Contents

Preface to the Extended Edition

Approximation Theory and Approximation Practice (*“ATAP”*), originally published in 2013, concerns approximation of nonperiodic functions on the interval $[-1,1]$, the *Chebyshev* setting of constructive analysis. But this is just one of three essentially equivalent situations:

Chebyshev, for nonperiodic functions of $x \in [-1,1]$,

Fourier, for periodic functions of $\theta \in [-\pi,\pi]$,

Laurent/Taylor, for functions of z on the unit circle $|z| = 1$.

As discussed on p. 14, my original plan had been to give equal treatment to all three settings, but it became clear that a book written to that plan would be ponderous. So Chebyshev became the subject, since it was the most important of the three for numerical computation and also the least understood.

In this extended edition, we have found a way to give a kind of summary of the parallel mathematics of the Fourier and Laurent/Taylor cases without changing the main text. We have added appendices reprinting two papers published recently in SIAM journals:

Appendix B. Fourier.
G. B. Wright, M. Javed, H. Montanelli and L. N. Trefethen (2015), Extension of Chebfun to periodic functions, *SIAM J. Sci. Comput.* 37, C554–C573.

Appendix C. Laurent/Taylor.
A. P. Austin, P. Kravanja and L. N. Trefethen (2014), Numerical algorithms based on analytic function values at roots of unity, *SIAM J. Numer. Anal.* 52, 1795–1821.

In the first of these papers, readers will see a list of 10 parallels between Fourier and Chebyshev spelled out in Table 1, followed by a discussion of 10 respects in which the two cases differ. These differences are not deep mathematically (there are no deep mathematical differences between Fourier and Chebyshev!), but they involve conceptual twists. Readers may also enjoy Figure 9, highlighting the continuity from Runge in 1904 to his descendants in 2015. One may think of Wright et al. as a kind of "*ATAP* for periodic functions." Specifically, it is noted on p. 275 that Section 2 of the paper corresponds to Chapter 3 of this book, Section 3 to Chapters 2, 4, and 5, and Section 4 to Chapters 6, 7, 8, 10, and 19.

In the second paper, a notable feature of the presentation is a sequence of 13 short MATLAB code segments (summarized on pp. 320–321), many just one line long, for interpolation, differentiation, integration, and zero- and polefinding from data values at roots of unity. The links to Chapters 2–5, 18, and 19 of this book are very close. The history of this mathematics is collected in Figure 1 of the paper, and Figure 4 shows a striking image related to the Jentzsch, Walsh, and Blatt–Saff theorems discussed here on pp. 140–141. Austin et al. is a kind of "*ATAP* for the unit disk."

Since the first edition of *ATAP* was published, the use of Chebfun has grown around the world. The code was completely rewritten during 2013–14 by a team led by Nick Hale, and version 5 was released in June 2014. By the combined efforts of Alex Townsend, Behnam Hashemi, Grady Wright, Heather Wilber, and Nicolas Boullé, Chebfun has been expanded to compute with functions not only on intervals but also on rectangles, boxes, spheres, disks, and balls. Multidimensional approximation is not treated in this book, but it is a hot topic in the era of big data, and these new capabilities of Chebfun make use of low-rank compression ideas that are familiar to today's engineers and data scientists.

I have taken the opportunity of the new edition to correct a number of errors pointed out by readers and colleagues, of whom I would like in particular to acknowledge Folkmar Bornemann, Behnam Hashemi, Mohsin Javed, Yuji Nakatsukasa, Grady Wright, and Kuan Xu. The corrections have included updates to Chebfun syntax such as the replacement of `interp1`, `chebpolyplot`, and `chebellipseplot` by `chebfun.interp1`, `plotcoeffs`, and `plotregion`, respectively.

Although this is not a thoroughgoing revision of *ATAP,* about a dozen new references have been added. Let me mention two areas of particularly interesting developments since 2013. One is Chebyshev spectral discretization of ordinary differential equations, a subject reflected in the new SIAM book *Exploring ODEs* [Trefethen, Birkisson & Driscoll 2018]. *ATAP* and *Exploring ODEs* are siblings, two mathematical textbooks based on Chebfun, and a PDF file of the latter is freely available at `http://people.maths.ox.ac.uk/trefethen/ExplODE`. Chebfun now makes systematic use of the rectangular spectral discretizations introduced by Driscoll and Hale [2015]; a leader of this effort was Ásgeir Birkisson. Details can be found in [Aurentz & Trefethen 2017a] and in Appendix A of *Exploring ODEs.*

The other area to highlight is rational approximation. As described in the last six chapters of this book, rational approximations are particularly effective for functions that have singularities since they can cluster poles and zeros near them. On the other hand, they are traditionally hard to compute. A few years after this book was published, a flexible new method for computing rational approximations on arbitrary real or complex domains, the *AAA algorithm,* was introduced by Nakatsukasa, Sète, and myself [2018]. The AAA algorithm gets its power from combining a barycentric rational representation with an adaptive choice of support points. Most recently, attempts to extend AAA have led to the introduction of *lightning Laplace solvers,* which exploit rational functions to solve certain partial differential equations with remarkable speed and accuracy on regions with corners, where solutions almost invariably have singularities [Gopal & Trefethen 2019a,b]. The approximation theory underlying these new methods is built on generalizations for rational functions of the Hermite integral formula and the potential theory presented in Chapters 11–12 for polynomials. Together, these encouraging developments in the numerical use of rational functions suggest, as we like to say (see the references labeled 1885a and 1964), that "Runge" is joining forces with "Newman."