Notes of a Numerical Analyst

Randomness and Certainty

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Here is a curious symmetry. To achieve randomness, our standard strategy is exponentials. You can toss a coin, but the outcome isn't so random because it is only algebraically sensitive to the initial conditions. For truer randomness you need a chaotic system with exponential sensitivities, like the logistic equation or the Lorenz equations or a pinball machine (Figure 1). Run such a system for a moment and your randomness might be 99%. If that's not enough, run it a little longer to get 99.99%. With each step, your knowledge of the state of the system shrinks by a multiplicative factor, soon reaching zero for practical purposes.

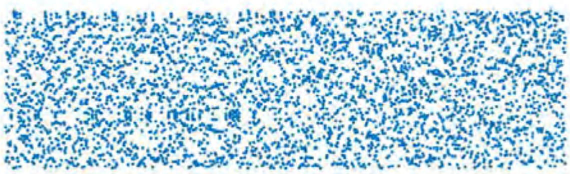

Figure 1. 4000 numbers $y_k \in [0,1]$. The first 2000 are the values $\pi^{-1}\cos^{-1}(y_k)$ for 2000 samples of the logistic equation $y_{n+1} = 1 - 2y_n^2$. The next 2000 are uniformly distributed random numbers in $[0,1]$. One can hardly tell the difference.

And to achieve certainty, our standard strategy is exponentials again! At the level of physics, almost anything can happen because of quantum tunnelling. But some things "never" happen in practice, such as the radioactive decay of an iron-56 atom or an error in a integrated circuit gate. Why? Because the frequency of quantum events shrinks exponentially with the width of a potential barrier. With each slight thickening of a circuit element, your uncertainty about the state of the system shrinks by a multiplicative factor, soon reaching zero for practical purposes.

Figure 2 illustrates the approach to certainty with a solution to the bistable ODE

$$y' = y - y^3 + Cf(t) \tag{1}$$

driven by f, a smooth random function which approximates standard white noise. Transition from one metastable state to the other depends, loosely speaking, on a succession of random impulses happening to align. As C is reduced, the odds of sufficient alignment shrink exponentially and the half-lives increase to ∞.

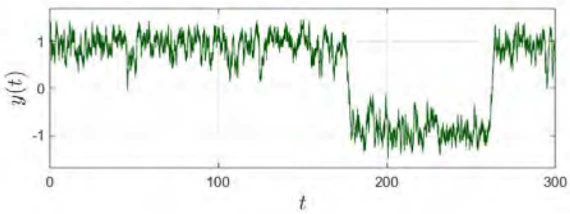

Figure 2. Solution to the random ODE (1) with $C = 0.4$. By reducing C, one lengthens the metastable periods exponentially.

These principles have roots long before chaos theory or quantum mechanics. Think of a combination lock with wheels from 0 to 9. Three wheels gives some security, but four or five are better. The challenge of risk assessment ranges from simple stories like this to large deviations, anomalous diffusion, fat tails and black swans.

Figure 1 generates randomness from certainty, and Figure 2 generates certainty from randomness—or would, with a smaller value of C. The fact that one happens to be set in discrete time and the other in continuous time is not important.

FURTHER READING

[1] P. Hänggi, P. Talkner and M. Borkovec, Reaction-rate theory: fifty years after Kramers, *Rev. Modern Phys.* 62 (1990), 251–341.
[2] G. 't Hooft and S. Vandoren, *Time in Powers of Ten: Natural Phenomena and Their Timescales*, World Scientific, 2014.

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