It would be difficult to pass a day on this planet without encountering capillary phenomena. Capillarity determines the shape of raindrops, the structure of spider webs, the way a candle burns, and the absorption of ink on paper; it controls many essential life processes. The mathematics is the same as in the *minimal surface equation* ($\rightarrow$ *ref*), but with new boundary conditions and a nonzero right-hand side corresponding to a pressure change across the interface.

Early discourses on capillary surfaces appeared in the writings of Leonardo and Galileo. Following the advent of the Calculus there was great activity on the topic, and in 1805 Thomas Young introduced the *mean curvature* H of a surface $\mathcal{S}$ and related it to the pressure change δp across $\mathcal{S}$: $\delta p = 2\sigma H$, with $\sigma =$ surface tension. Young also gave an argument suggesting that the *contact angle* γ with which $\mathcal{S}$ meets rigid supporting walls depends only on the physical materials, not on the gravity field or the geometry. In the following year, Laplace showed that for a surface $u(x, y)$ the mean curvature is given by $2H = \nabla\cdot(\nabla u/(1+|\nabla u|^2)^{1/2})$, from which he obtained the equation

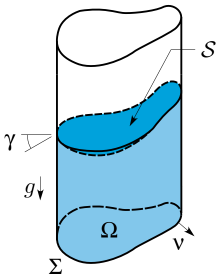

Fig. 1: Capillary surface

$$\boxed{\nabla\cdot\left(\nabla u\Big/(1+|\nabla u|^2)^{1/2}\right) = \kappa u + \lambda} \tag{1}$$

(in Ω) and the equally important contact angle boundary condition,

$$\nu\cdot\left(\nabla u\Big/(1+|\nabla u|^2)^{1/2}\right) = \cos\gamma \tag{2}$$

(on Σ) for the height u of a liquid in a vertical cylinder of cross-section Ω with boundary Σ. Here $\kappa = \rho g/\sigma$ is the capillarity constant, where ρ is the density change across $\mathcal{S}$ and g is the gravitational acceleration; ν is the unit exterior normal to Σ; and λ is a Lagrange parameter, a pressure difference that will adjust to satisfy a volume constraint.

In 1830, Gauss rederived (1) and (2) by a variational procedure, and his approach has become the basis for much of current theory.

During the nineteenth century most studies of (1)–(2) were restricted to symmetric configurations, such as the familiar case of a circular capillary tube. Such studies overlook dramatic behaviour that capillary surfaces can exhibit. This behaviour is particularly striking when gravity is absent ($\kappa = 0$). Consider for example a domain Ω containing a corner of opening angle 2α. If $\alpha < |(\pi/2)-\gamma|$, there is no solution of (1)–(2). Figure 2 shows an experiment by W. Masica in the 132-meter drop tower at the NASA Glenn Research Center, USA. Two regular hexagonal cylinders were partly filled with alcohol-water mixtures of different strengths, leading to contact angles above and below critical. With $\alpha > |(\pi/2)-\gamma|$, the experiments reveal the explicit solution, known as a spherical cap. With $\alpha < |(\pi/2)-\gamma|$, the liquid fills out the corners and partially covers the top of the container; the surface folds over itself, and cannot be expressed as a solution of (1).

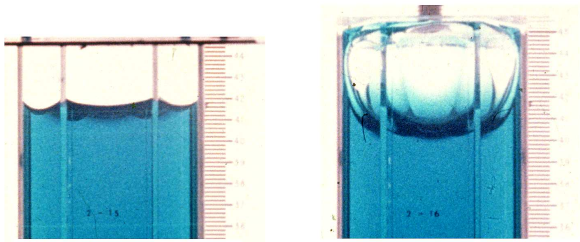

Fig. 2: Hexagonal container, $\kappa = 0$: $\alpha > |(\pi/2)-\gamma|$ (left) and $\alpha < |(\pi/2)-\gamma|$ (right)

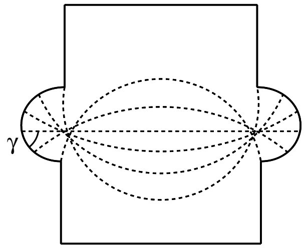

Fig. 3: Unstable equilibria

In the configuration of Figure 1, for which $\mathcal{S}$ meets only vertical walls, $\mathcal{S}$ is uniquely determined by the liquid volume and contact angle. In general, however, uniqueness can fail, and in fact configurations can be given that admit entire continua of equilibrium free surfaces, all with the same volume, contact angle, and mechanical energy. In the example of Figure 3, the rotationally symmetric equilibria sketched are unstable; the actual energy minimising surfaces are asymmetric. Figure 4a shows asymmetric local energy minimisers from computer calculations, which correspond closely to the experimental observations of Figure 4b, made by S. Lucid in the microgravity environment of the *Mir* Space Station.

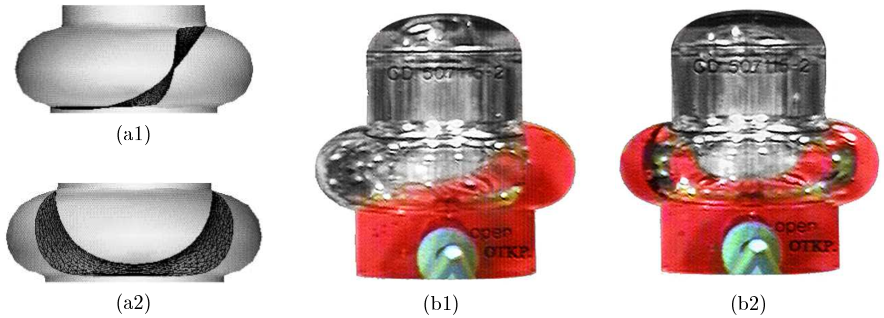

Fig. 4: Stable configurations: (a) computed, (b) observed on the *Mir* Space Station

References

P. Concus, R. Finn, and M. Weislogel, *Capillary surfaces in an exotic container: results from Space experiments*, J. Fluid Mech., **394** (1999), pp. 119–135.

R. Finn, *Equilibrium capillary surfaces*, Springer-Verlag, 1986.

R. Finn, *Capillary surface interfaces*, Notices Amer. Math. Soc., **46** (1999), pp. 770–781.

P. S. Laplace, *Traité de mécanique céleste; suppléments au livre X*, Gauthier-Villars, 1805–06.

T. Young, *An essay on the cohesion of fluids*, Philos. Trans. Roy. Soc. London, **95** (1805), pp. 65–87.