Algebraic Topology
Michaelmas 2013
Question Sheet 3

Long exact relative homology sequence, excision, Five Lemma.

1. Let $X = S^1 \times S^1$ be the torus, and $A = S^1 \times \{1\}$. Compute $H_*(X, A)$.

2. (Naturallity of the connecting homomorphism) Let $f\colon (X, A) \to (Y, B)$ be a map of pairs, i.e. f is a continuous map from X to Y such that $f(A) \subset B$. Let δ denote the connecting homomorphism in the relative homology long exact sequence. Prove that $f_* \circ \delta = \delta \circ f_*$, and hence that f induces a map of long exact sequences.

   Prove that if any two of the three maps $H_*(X) \to H_*(Y)$, $H_*(A) \to H_*(B)$ and $H_*(X, A) \to H_*(Y, B)$ induced by f are isomorphisms then so is the third. Hence, prove that the inclusion $(D^k, S^{k-1}) \to (D^k, D^k - \{0\})$ induces an isomorphism on homology groups.

3. If A is a retract of X and (X, A) is good, show that $\tilde{H}_n(X) \simeq \tilde{H}_n(A) \oplus \tilde{H}_n(X/A)$ for all $n \geq 0$. Deduce that $\tilde{H}_n(X \vee Y) = \tilde{H}_n(X) \oplus \tilde{H}_n(Y)$ for all $n \geq 0$ when the base point has a contractible neighborhood in Y or X.

4. Let Y be obtained from X by attaching an m-dimensional disk $D^m, m > 0$ via the continuous map $f : S^{m-1} \to X$. In other words, Y is the quotient space of $X \cup D^m$ where points $x \in \partial D^m = S^{m-1}$ are identified with their image $f(x) \in X$. Prove that $H_n(Y)$ is equal to

   $H_{m-1}(X)/\mathrm{Im}\ (f_*)$ if $n = m - 1$;

   $H_m(X) \oplus \mathrm{Ker}\ (f_*)$ if $n = m$;

   $H_n(X)$ otherwise.

   [*Consider the long exact sequence for the pair* (Y, V) *where* V *is the complement of a closed disk in the interior of* D^m.]

5. Compute the homology of an oriented surface F_g of genus g, which can be constructed from a $4g$-gon with sides labeled

$$a_1, b_1, a_1^{-1}, b_1^{-1}, \ldots, a_g, b_g, a_g^{-1}, b_g^{-1}.$$

   Also compute the homology of a non-orientable surface N_g of genus g, which can can be constructed from a $2g$-gon with sides labeled

$$a_1, a_1, a_2, a_2, \ldots, a_g, a_g.$$

   Finally, compute the homology of $F_{g,n}$ and $N_{g.n}$, where $F_{g,n}$ and $N_{g,n}$ denote the surfaces F_g and N_g with n punctures.