The Glass Bead Game

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In 1943, the novelist Hermann Hesse completed the novel which represents the quintessence of his imagination and vision, and for which, three years later, he was awarded the Nobel Prize for Literature. The novel is *Das Glasperlenspiel*, or *The Glass Bead Game*. It is set some four or five centuries hence in a state called Kastalien. This state dedicates itself to the cultivation of scholarship, and the members of its ruling elite live in a quasi-monastic society and develop their minds by studying and playing the glass bead game. The twentieth century is regarded by them as an age neither culturally nor intellectually impoverished but without much idea of the role of culture; an age in which philosophy and literature were thought to be the greatest achievements of the era from the end of the Middle Ages. The scholars of Kastalien, on the other hand, 'have for generations given the palm to mathematics and music'.

Mathematics and music have been been linked together since the time of Pythagoras. Mathematics has been described as the science of pattern and music as the art of pattern, and the two disciplines have many common features. Both require a high degree of creative thought, and in both cases, creation is subject to rigorous constraints. For the composer, these constraints ensure that the patterns to which the listener responds have appropriate complexity and prevalence, are not obscured by incidental details, and can be related to previous musical experience. They enable creativity, and facilitate rather than preclude musical expression. In the twentieth century, composers introduced a variety of new rigorous techniques yielding scope for artistic inspiration, and composers such as Berio and Schnittke demonstrated that a very tightly structured composition can accommodate remarkably diverse elements. The fundamental test of the success of a musical composition remains, of course, its effectiveness in performance.

The glass bead game is a highly developed language drawing on several disciplines, principally mathematics and music. It originated, in the middle of the 21'st century, in a game played by musicians and musicologists in England and Germany to develop their memory and contrapuntal skill. One player would call out in musical notation a theme from a classical composition, and another would answer by continuing it or adding a contrasting theme, and so on. The progress of the game was recorded on a frame modelled on the abacus, with metal wires corresponding to lines of the musical stave, and coloured beads to the notes. The idea was then adopted by mathematicians; instead of notes, they used strings of mathematical symbols expressing mathematical statements. Then it was taken up by other scholarly disciplines. It was used by philologists to shed light on problems in linguistics, and within architecture to establish links between the visual arts and mathematics. The final phase in its development was the creation of a language of symbols and formulas unifing mathematics and music, and it became contemplative rather than competitive. The game might start from one, two or

three themes, for example, a theme from Bach, a mathematical statement, an astronomical configuration or perhaps or some text from the Latin Mass; parallels and relationships would be established between concepts from different disciplines.

Back in the twentieth century, the borders between different disciplines are rather sharply delineated and they are reinforced by the administrative structures of the institutions and societies dedicated to scholarship. The barriers of communication separating scientists and non-scientists are certainly no less now than they were forty years ago, when C. P. Snow identified the concept of the 'two cultures'. Four or five centuries seems a long time to wait for a reconciliation of ideas from different disciplines. So, being not convinced that Hesse's glass bead game is only a metaphor, I have tried to accelerate progress by playing the glass bead game. I am probably the first person to have done so, and because of my lack of experience I have used only two themes, a theme from Bach and a statement in group theory.

The short piece of music below is the 'Order Fughetta'. It sounds quite effective in performance. It makes good use of its fugal subject and is convincing Bach pastiche: apart from its brevity, there is little to betray that was not written by Bach.

What has the 'Order Fughetta' to do with Hesse's game? The first two bars come from material in bars 1 and 3 of the fugue from Bach's famous Toccata and Fugue in D minor (BWV 565), transposed up a fifth. Now consider the mathematical statement

<center>'the group has order at most six'.</center>

This can be expressed in various ways in the first-order language of group theory, and one of these is as follows:

$$\exists D\, \exists C\, \exists H\, \exists A\, \forall G\, \forall F\, \forall E \quad D = F\, \vee\, E = H\, \vee\, E = A\, \vee\, E = D\, \vee\, E = C$$
$$\vee\, F = C\, \vee\, F = H\, \vee\, F = A\, \vee\, F = H\, \vee\, G = D\, \vee\, G = C\, \vee\, G = H\, \vee\, G = A.$$

This expresses the fact that there is a non-empty subset which has at most four elements and which has non-trivial intersection with every subset having three elements. There are more elegant ways of expressing this fact, but the above choice is made because of the musical exigencies.

The German nomenclature for notes, according to which B♭ is called B and B♮ is called H, was conventional long before Bach used BACH as a fugal subject. The symbols $\exists$ and $\forall$ are mathematical variants of the ordinary letters E and A, and the symbol $\vee$, which to a mathematician means 'or', means to a musician that notes are to be detached. The statement C = D surely means that the notes C, D are to be played simultaneously. The music of the top two voices of the fughetta, up to the first notes of bar 6, omitting the four notes ringed in bar 2, is precisely the statement that the group has order at most six.

Order six fughetta