Notes of a Numerical Analyst

Dribbling a Ball

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During a recent visit of mine to the University of Delaware, Toby Driscoll told me of a computational problem he'd assigned his students. Solve the slightly damped linear oscillator equation

$$y'' + 0.04y' + y = 0.1\sin(\omega t) \tag{1}$$

with initial data $u = u' = 0$ and measure the amplitude of the resulting oscillation as a function of the forcing frequency ω. You'll see a nice amplitude peak at the resonant frequency $\omega_0 = 1$. Now do the same for the nonlinear pendulum

$$y'' + 0.04y' + \sin(y) = 0.1\sin(\omega t). \tag{2}$$

What happens to the response curve?

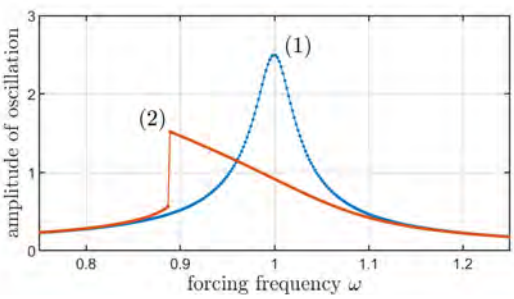

Figure 1. The linear oscillator (1) has a simple response with a peak at $\omega_0 \approx 1$. What's going on with the nonlinear pendulum (2)?

This is an easy calculation with your favourite ODE solver (I used Chebfun). The effect that turns up is striking, shown in Figure 1. The response for (2) suddenly jumps at around $\omega = 0.89$, more than doubling in amplitude.

With further numerical explorations, not shown, it's not hard to figure out what's going on. The linear oscillator (1) has natural frequency $\omega_0 \approx 1$, independent of amplitude. For $\omega \ll \omega_0$, the amplitude is small, so $\sin(y) \approx y$ and (2) has about the same behaviour. As ω increases toward ω_0, however, the amplitude increases and the natural frequency of (2) decreases, bringing ω and ω_0 closer together and enhancing the resonance. Around $\omega = 0.89$ the process becomes unstable and takes off, with the amplitude jumping up and ω_0 plunging down to a new value less than ω. Exercises 8.4.9–8.4.10 of [2] analyze such an event as a *cusp catastrophe* for the Duffing equation, a familiar effect to experts in nonlinear oscillations.

Figure 2. Toby Driscoll with a bouncy ball. As the amplitude grows, the natural frequency decreases, so to get the ball off the ground, you have to dribble fast at first and then slower.

As Toby and I chatted about (2), we were playing with a big green ball. Suddenly we noticed, this was a variation on the same nonlinear theme! You can't get the ball off the ground by patting it at a fixed frequency, but the sky's the limit if you start patting fast and then slow down as the bounce gets higher and the natural frequency falls. We are sure LeBron James knows this.

FURTHER READING

[1] A. H. Nayfeh and D. T. Mook, *Nonlinear Oscillations*, Wiley, 2008.
[2] S. H. Strogatz, *Nonlinear Dynamics and Chaos*, CRC Press, 2018.
[3] L. N. Trefethen, Á. Birkisson and T. A. Driscoll, *Exploring ODEs*, SIAM, 2018.

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