Quasi-Lorentz group acting on a Minkowski space-time lattice

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Abstract

We use the notion of structural approximation to represent the Lorentz-invariant Minkowski space-time as a limit of finite cyclic lattices with the action of finite quasi-Lorentz groups

1 Introduction

1.1 A physical theory is an approximation to reality. But what is an approximation? In [1] we discussed this problem from the perspective of model theory. This results in the definition of **structural approximation** which we use here along with a more advanced recent paper [2] which sets the general background for applications in Foundations of Physics.

The idea behind the definition is that a structural approximation preserves the structure, e.g. a sequence of finite groups approximates a continuous group (or more often a "compactified" version of a group).

In the paper we construct a pseudo-finite group, a model-theoretic limit (= an ultraproduct) of a sequence of finite groups, which approximates the Lorentz group, along with a pseudo-finite cyclic lattice on which this quasi-Lorentz group acts, which approximates Minkowski space.

We suggest this construction as a form of discretisation of space-time with Lorentzian symmetry, a problem discussed in various publications, see e.g. [3]. In some sense our mathematical techniques is not dissimilar to ones proposed in [4] and some other publications relying on the p-adic number system. The pseudo-finite resudue ring K underlying our construction, is quite similar to the ring of adeles.

1.2 A structural approximation is a surjective map between two structures in the same language

$$\mathsf{Im} : {}^*\mathbf{M} \twoheadrightarrow \mathbf{M} \tag{1}$$

which has the property

$$S \subset {}^*\mathbf{M}^n \text{ closed} \Rightarrow \mathsf{Im}(S) \subset \mathbf{M}^n \text{ closed},$$

where "closed" means defined by positive quantifier-free formulas. Two elements $a, a' \in \mathbf{M}^n$ are seen to be "infinitesimally close" if $\mathsf{Im}\, a = \mathsf{Im}\, a'$.

Here ${}^*\mathbf{M}$ is a structure obtained by taking an ultraproduct of a sequence $\mathbf{M}_i$ of structures along a certain ultrafilter D. As it happens, below, most of the time $\mathbf{M}_i$, $\mathbf{M}$ and ${}^*\mathbf{M}$ are rings or groups and "closed" equivalently means closed in **Zariski topology**: a subset $P \subset \mathbf{M}^n$ is closed if it is the set of solutions of a system of algebraic equations in n-variables with parameters in $\mathbf{M}$.

In fact, as established in [1], $\mathbf{M}$ has to be **quasi-compact** in order for it to appear in (1) for non-trivial sequences $\mathbf{M}_i$. In particular, the field $\mathbb{C}$ is not quasi-compact but its compactification $\bar{\mathbb{C}} := \mathbb{C} \cup \{\infty\} = \mathbf{P}^1(\mathbb{C})$ is. Theorem 5.2(i) of [1] proves that for any uncountable zero-characteristic pseudo-finite field E there exists a structural approximation

$$\mathsf{Im}_{\mathrm{E}} : \mathrm{E} \twoheadrightarrow \bar{\mathbb{C}}. \tag{2}$$

Such an approximation *can not be explained in terms of non-standard analysis*. It is far of being unique but we can pick ones with some specific and useful properties as in [2].

On the other hand, the notion is quite restrictive in another sense: it is proved in Theorem 5.2(ii) of [1] that $\mathbb{C}$ is the only locally compact field for which an approximation by finite fields is possible. This part of the theorem is much more subtle.

1.3 Scales and scale-dependenve of approximation.

The interplay between the domain and the range of the approximation map Im_{E} as in (2) brings in some features not encountered in the limit construction with inherent metrics. By its nature field E is of pseudo-finite characteristic $\mathfrak{p}$ (an infinite non-standard prime $\mathfrak{p}$, and more generally we also consider pseudo-finite residue rings $\mathrm{E} = {}^*\mathbb{Z}/\mathcal{N}$) while $\mathbb{C}$ is characteristic zero field with a natural metric.

It is clear by algebraic considerations that

$$\mathsf{Im}_{\mathrm{E}} : \{1,2,3,\ldots\} \mapsto \{1,2,3,\ldots\}$$

where $\{1,2,3,\ldots\}$ include all usual (standard) integers and maybe some more. Moreover,

$$\mathsf{Im}_{\mathrm{E}} : \{\frac{1}{2},\frac{1}{3},\frac{1}{4},\ldots\} \mapsto \{\frac{1}{2},\frac{1}{3},\frac{1}{4}\ldots\}$$

and in the limit we can see the approximate reals emerging in E. In other words, an observer which has only access to small scale elements of E can think of E as being $\mathbb{R}$. Define

$$\mathrm{E}_{\mathrm{real}} = \{x \in \mathrm{E} : \mathsf{Im}_{\mathrm{E}}(x) \in \mathbb{R}\}.$$

So the remark we made is that small-scale elements of E are in $\mathrm{E}_{\mathrm{real}}$.

However, as we continue along the natural cyclic order $1,2,3,\ldots$ of E, inevitably, we will encounter an element $\mathbf{i} \in \mathrm{E}$ such that

$$\mathsf{Im}_{\mathrm{E}} : \mathbf{i} \mapsto \imath = \sqrt{-1}; \quad \mathbf{i}\cdot \mathrm{E}_{\mathrm{real}} \to \imath\,\mathbb{R}$$

and so with other complex numbers.

Again by (2), we will also have a non-empty domain

$$\mathrm{E}_{\infty} = \{x \in \mathrm{E} : \mathsf{Im}_{\mathrm{E}}(x) = \infty\}.$$

So, an observer which has tools to explore the global characteristics of E has to think of E as a Riemann sphere $\bar{\mathbb{C}}$.

It is clear that $\mathrm{E}_{\mathrm{real}}$, $\mathbf{i}\mathrm{E}_{\mathrm{real}}$ and E_{∞} should be considered of "different scales", perhaps in some context related to "low energy – high energy" philosophy. In [2], section 3, we introduce a formal notion which allows us to speak about scales and use it in constructing approximations with prescribed properties.

1.4 Below we construct a structural approximation of the Minkowski space by finite 4-dim lattices along with an approximation of the Lorentz group $\mathrm{SO}^+(1,3)$ by finite groups acting on the lattices respectively and preserving the Minkowski metric.

Note that unlike other approximations of Lorentz action, we have an actual group G acting on the discrete space $\mathcal{M}$ so that the group G approximates the Lorentz group and the space $\mathcal{M}$ approximates the Minkowski space with Minkowski metric. See 2.8 and (9).

For the reasons explained above the limit Minkowski space has to be compactified and also complexified which happens in a natural way somewhat similar to the construction of the twistor space,

2 Pseudo-finite rings and groups and their limits

2.1 It is well-known that $\mathrm{SL}(2,\mathbb{C})$ is a double cover of the Lorentz group $\mathrm{SO}^+(1,3)$ and it acts in agreement with this on the Minkowski space .

More precisely (see e.g. [5]): represent a vector with components $(t,x,y,z)\in\mathbb{R}^4$ (Minkowski space) as a 2×2 matrix

$$X := \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix}$$

with $X^\dagger = X$ and $\det(X) = t^2 - x^2 - y^2 - z^2$, consider

$$X \mapsto MXM^\dagger \text{ with } M \in \mathrm{SL}(2,\mathbb{C}). \tag{3}$$

This preserves $\det X$ and thus the Minkowski metric, which leads to the proof that (3) is a Lorentz transformation and all Lorentz transformations can be expressed in this way. The fact that $\pm M$ both give the same transformation of X corresponds to the fact that $\mathrm{SL}(2,\mathbb{C})$ is the double cover of the Lorentz group, that is

$$\mathrm{SO}^+(1,3) \cong \mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2 \tag{4}$$

We denote

$$(\mathcal{M}(\mathbb{R}), \mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2)$$

the structure which consists of $\mathbb{R}$-linear Minkowski space $\mathcal{M}(\mathbb{R})$ with metric given by $X \mapsto \det X$ along with the group $\mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2$ acting on the space as describe in (3).

We note that the isomorphism of groups induces the isomorphism of structures

$$(\mathcal{M}(\mathbb{R}), \mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2) \cong \big(\mathcal{M}(\mathbb{R}), \mathrm{SO}^+(1,3)\big) \tag{5}$$

As in [2] let ${}^*\mathbb{Z}$ be an $\aleph_0$-saturated model of arithmetic, $\mathcal{N} \in {}^*\mathbb{Z}$ divisible by all standard integers and $\mathrm{K} = \mathrm{K}_\mathcal{N} := {}^*\mathbb{Z}/\mathcal{N}$ be the (non-standard) residue ring.[1]

[1]Note that

$$\mathrm{K} \cong \prod_{p|\mathcal{N} \text{ primes}} {}^*\mathbb{Z}/p^{\eta_p}$$

where $\eta_p \in {}^*\mathbb{Z}$ positive, and so, for all standard primes $\eta_p >> 1$. It follows that in the limit ${}^*\mathbb{Z}/p^{\eta_p}$ will be seen as the ring $\mathbb{Z}_p$ of p-adic integers (see [1]) and the whole K as the ring $\mathcal{A}_{\mathbb{Z},\mathrm{fin}}$ of finite integral adeles.

Let $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ which we will treat as a Zariski structure, that is the set with Zariski closed relations $R \subset \bar{\mathbb{C}}^n$ on it.

2.2 *There is a surjective homomorphism* Im_{K} *of Zariski structures*

$$\mathsf{Im}_{\mathrm{K}} : \ \mathrm{K} \to \bar{\mathbb{C}}.$$

In particular,

$$\mathsf{Im}_{\mathrm{K}}(x+y) = \mathsf{Im}_{\mathrm{K}}\, x + \mathsf{Im}_{\mathrm{K}}\, y, \ \textit{if} \ \ \mathsf{Im}\, x \neq \infty \ \textit{and} \ \ \mathsf{Im}\, y \neq \infty$$

$$\mathsf{Im}_{\mathrm{K}}(x \cdot y) = \mathsf{Im}_{\mathrm{K}}\, x \cdot \mathsf{Im}_{\mathrm{K}}\, y, \ \textit{if} \ \ \mathsf{Im}_{\mathrm{K}}\, x \neq \infty \ \textit{and} \ \ \mathsf{Im}_{\mathrm{K}}\, y \neq \infty$$

Im_{K} *is the composition of two Zariski homomorphisms*

$$\mathrm{pr}_{\mathrm{K,E}} : \mathrm{K} \twoheadrightarrow \mathrm{E} \ \textit{and} \ \mathsf{Im}_{\mathrm{E}} : \mathrm{E} \twoheadrightarrow \bar{\mathbb{C}}$$

where E *is a pseudo-finite field.*

The subsets

$$\mathrm{K}_{\mathrm{fin}} = \{x \in \mathrm{K} : \ \mathsf{Im}\, x \neq \infty\} \ \textit{and} \ \mathrm{K}_{\mathrm{real}} = \{x \in \mathrm{K} : \ \mathsf{Im}\, x \in \mathbb{R}\}$$

are subrings of K.

For every positive $n \in \mathbb{N}$

$$nx = 0 \Rightarrow \mathsf{Im}\, x = 0$$

Proof. Since $\mathcal{N}$, the order of K, is divisible by every standard prime q, there is a ring-homomorphism $\mathrm{pr} : \mathrm{K} \twoheadrightarrow \mathrm{E}_{\mathbf{q}}$, for an infinite non-standard $\mathbf{q}$. It follows that if a polynomial $P(X)$ over $\mathbb{Z}$ has a zero in K then it has a zero in $\mathrm{E}_{\mathbf{q}}$, a field of characteristic 0, and so in $\mathbb{C}$. Now one constructs Im by the same back-and-forth procedure as in the proof of Proposition 5.2(i) of [1] using the fact that the cardinality of K is not smaller than that of $\mathbb{C}$.

The statements about $\mathrm{K}_{\mathrm{fin}}$ and $\mathrm{K}_{\mathrm{real}}$ follow from the fact that Im preserves $+$ and $\cdot$ of K.

Finally, assume that $nx = 0$ for $x \in \mathrm{K}$. Note that since Im is surjective $\mathsf{Im}\, 0 = 0$ and $\mathsf{Im}\, n \cdot 1 = n$ for $1 \in \mathrm{K}$. Clearly, if $\mathsf{Im}\, x \neq \infty$ then $0 = \mathsf{Im}\, nx = n\mathsf{Im}\, x$ and so $\mathsf{Im}\, x = 0$. But if $\mathsf{Im}\, x = \infty$ then by the law on multiplication $\mathsf{Im}\,(n \cdot 1 \cdot x) = \infty$ which contradicts the fact that $n \cdot 1 \cdot x = 0$.

□

2.3 Complexification of a ring. Let A be a commutative unitary ring. Define

$A^{(2)}$ be the unitary ring obtained from the ring A as follows:

$$A^{(2)} := \{(a,b) \in A \times A\};$$

$$(a_1,b_1)+(a_2,b_2) := (a_1+a_2, b_1+b_2), \;\; (a_1,b_1)\cdot(a_2,b_2) := (a_1a_2-b_1b_2, a_1b_2+a_2b_1).$$

Clearly, $a \mapsto (a,0)$ is an embedding of A into $A^{(2)}$ as a subring $(A,0)$ and

$$(a,b) \mapsto (a,-b) \text{ an automorphism of } A^{(2)}.$$

2.4 Let $\mathrm{M}(2,A^{(2)})$ be the set of 2×2 matrices over $A^{(2)}$ which we treat as an 8-dim A-module and let $\mathrm{SL}(2,A^{(2)})$ be the group of matrices of determinant 1.

A **Minkowski** A**-lattice** is the A-submodule $\mathcal{M}(A)$ of $\mathrm{M}(2,A^{(2)})$ consisting of matrices $X_{t,x,y,z}$ over $A^{(2)}$ of the form

$$X_{t,x,y,z} = X := \begin{pmatrix} (t+z,0) & (x,-y) \\ (x,y) & (t-z,0) \end{pmatrix}, \quad t,x,y,z \in A.$$

We have

$$\det(X) = (t^2 - x^2 - y^2 - z^2, 0) \in A \times \{0\}$$

and this defines Minkowski A-metric length of (t,x,y,z).

For the general $A^{(2)}$-matrix

$$Y = \begin{pmatrix} (a_1,a_2) & (b_1,b_2) \\ (c_1,c_2) & (d_1,d_2) \end{pmatrix}$$

define the adjoint matrix

$$Y^\dagger := \begin{pmatrix} (a_1,-a_2) & (c_1,-c_2) \\ (b_1,-b_2) & (d_1,-d_2) \end{pmatrix}$$

Clearly, $X^\dagger = X$ for $X \in \mathcal{M}(A)$. In general

$$(YZ)^\dagger = Z^\dagger Y^\dagger.$$

In particular, Y is self-adjoint ($Y = Y^\dagger$) iff $a_2 = 0 = d_2$ and $b_1 = c_1$, $b_2 = -c_2$.

It follows that for any $M \in \mathrm{SL}(2, A^{(2)})$, $X \in \mathcal{M}(A)$

$$MXM^\dagger \in \mathcal{M}(A) \text{ and } \det X = \det MXM^\dagger \qquad (6)$$

Let

$$C = \{M \in \mathcal{M}(A) : MXM^\dagger = X \text{ for all } X \in \mathcal{M}(A).\}$$

Let $M_0 \in C$. In particular, $M_0 M_0^\dagger = \mathrm{I}$. It is equivalent to $M_0^\dagger = M_0^{-1}$ and thus $M_0 X M_0^{-1} = X$ for all $X \in \mathcal{M}(A)$. This readily implies that M_0 is diagonal, in the centre of $\mathrm{SL}(2, A^{(2)})$ and so

$$C = \{M = \begin{pmatrix} (a_1, a_2) & 0 \\ 0 & (a_1, a_2) \end{pmatrix}; \quad a_1^2 - a_2^2 = 1 \ \& \ a_1 a_2 = 0\} \qquad (7)$$

Thus we have established:

2.5 Proposition. *The 2-sorted structure*

$$\Big(\mathcal{M}(A), \mathrm{SL}(2, A^{(2)})/C\Big)$$

is interpretable in the ring A along with the group action $X \mapsto MXM^\dagger$ and A-Minkowski metric.

The action and Minkowski metric are defined by systems of polynomial equations over $\mathbb{Z}$.

In particular, $\mathrm{SL}(2, \mathrm{K}^{(2)})/C$ *is the group of K-linear transformations of* $\mathcal{M}(\mathrm{K})$ *preserving Minkowski* K*-valued metric.*

2.6 Lemma.

$$\mathrm{SL}(2, \mathbb{C}^{(2)})/C \cong \mathrm{SO}(4, \mathbb{C})$$

where C is the centre of $\mathrm{SL}(2, \mathbb{C}^{(2)})$ and

$$C \cong \mathbb{Z}_2 \times \mathbb{Z}_2.$$

Proof. By the Proposition $\mathrm{SL}(2, \mathbb{C}^{(2)})/C$ is the group of transformations of $\mathcal{M}(\mathbb{C})$ preserving Minkowski $\mathbb{C}$-valued metric, that is the form $x_0^2+x_1^2+x_2^2+x_3^2$. But this is also the definition of group $\mathrm{SO}(4, \mathbb{C})$.

The form of C is determined by (7). □

2.7 Compactification of $\mathbb{C}$-structures. Consider $\mathcal{M}(\mathbb{C})$ and $\mathrm{SO}(4,\mathbb{C})$ as complex quasi-projective algebraic varieties, in particular we have

$$\mathcal{M}(\mathbb{C}) \times \mathrm{SO}(4,\mathbb{C}) \subset \mathbf{P}$$

where $\mathbf{P}$ is a projective variety (not uniquely determined). Note that

$$\mathcal{M}(\mathbb{C}) \times \mathrm{SO}(4,\mathbb{C}) \times \mathcal{M}(\mathbb{C}) \hookrightarrow \mathbf{P} \times \mathbf{P}$$

and so the graph of the action of $\mathrm{SO}(4,\mathbb{C})$ on $\mathcal{M}(\mathbb{C})$ is also a quasi-projective subvariety of $\mathbf{P} \times \mathbf{P}$.

Define the **compactification of the structure** $(\mathcal{M}(\mathbb{C}), \mathrm{SO}(4,\mathbb{C}))$,

$$(\mathcal{M}(\mathbb{C}), \mathrm{SO}(4,\mathbb{C}))^{\mathbf{P}} \supseteq (\mathcal{M}(\mathbb{C}), \mathrm{SO}(4,\mathbb{C}))$$

to be the structure defined by the relevant Zariski closed subsets and relations in cartesian powers of $\mathbf{P}$.

2.8 Theorem. *There is a Zariski homomorphism of structures*

$$\mathrm{Lm} : \Big(\mathcal{M}(\mathrm{K}), \mathrm{SL}(2, \mathrm{K}^{(2)})/C\Big) \twoheadrightarrow (\mathcal{M}(\mathbb{C}), \mathrm{SO}(4,\mathbb{C}))^{\mathbf{P}} \qquad (8)$$

Its restriction to the structure over $\mathrm{K}_{\mathrm{real}}$ *is a Zariski homomorphism*

$$\mathrm{Lm} : \Big(\mathcal{M}(\mathrm{K}_{\mathrm{real}}), \mathrm{SL}(2, \mathrm{K}^{(2)}_{\mathrm{real}})/C\Big) \twoheadrightarrow \Big(\mathcal{M}(\mathbb{R}), \mathrm{SO}^{+}(1,3)\Big) \qquad (9)$$

Proof. By 2.2 we have an induced Zariski homomorphism

$$\mathrm{lm}_{\mathrm{K}} : \Big(\mathcal{M}(\mathrm{K}), \mathrm{SL}(2, \mathrm{K}^{(2)})/C\Big) \twoheadrightarrow \Big(\mathcal{M}(\mathbb{C}), \mathrm{SL}(2, \mathbb{C}^{(2)})/C\Big)^{\mathbf{P}}$$

which by 2.6 is the same as (8).

The restriction of limit maps to the structure over $\mathrm{K}_{\mathrm{real}}$ by construction has the form

$$\Big(\mathcal{M}(\mathrm{K}_{\mathrm{real}}), \mathrm{SL}(2, \mathrm{K}^{(2)}_{\mathrm{real}})/C\Big) \twoheadrightarrow (\mathcal{M}(\mathbb{R}), \mathrm{SL}(2,\mathbb{C})/C)$$

which becomes (9) when one takes into account (4). □

2.9 Commentary.

The statement in (9) can be interpreted as the statement that *at low scale* the pseudo-finite space looks like the canonical Minkowski space $\mathcal{M}(\mathbb{R})$ with the action of the Lorentz group $\mathrm{SO}^{+}(1,3)$.

2.10 Relation between the discrete Minkowski space $\mathcal{M}(\mathrm{K})$ and the discrete universe $\mathbb{U}$ of [2].

Recall that the 1-dimensional universe $\mathbb{U}$ of [2] is defined as the additive group of the residue ring

$$\mathrm{K} = {}^*\mathbb{Z}/\mathcal{N}, \text{ where } \mathcal{N} = (\mathfrak{p}-1)\mathfrak{l},$$

$\mathfrak{p}$ non-standard prime and $\mathfrak{l}$ a highly divisible non-standard integer. Thus $\mathbb{U}$ can also be considered a 1-dimensional K-module, where we can now identify K with the one from previous sections, introduced in 2.1.

Thus, for the Minkowski K-space $\mathcal{M}(\mathrm{K})$ one establishes an isomomrphism

$$\mathcal{M}(\mathrm{K}) \cong \mathbb{U}^4$$

as K-modules, and the constructions above define the action of the quasi-Lorentz group $\mathrm{SL}(2, \mathrm{K}^{(2)})/C$ on $\mathbb{U}^4$ along with the Minkowski K-valued metric invariant under $\mathrm{SL}(2, \mathrm{K}^{(2)})/C$.

In [2] we identified in the universe $\mathbb{U}$ and its cartesian powers $\mathbb{U}^n$ subdomains which correspond to the scales of quantum mechanics and statistical mechanics and developed elements of these theories in the model on $\mathbb{U}$ which unified the two theories. The current work demonstrates that the same model can incorporate special relativity.

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