On the unreasonable effectiveness of mathematics

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Why science is possible?

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Why and how?

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Eugene Wigner called it *"the unreasonable effectiveness of mathematics"*.

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Can this happen indeed?

Examples and counter-examples

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is countable and has countable formal theory

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the theory of integers **does have** uncountable models too, but there is no uniqueness, so **not uncountably categorical.**

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If physical universe is *co-ordinatizable in terms of (the algebra of) complex numbers*, then this explains that the comprehensive physical science is possible.

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Algebraic geometry studies spaces with complex co-ordinates, or more generally, co-ordinates over **algebraically closed fields** in which all polynomial equations have solutions.
Trajectories in algebraic geometry are given by polynomial equations, e.g.

$$y^2 = x^3 + x + 1,$$

the equation for an elliptic curve. Those are expressible in the same formal language as the field of complex numbers.

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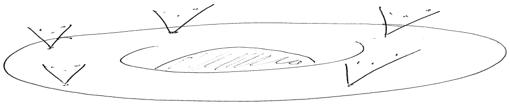
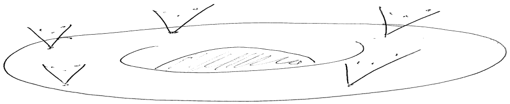
Question (1981). *Is it true that every categorical theory can be reduced to algebraic geometry? So eventually to the theory of an algebraically closed field?*

Classification Theorem

1993: (Hrushovski and Z.) Under a natural extra assumption the only uncountably categorical structures are algebraic-geometric with possibly "finite fibres" over them.





2004: In general, finite fibres over algebraic-geometric objects have quantum nature.

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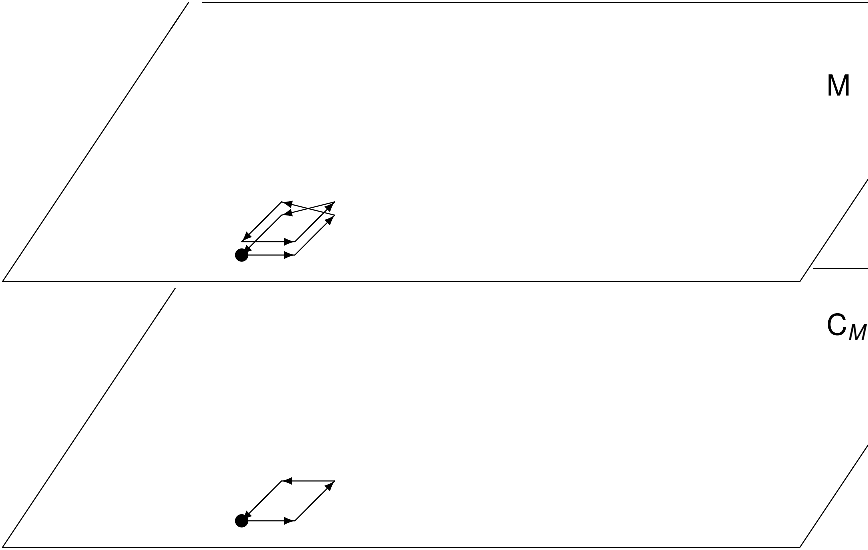
Classifying uncountable categorical structures model theory closes in on solving this problem.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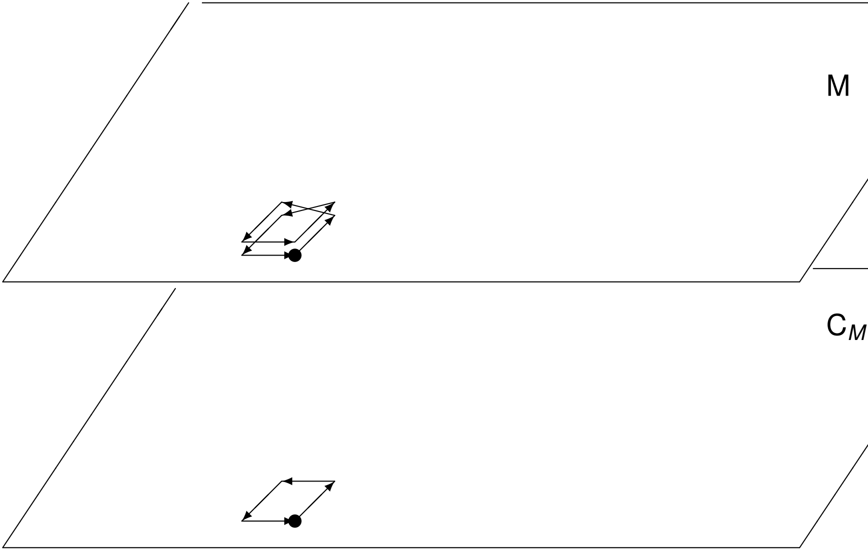
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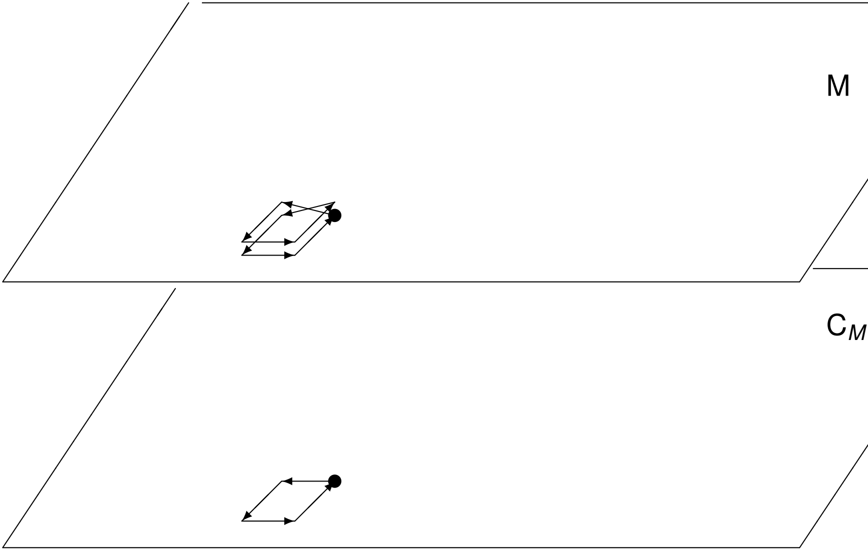
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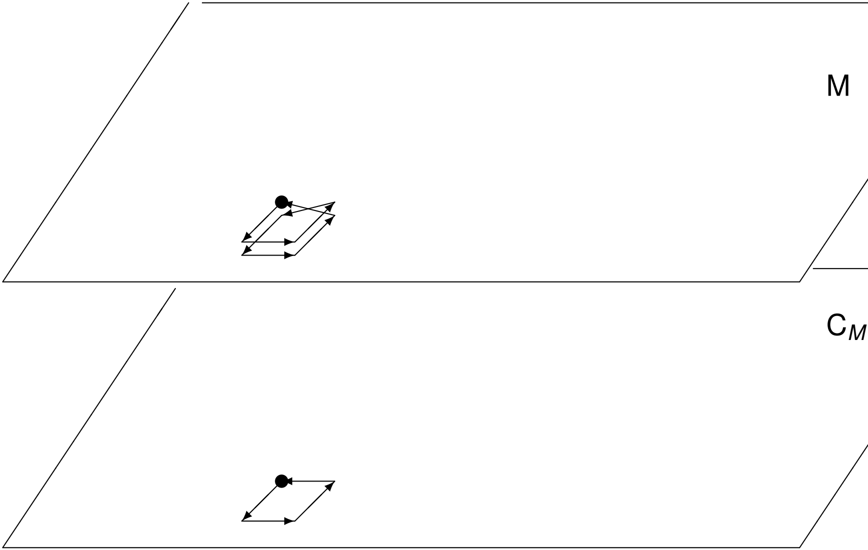
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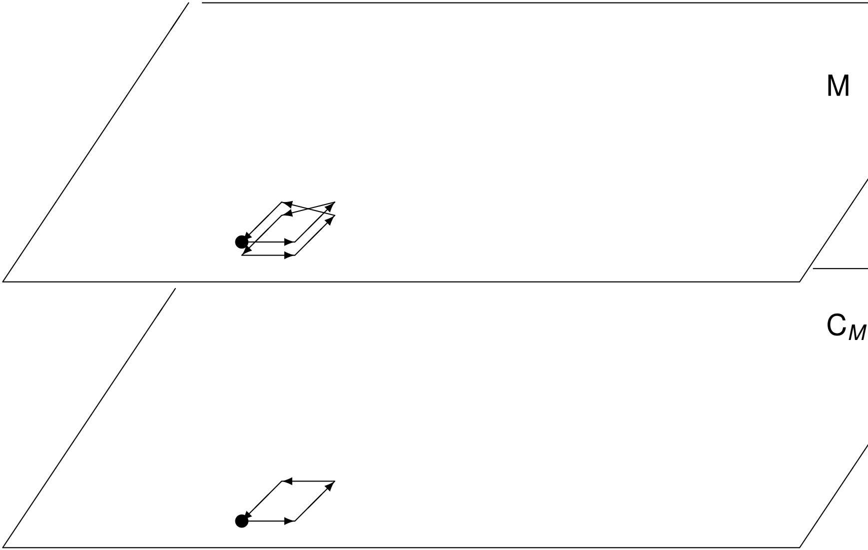
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