Notes of a Numerical Analyst

Designer Non-uniqueness

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I've been teaching ODEs, and we show students that some problems have non-unique solutions. For example,

$$y' = y^{1/2}, \qquad y(0) = 0 \tag{1}$$

is satisfied for any $t_0 \geq 0$ by

$$y(t) = \begin{cases} 0, & t \leq t_0, \\ \frac{1}{4}(t - t_0)^2, & t \geq t_0. \end{cases} \tag{2}$$

In reversed time, this gives a consequence of Torricelli's Law of 1643: if a leaky bucket is empty, you can't tell when the last drop drained away. The reason is that the fundamental existence and uniqueness theorem for $y' = f(t, y)$ assumes that f is Lipschitz continuous with respect to y, which does not hold for $f(t, y) = y^{1/2}$.

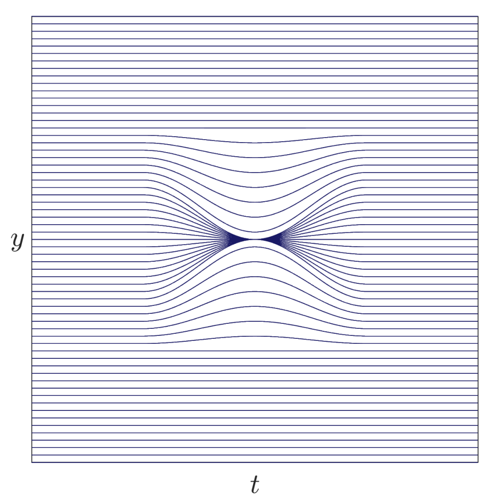

Figure 1. A flow field with a pinch point gives non-unique solutions to an ODE at that point.

What may seem surprising for an ODE becomes elementary when you plot the field of solution curves $(t, y(t))$. A point of non-uniqueness is just a point where the curves pinch together, as in Fig. 1. The set of all these trajectories defines an ODE in which $f(t, y)$ is simply the value $y'(t)$ at each point.

Once we note that non-uniqueness is a matter of pinch points, we can design ODEs with non-uniqueness wherever we like. One idea, going back to Lavrentieff and Hartman [1, 2], is to have a dense infinity of pinch points with decreasing spatial scales. Figure 2 suggests the first step of such a construction.

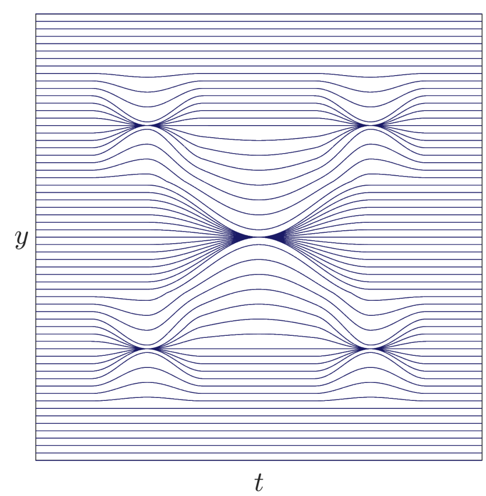

Figure 2. More points of non-uniqueness can be added.

Taking the process to the limit, we see that there exists an ODE $y' = f(t, y)$ with the property that for every initial point (t_0, y_0), there is more than one solution on every interval $[t_0, t_0 + \varepsilon]$.

FURTHER READING

[1] P. Hartman, A differential equation with non-unique solutions, *Am. Math. Mon.*, 70 (1963) 255–259.

[2] M. Lavrentieff, Sur une équation différentielle du premier ordre, *Math. Z.*, 23 (1925) 197–209.

[3] L.N. Trefethen, Á. Birkisson, T.A. Driscoll, *Exploring ODEs*, SIAM, 2017.

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