A topological $L_{\omega_1,\omega}$-invariant.

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Abstract

We suggest to look at formal sentences describing complex algebraic varieties together with their universal covers as topological invariants. We prove that for abelian varieties and Shimura varieties this is indeed a complete invariant, i.e. it determines the variety up to complex conjugation.

1 Introduction

1.1 The universal (and more general) cover of a complex algebraic variety as an abstract structure has been a subject of study in model theory in recent years, see e.g. [2], [3], [5], [6]. More specifically, given a smooth complex variety $\mathbb{X}$ and its cover $\mathbf{p} : \mathbb{U} \to \mathbb{X}$ presented in a natural universal language (see 1.2 below), the aim has been to find an $L_{\omega_1,\omega}$-sentence $\Sigma_{\mathbb{X}}$ which axiomatises $\mathbf{p} : \mathbb{U} \to \mathbb{X}$ categorically, that is determines the isomorphism type of an uncountable model of $\Sigma_{\mathbb{X}}$ uniquely in a given cardinality, up to an abstract isomorphism (ignoring topology).

We suggest here that, in cases that the cover is sufficiently non-degenerate, such a $\Sigma_{\mathbb{X}}$ can serve as a complete topological invariant of respective $\mathbb{X}$. More precisely, assuming that in a given class of varieties $\mathbb{X}$ such a categorical $L_{\omega_1,\omega}$-sentence $\Sigma_{\mathbb{X}}$ exists, then:

if for a complex variety $\mathbb{X}'$ its cover structure $\mathbf{p}' : \mathbb{U}' \to \mathbb{X}'$ is a model of the same sentence $\Sigma_{\mathbb{X}}$ then there is a biholomorphic isomorphism either between $\mathbb{X}$ and $\mathbb{X}'$ or between $\mathbb{X}^c$, the complex conjugate of $\mathbb{X}$, and $\mathbb{X}'$. (Compare this with interesting speculations in a similar vein presented by M.Gavrilovich in [8].)

We elaborate the cases of abelian varieties and Shimura varieties.

1.2 For the general case of cover structure, let $\mathrm{k} \subseteq \mathrm{k}^\sharp \subseteq \mathbb{C}$ be a countable subfield of $\mathbb{C}$ and its algebraic extension invariant under $\mathrm{Aut}(\mathbb{C}/\mathrm{k})$. Let $\{\mathbb{X}_i : i \in I\}$ be a collection of non-singular irreducible complex algebraic varieties (of dim > 0) defined over $\mathrm{k}^\sharp$ and $I := (I, \geq)$ a lattice with the minimal element 0 determined by unramified $\mathrm{k}^\sharp$-rational epimorphisms $\mathrm{pr}_{i',i} : \mathbb{X}_{i'} \to \mathbb{X}_i$, for $i' \geq i$. $\mathbb{X}_0 = \mathbb{X}$ defined over k. It is assumed that the lattice is invariant under $\mathrm{Aut}(\mathbb{C}/\mathrm{k})$.

Let $\mathbb{U}(\mathbb{C})$ be a connected complex manifold and $\{\mathbf{p}_i : i \in I\}$ a collection of holomorphic covering maps (local bi-holomorphismss)

$$\mathbf{p}_i : \mathbb{U}(\mathbb{C}) \twoheadrightarrow \mathbb{X}_i(\mathbb{C}), \quad \mathrm{pr}_{i',i} \circ \mathbf{p}_{i'} = \mathbf{p}_i.$$

as illustrated by the picture:

$$\begin{tikzcd}
& \mathbb{U} \arrow[dl, "\mathbf{p}_m"'] \arrow[ddd, "\mathbf{p}"] \arrow[ddr, "\mathbf{p}_n"] & \\
\mathbb{X}_m \arrow[drr, "\mathrm{pr}_{m,n}"] \arrow[ddr, "\mathrm{pr}_{m,0}"'] & & \\
& & \mathbb{X}_n \arrow[dl, "\mathrm{pr}_{n,0}"] \\
& \mathbb{X}_0 &
\end{tikzcd}$$

A model-theoretic treatment of the general case is discussed in [1].

However, in the cases of interest to us we may assume that $\mathbb{X}_0(\mathrm{k}) \neq \emptyset$ (the variety has a k-rational point) and then the structure can be simplified (bi-inerpretable, in an appropriate language) to one with $(I, \geq)$ a linear order, that is the lattice is reduced to a tower, see [14] for detail.

This multisorted structure we call the **cover structure** of $\mathbb{X}$ (here $\mathbb{X} = \mathbb{X}_0$) and often write as $(\mathbf{p} : \mathbb{U} \to \mathbb{X})$. The constants and basic relations of $(\mathbf{p} : \mathbb{U} \to \mathbb{X})$ are:

- the set of constant symbols C naming elements of k, the field of definition of $\mathbb{X}$;

- the Zariski closed relations on sorts $\mathbb{X}_i$ defined over k;

- the rational maps $\mathrm{pr}_{m,n}$ defined over k;

- the maps $\mathbf{p}_i$.

Examples.

1. $\mathbb{X} = \mathbb{A}$ an abelian variety. In this case all the $\mathbb{X}_i$ are equal to $\mathbb{A}$, I is a subset of $\mathbb{N}$ linearly ordered by the relation $n|m$ such that, for each prime power p^k there is $m \in I$, $p^k|m$, and $\mathrm{pr}_{m,n} : x \mapsto \frac{n}{m}x$.

2. $\mathbb{X} = \mathbb{Y}(1) = A^1 \cong \Gamma(1)\backslash\mathbb{H}$ the modular curve, $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$, $\mathrm{k} = \mathbb{Q}$. $\mathbb{X}_n = \mathbb{Y}(n) \cong \Gamma(1)\backslash\mathbb{H}$ and $\mathrm{pr}_{m,n}$ are induced by the embeddings $\Gamma(m) \subseteq \Gamma(n)$.

In such a cover structure, as noted by M.Gavrilovich [8] and used since multiple times, many more interesting relations and maps are definable. In particular, if $\mathbb{X}$ is an abelian variety then + on $\mathbb{X}$ can be lifted to $\mathbb{U}$ as a type-definble (and so $L_{\omega_1,\omega}$-definable) operation on $\mathbb{U}$. In case of $\mathbb{X}$ a Shimura variety the subgroup $\mathrm{G}(\mathbb{Q})$ of $\mathrm{G}(\mathbb{R})$ is definable along with its action on $\mathbb{U}$, for $\mathrm{G}(\mathbb{R})$ the reductive group of the Shumura data for $\mathbb{X}$. In general, the topological fundamental group of $\mathbb{X}$ acting on $\mathbb{U}$ is definable.

1.3 For each of the cases of $\mathbb{X} = \mathbb{X}(\mathbb{C})$ we are interested in an $L_{\omega_1,\omega}$-sentence $\Sigma_{\mathbb{X}}$ which describes the cover structure $\mathbf{p} : \mathbb{U} \to \mathbb{X}$.

We may assume that $\Sigma_{\mathbb{X}}$ is stronger than the first order theory $\mathrm{Th}_{\mathbb{X}}$ in the same language. Recall the following folklore fact mentioned e.g. in [13] and proved in detail in [2], Fact A.21.

1.4 Fact. Let F be an algebraically closed field and $\mathbb{X}(\mathrm{F})$ be F-points an algebraic variety over $\mathrm{k} \subset \mathrm{F}$ considered as the structure with relations corresponding to k-definable Zariski closed subsets of $\mathbb{X}^m(\mathrm{F})$, $m \in \mathbb{N}$.

Then $\mathbb{X}(\mathrm{F})$ is bi-interpretable with $(\mathrm{F}; +, \cdot, c_a)_{a \in \mathrm{k}}$.

Consequently,

if $\mathbb{X}'$ is a variety over $\mathrm{k}' \subset \mathrm{F}$ and $\mathbb{X}'(\mathrm{F})$ is a model of $\Sigma_{\mathbb{X}}^{f.o.}$, the first order part of axioms $\Sigma_{\mathbb{X}}$, then $\mathbb{X}'$ is conjugated to $\mathbb{X}$ by an automorphism σ of field F, that is $\mathrm{k}' = \mathrm{k}^\sigma$ and $\mathbb{X}^\sigma(\mathrm{F}) = \mathbb{X}'(\mathrm{F})$.

1.5 Let $\mathbb{X}_1$ and $\mathbb{X}_2$ be two complex algebraic varieties over k_1 and k_2 respectively. We will say that $\mathbb{X}_1$ and $\mathbb{X}_2$ have the **same $L_{\omega_1,\omega}$-type** if there is an uncountably categorical $L_{\omega_1,\omega}$-sentence Σ over parameters C interpreted as elements of k_1 in $\mathbb{X}_1$ and of k_2 in $\mathbb{X}_2$ such that the cover structures of both $\mathbb{X}_1(\mathbb{C})$ and $\mathbb{X}_2(\mathbb{C})$ are models of Σ.

Our aim is to prove that for certain classes of complex algebraic varieties:

if $\mathbb{X}_1$ and $\mathbb{X}_2$ have the same $L_{\omega_1,\omega}$-type then

$$\mathbb{X}_1 \cong \mathbb{X}_2 \text{ or } \mathbb{X}_1 \cong \mathbb{X}_2^c$$

where $\cong$ stands for biholomorphic isomorphism and $\mathbb{X}^c$ is the variety complex-conjugated to $\mathbb{X}$.

This is equivalent to the statement that there is a homeomorphism $\mathbb{X}_1 \to \mathbb{X}_2$ which respects the Zariski closed relations.

1.6 A **complementary problem** is also of interest:

Let $\mathbb{X} = \mathbb{X}(\mathbb{C})$ be a complex algebraic variety and $\mathbb{X}^\tau$ the variety obtained by applying an (abstract) automorphism τ of the field $\mathbb{C}$ to $\mathbb{X}(\mathbb{C})$ point-wise. Suppose there exists $\Sigma_{\mathbb{X}}$, a categorical $L_{\omega_1,\omega}$-sentence for the cover structure of $\mathbb{X}$.

Does a categorical $L_{\omega_1,\omega}$-sentence for the cover structure of $\mathbb{X}^\tau$ exist? How one obtains $\Sigma_{\mathbb{X}^\tau}$ from $\Sigma_{\mathbb{X}}$?

The latter question in application to Shimura varieties is essentially the content of Langlands' conjecture on conjugation of Shimura varieties [10] (without a categoricity statement), proved by M.Borovoi and independently by J.Milne and K.Shih.

The following citation from Langlands's [10] helps to appreciate difficulties in formulating $\Sigma_{\mathbb{X}}$, not to mention proving its categoricity:

The problem of conjugation is formulated in the sixth section as a conjecture, which was arrived at only after a long sequence of revisions. My earlier attempts were all submitted to Rapoport for approval, and found lacking. They were too imprecise, and were not even in principle amenable to proof by Shimuras methods of descent.

2 Abelian varities

Our references to the standard facts about abelian varieties is [9] and for more specific reference to the Weil pairing is [11].

We assume the abelian varieties below to be polarised, that is embedded into a complex projective space.

2.1 Recall that by [2] for each elliptic non-CM curve E over a field $\mathrm{k} \subset \mathbb{C}$ there is a categorical $L_{\omega_1,\omega}$-sentence Σ_{E} in the language with parameters C interpreted as elements of k, which models the cover structure $(\mathbf{p} : \mathbb{C} \to \mathrm{E}(\mathbb{C}))$:

$$(\mathbf{p} : \mathbb{C} \to \mathrm{E}(\mathbb{C})) \vDash \Sigma_{\mathrm{E}}.$$

The same is true for the general case of abelian varieties $\mathbb{A}$ by [3]. The sentence $\Sigma_{\mathbb{A}}$ therein is in the language naming generators $\bar{\lambda}$ of Λ, so is $\Sigma_{\mathbb{A}}(\bar{\lambda})$ in fact. But this can be eliminated by taking instead

$$\Sigma_{\mathbb{A}}^{\sharp} := \exists \bar{\lambda} \Sigma_{\mathbb{A}}(\bar{\lambda}) \ \& \ \mathrm{qftp}(\mathbf{p}(\mathbb{Q} \otimes \Lambda)).$$

(Note that the Σ_{E} in [2] is described in quite precise terms, which is important when discussing topological invariants expressed by the formulas.)

2.2 Theorem. *Let $\mathbb{A}_1$ and $\mathbb{A}_2$ be two complex abelian varieties.*

Suppose $\mathbb{A}_1$ and $\mathbb{A}_2$ have the same $L_{\omega_1,\omega}$-type. Then there is a biholomorphic isomorphism

$$\varphi : \mathbb{A}_1 \to \mathbb{A}_2, \ \textit{or} \ \varphi : \mathbb{A}_1 \to \mathbb{A}_2^c$$

where $\mathbb{A}_2^c$ is the abelian variety obtained by complex conjugation from $\mathbb{A}_2$.

Proof. We assume that $\mathbb{A}_1$ is over k_1 and $\mathbb{A}_2$ over k_2, subfields of $\mathbb{C}$. By categoricity and 1.4 above there is an abstract isomorphism ϕ between the cover structures, which include the isomorphism

$$\phi : \mathrm{k}_1 \to \mathrm{k}_2; \quad \mathrm{F} \to \mathrm{F},$$

the isomorphism

$$\phi : \Lambda_1 \otimes \mathbb{Q} \to \Lambda_2 \otimes \mathbb{Q}$$

because the isomorphism type of the respective lattice Λ is definable.

Now we recall that for an abelian variety $\mathbb{C}^{2g}/\Lambda$ there exists an alternating **Riemann form**.

2.3 Recall the definition. Let $\mathbb{U}$ be a finite dimensional vector space over $\mathbb{C}$ of dimension g. A Riemann form for $(\mathbb{U}, \Lambda)$ is a skew symmetric $\mathbb{Z}$-bilinear map

$$\Phi : \Lambda \times \Lambda \to \mathbb{Z}$$

such that the map $(u_1, u_2) \mapsto \Phi_R(\sqrt{-1} \cdot u_1, u_2) \quad \forall u_1, u_2 \in V$ is a symmetric positive definite $\mathbb{R}$-bilinear form on $\mathbb{U}$, where $\Phi_R : \mathbb{U} \times \mathbb{U} \to \mathbb{R}$ is the unique skewsymmetric $\mathbb{R}$-bilinear which extends Φ.

Remark. The map $H : \mathbb{U} \times \mathbb{U} \to \mathbb{C}$ defined by

$$H(u_1, u_1) = \Phi_{\mathbb{R}}(\sqrt{-1} \cdot u_1, u_2) + \sqrt{-1} \cdot \Phi_{\mathbb{R}}(u_1, u_2)$$

is a positive definite Hermitian form whose imaginary part is equal to Φ_R

$$\Phi : \Lambda \otimes \mathbb{R} \;\times \Lambda \otimes \mathbb{R} \to \mathbb{R}$$

which is integer-valued on $\Lambda \times \Lambda$ and thus gives rise to a **Weil pairing**

$$w : \mathbb{A}(n) \times \mathbb{A}(n) \to \mu_n; \quad (\mathbf{p}(\frac{\lambda_1}{n}), \mathbf{p}(\frac{\lambda_2}{n})) \mapsto \exp(\frac{2\pi i}{n}\Phi(\lambda_1, \lambda_2))$$

for each n. The Weil pairing $w : \mathbb{A}(n) \times \mathbb{A}(n) \to \mu_n$ is $\mathrm{Gal}_{\mathrm{k}}$-invariant, that is $w(\alpha_1, \alpha_2)$ is definable in the cover structure. Thus, for a Weil pairing $w_1 : \mathbb{A}_1(n) \times \mathbb{A}_1(n) \to \mu_n$ and the corresponding Riemann form $\Phi_1 : \Lambda_1 \times \Lambda_1 \to \mathbb{Z}$ there is a well-defined Weil pairing $w_1^\phi := w_2 : \mathbb{A}_2(n) \times \mathbb{A}_2(n) \to \mu_n$ defined by the condition

$$w_1(\alpha_1, \alpha_2)^\phi = w_2(\alpha_1^\phi, \alpha_2^\phi).$$

Note that $(\alpha_1, \alpha_2) \mapsto w_2(\alpha_1, \alpha_2)^{-1}$ is a Weil pairing too since $\zeta \mapsto \zeta^{-1}$ is an automorphism on μ.

w_2 applyed to $\mathbb{A}_2(n)$ for all n determines a bilinear map

$$\Phi_2 : \Lambda_2 \times \Lambda_2 \to \mathbb{Z}$$

(where $\mathbb{Z}$ is the additive group without order) such that, depending on the choice of the generator of $\ker \exp$,

$$\exp(\frac{2\pi i}{n}\Phi_1(\lambda_1, \lambda_2))^\phi = \exp(\frac{2\pi i}{n}\Phi_2(\lambda_1^\phi, \lambda_2^\phi))$$

or

$$\exp(\frac{2\pi i}{n}\Phi_1(\lambda_1, \lambda_2))^\phi = \exp(-\frac{2\pi i}{n}\Phi_2(\lambda_1^\phi, \lambda_2^\phi))$$

for all n.

Thus

$$\Phi_2(\lambda_1^\phi, \lambda_2^\phi) = \Phi_1(\lambda_1, \lambda_2), \tag{1}$$

or

$$\Phi_2(\lambda_1^\phi, \lambda_2^\phi) = -\Phi_1(\lambda_1, \lambda_2) \tag{2}$$

for all $\lambda_1, \lambda_2 \in \Lambda_1$.

In the first case Φ_2 extends to a positive definite $\mathbb{R}$-linear form $\mathbb{U} \times \mathbb{U} \to \mathbb{C}$, since so was Φ_1. In the second case $-\Phi_2$ is such a form.

2.4 Proof continued.

In case (1) Φ_1 and Φ_2 satisfy the Riemann form assumptions which give rise, by the Appell-Humbert theorem, to isomorphic line bundles $L_1 \to \mathbb{A}_1$ and $L_2 \to \mathbb{A}_2$ respectively.

By the Lefschetz Embedding Theorem the bundles L_1^m and L_2^m determine holomorphic maps of $\mathbb{A}_1$ and $\mathbb{A}_2$, respectively, into a projective space $\mathbf{P}^{N_m}$ and, for $m \geq 3$ the maps are embeddings. The isomorphism over $\mathbb{C}$ between line bundles gives rise to isomorphism of the complex projective varieties $\mathbb{A}_1$ and $\mathbb{A}_2$.

If $\Phi_1' = -\Phi_2'$ then we consider the complex conjugated abelian variety $\mathbb{A}_2^c$, which will have $-\Phi_2$ as the respective Riemann form. By the above $\mathbb{A}_1 \cong \mathbb{A}_2^c$. $\square$

2.5 Remark. In fact, we have proved that $c_1(L) \in \mathrm{H}^2(\mathbb{A}, \mathbb{Z})$, the first Chern class of a line bundle (corresponds to the Riemann form Φ), is $L_{\omega_1,\omega}$ definable over parameters in Λ, up to the sign. And, of course, $\Lambda \cong \mathrm{H}_1(\mathbb{A}, \mathbb{Z})$ and $\mathrm{H}_1(\mathbb{A}, \mathbb{Q})$ are defined up to isomorphism.

3 Shimura varieties

3.1 Let $\mathbb{X}_1$, $\mathbb{X}_2$ be connected Shimura varieties (see [12], Def.4.4) over their reflex fields $\mathrm{k}_1, \mathrm{k}_2$ given by the Shimura datum $(\mathrm{G}_1, \mathcal{H}_1)$, $(\mathrm{G}_2, \mathcal{H}_2)$ and congruence subgroups $\Gamma_1 \subset \mathrm{G}_1(\mathbb{Q})$, $\Gamma_2 \subset \mathrm{G}_2(\mathbb{Q})$, respectively.

$$\mathbb{X}_1 = \Gamma_1 \backslash \mathcal{H}_1 \text{ and } \mathbb{X}_2 = \Gamma_2 \backslash \mathcal{H}_2$$

with respective covering maps

$$j_1 : \mathcal{H}_1 \to \mathbb{X}_1 \text{ and } j_2 : \mathcal{H}_2 \to \mathbb{X}_2.$$

A Shimura datum $(\mathrm{G}, \mathcal{H})$, also associates with each point $h \in \mathcal{H}_1$ a homomorphisms $h : U(1) \to \mathrm{G}^{\mathrm{ad}}_{\mathbb{R}}$ from the 1-dimensional real torus $U(1)$, and all the homomorphisms are linked by conjugation by elements of $\mathrm{G}^{\mathrm{ad}}(\mathbb{R})^+$. The image $h(U(1)) = T_h$ is a torus subgroup of $\mathrm{G}^{\mathrm{ad}}_{\mathbb{R}}$ which can be identified as a subgroup fixing the point h. Since there is only one non-identity continuous automorphism of $U(1)$, $u \mapsto u^{-1}$, there are exactly two $\mathrm{G}^{\mathrm{ad}}_{\mathbb{R}}$-conjugacy classes of homomorphisms $h : U(1) \to \mathrm{G}^{\mathrm{ad}}_{\mathbb{R}}$ which can be called $\mathcal{H}^+$ and $\mathcal{H}^-$.

It follows from Shimura theory, see [4], that $\mathbb{X}$, a complex Shimura variety and $\mathbb{X}^c$ its complex conjugate,

$$\mathbb{X} = \Gamma\backslash\mathcal{H}^+ \Leftrightarrow \mathbb{X}^c = \Gamma\backslash\mathcal{H}^- \tag{3}$$

and the action of complex conjugation corresponds to the automorphism $u \to u^{-1}$ on $U(1)$.

The issue of categorical $L_{\omega_1,\omega}$-axiomatisation of some classes of Shimura varieties was addressed in [5], [6] and some others. Similar to [3] for abelian varieties, [5], [6] prove categoricity statements in the language that has constants naming special points in Shimura varieties. One can eliminate the constant symbols using existential quantifiers as is done in 2.1 for abelian varieties.

Below we use the notation $\mathcal{H}_1$ and $\mathcal{H}_2$ for sets and $\mathcal{H}_1^{\pm}$ and $\mathcal{H}_2^{\pm}$ for their presentations as classes of homomomorphisms from $U(1)$.

3.2 Fact (Th 5.4,[12]). Let G be a connected algebraic group over $\mathbb{Q}$. Then $\mathrm{G}(\mathbb{Q})$ is dense in $\mathrm{G}(\mathbb{R})$.

3.3 Theorem. *Suppose there is a categorical $L_{\omega_1,\omega}$-sentence Σ over parameters C such that both structures $(\mathrm{G}_1, \mathcal{H}_1, \Gamma_1, \mathbb{X}_1, j_1)$ and $(\mathrm{G}_2, \mathcal{H}_2, \Gamma_2, \mathbb{X}_2, j_2)$ are models of Σ when one interprets C as k_1 and k_2 respectively. Then there is a biholomorphic isomorphism*

$$\varphi : \mathbb{X}_1(\mathbb{C}) \to \mathbb{X}_2(\mathbb{C}), \text{ or } \varphi : \mathbb{X}_1(\mathbb{C}) \to \mathbb{X}_2^c(\mathbb{C}). \tag{4}$$

Proof. First we note that it follows from the categoricity assumption and the Fact of section 2 that there exists an abstract isomorphism ϕ between the two structures which is realised as ismorphisms

$$\phi : \mathrm{k}_1 \to \mathrm{k}_2 \tag{5}$$

$$\phi : \mathbb{X}_1 \to \mathbb{X}_2 \tag{6}$$

$$\phi : \mathcal{H}_1 \to \mathcal{H}_2 \tag{7}$$

$$\phi : \mathrm{G}_1(\mathbb{Q}) \to \mathrm{G}_2(\mathbb{Q}); \tag{8}$$

$$\phi : \Gamma_1 \to \Gamma_2 \tag{9}$$

$$\forall g \in \mathrm{G}_1(\mathbb{Q}), \forall x \in \mathcal{H}_1 \quad \phi(g \cdot x) = \phi(g) \cdot \phi(x) \tag{10}$$

and

$$j_2 \circ \phi = \phi \circ j_1. \tag{11}$$

Since $\mathrm{G}_1(\mathbb{Q})$ is dense in $\mathrm{G}_1(\mathbb{R})$ and $\mathrm{G}_2(\mathbb{Q})$ in $\mathrm{G}_2(\mathbb{R})$ there is a unique extension of ϕ to the continuous isomorphism

$$\phi_{\mathbb{R}} : \mathrm{G}_1(\mathbb{R}) \to \mathrm{G}_2(\mathbb{R}),\ \mathrm{G}^{\mathrm{ad}}1(\mathbb{R}) \to \mathrm{G}^{\mathrm{ad}}2(\mathbb{R})$$

together with the isomorphism of the action (10) since it is continuous both in g and x :

$$\forall g \in \mathrm{G}_1(\mathbb{R}), \forall x \in \mathcal{H}_1 \quad \phi_{\mathbb{R}}(g \cdot x) = \phi_{\mathbb{R}}(g) \cdot \phi_{\mathbb{R}}(x)$$

(where $\mathrm{G}^{\mathrm{ad}}(\mathbb{R}) = \mathrm{G}(\mathbb{R})/Z$, Z subgroup acting trivially).

Thus, ϕ induces

$$\phi_{\pm} : \{h : U(1) \to \mathrm{G}_1^{\mathrm{ad}}(\mathbb{R})\} \to \{\phi_{\pm}(h) : U(1) \to \mathrm{G}_2^{\mathrm{ad}}(\mathbb{R})\}$$

Since $h^{-1} \circ \phi_{\mathbb{R}}^{-1} \circ \phi_{\pm}(h) : U(1) \to U(1)$ is either identity or $u \to u^{-1}$, there are two possibilities:

$$\phi_{\pm} : \mathcal{H}_1^+ \to \mathcal{H}_2^+$$

and

$$\phi_{\pm} : \mathcal{H}_1^- \to \mathcal{H}_2^+$$

Respectively, by fact (3), we have two biholomorphic isomorphisms (4). □

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