29. Blow-up equation with u^p nonlinearity

There are many mathematical models of physical problems which result in solutions that develop singularities after a finite time. Sometimes it is a derivative of the solution that diverges to infinity, as with the formation of a shock with the inviscid Burgers' equation ($\rightarrow$ *ref*), and sometimes the function value itself may "*blow up*". One of the simplest equations of the latter kind is the nonlinear heat (or reaction-diffusion) equation

$$u_t = \Delta u + f(u), \tag{1}$$

where Δ is the Laplacian and the nonlinear term $f(u)$ satisfies $f(u) \to \infty$ as $u \to \infty$.

An extensively studied special case of (1) is the 1D nonlinear heat equation

$$\boxed{u_t = u_{xx} + u^p} \tag{2}$$

with $p > 1$. For definiteness let us suppose (2) is posed on a finite closed interval with $u = 0$ at the boundaries and initial condition $u(x,0) = u_0(x)$. The equation is governed by two opposing forces, one of explosive growth and one of diffusion. The linear 1D heat equation ($\rightarrow$ *ref*) only models diffusion, and all solutions smooth into a hump that decreases to 0. The nonlinear term, u^p, is reminiscent of the ODE $u' = u^p$, which blows up in finite time for positive u_0. Whether or not the solution of (2) blows up depends on whether the diffusive term or the nonlinear term dominates. If u_0 is small enough, the energy will dissipate, and, after a finite time the solution will behave as if governed only by the linear heat equation. However, if u_0 is large enough the solution blows up at some finite time t^* at some point x^*:

$$u(x^*, t) \to \infty \quad \text{as} \quad t \to t^*$$

The singularity that develops about x^* narrows as $t \to t^*$. Much is known about the structure of this blow-up process. The position and time of blow-up depend upon the initial data, but close to blow-up the solution is almost independent of $u_0(x)$.

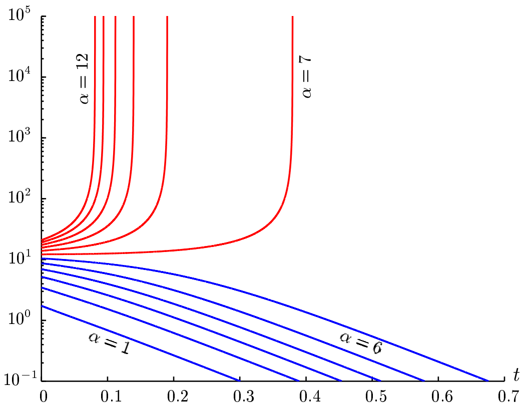

Fig. 1: $\max_x u(x,t)$ as a function of t

Figure 1 demonstrates the growth or decay of the solution to (2) on the interval $[0,1]$ with $p = 2$ and initial data

$$u(x,0) = \alpha\, e^x \sin \pi x, \tag{3}$$

for various values of α. When α is between 1 and 6, the behaviour is eventually dominated by diffusion. However, as α increases the time it takes for the solution to smooth out increases. At about $\alpha = 6.8485$ the balance shifts, as illustrated in Figures 2 and 3.

If (2) is extended to the entire real line the behaviour may change, since

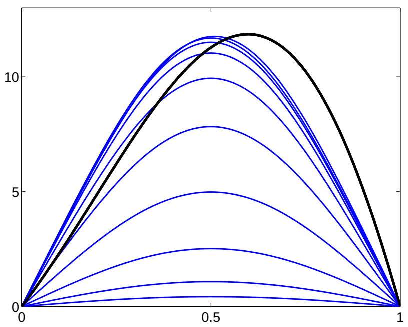

Fig. 2: $\alpha = 6.84$: decay to 0 ($t = 0, 0.1, 0.2, \ldots, 1$ from top to bottom)

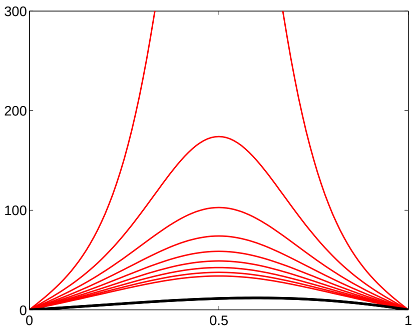

Fig. 3: $\alpha = 6.85$: blow-up ($t = 0, 0.8, 0.805, 0.810, \ldots, 0.840$ from bottom to top)

now sufficiently broad solutions may diffuse arbitrarily slowly. Fujita and Weissler showed that for the n-dimensional generalisation of (2), the behaviour depends on the relationship between p and n. For $p > 1 + 2/n$, sufficiently small initial data may generate bounded solutions valid for all time. For $1 < p \le 1 + 2/n$, on the other hand, any positive initial condition leads to blow-up in finite time. In one dimension, the critical exponent is 3, so the behaviour of Figure 2 could not have been observed for this value $p = 2$ if the domain had been unbounded.

Along with (2), the other special case of (1) that has been most extensively studied is the equation

$$u_t = u_{xx} + e^u \tag{4}$$

($\rightarrow$ *ref*). The behaviours of (2) and (4) are similar, except that the nonlinear term of (4) does not "shut off" as $u \to 0$, with the consequence that blow-up occurs in general even on an unbounded domain.

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