrightarrow{\beta} K \longrightarrow 0$ *splits* if either

   (i) there is a homomorphism $\gamma\colon K \to H$ such that $\beta \circ \gamma = \mathrm{id}_K$, or

   (ii) there is a homomorphism $\delta\colon H \to G$ such that $\delta \circ \alpha = \mathrm{id}_G$.

   Show that in this case $H \cong G \oplus K$.

   Show that if $K = \mathbb{Z}^n$, then the sequence always splits. Deduce that if $0 \to \mathbb{Z}/2\mathbb{Z} \to G \to \mathbb{Z} \to 0$ is exact, then G is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}$. If $0 \to \mathbb{Z} \to G \to \mathbb{Z}/2\mathbb{Z} \to 0$ is exact, show that G does not have to be isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}$.

4. Define $\tilde{H}_0(X)$ to be the kernel of the map induced by the canonical map from X to a point p. For a continuous map $f : X \to Y$ show that f_* restricts to a map from $\tilde{H}_0(X)$ to $\tilde{H}_0(Y)$. Proof that $H_0(X) \simeq \tilde{H}_0(X) \oplus \mathbb{Z}$.

5. Let $i : A. \to B.$ be an inclusion of chain complexes.

   (i) Prove that the boundary in $B.$ induces a boundary map $B_n/A_n \to B_{n-1}/A_{n-1}$ so that the quotients B_n/A_n form again a chain complex.

   (ii) Define the connecting homomorphism $\delta : H_n(B./A.) \to H_{n-1}(A.)$ in this context, and prove exactness of the resulting long sequence at $H_n(B.)$.

6. (Optional) Let γ, δ be two loops based at $x_0 \in X$ which are path-homotopic via the homotopy F. By collapsing the right-hand edge of the square domain of $F : I \times I \to X$, construct a 2-simplex $\sigma : \triangle^2 \to X$ such that $\partial\sigma = \gamma - \delta + \epsilon$, where ϵ is the constant 1-simplex that maps all of $\triangle^1$ to the point x_0. Deduce that there is a well-defined map $\psi : \pi_1(X, x_0) \to H_1(X)$ with $\psi([\gamma]) = [\gamma]$.