Geometric stability and Zariski geometries

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July 28, 2010

Lecture I

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- Model theory allows us to explore the landscape of mathematics and beyond.
- **Zariski geometries** is the class of structures discovered in this exploration.
- Zariski geometries are on the very top of stability hierarchy, so, in the very centre of mathematics.

Noetherian Zariski structures: The idea

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Example. Algebraic Geometry is a model theory of (algebraically closed) fields with the emphasis on positively quantifier-free definable sets (**Zariski-closed** sets).

Noetherian Zariski structures: Definition and Axioms

Let **M** be a structure and let $\mathcal{C}$ be a distinguished sub-collection of the definable subsets of M^n, $n = 1, 2, \ldots$. The sets in $\mathcal{C}$ are called (definable) **closed.** The relations corresponding to the sets are the basic (primitive) relations of the language we will work with. $\langle M, \mathcal{C} \rangle$, or **M**, is **a topological structure** if it satisfies axioms:

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(L) Topological Language: The primitive n-ary relations of the language are exactly the ones that distinguish definable closed subsets of M^n, all n (that is the ones in $\mathcal{C}$), and every quantifier-free positive formula in the language defines a closed set (so is equivalent to an atomic one).

Noetherian Zariski structures: Definition and Axioms

More precisely:

1. the intersection of a finite family of closed sets is closed;
2. finite unions of closed sets are closed;
3. the domain of the structure is closed;
4. the graph of equality is closed;
5. any singleton of the domain is closed;
6. Cartesian products of closed sets are closed;
7. the image of a closed $S \subseteq M^n$ under a permutation of coordinates is closed;
8. for $a \in M^k$ and S a closed subset of M^{k+l} defined by a predicate $S(x, y)$ the *fibre over a*

$$S(a, M^l) = \{b \in M^l : M \models S(a, b)\}$$

is closed.

Noetherian Zariski structures: Definition and Axioms

Remarks

L6 assumes that, for $S_1 \subseteq M^n$ and $S_2 \subseteq M^m$ closed, $S_1 \times S_2$ is canonically identified with a subset of M^{n+m} which is closed in the latter.

The canonical identification is

$$\langle\langle x_1, \ldots, x_k\rangle, \langle y_1, \ldots, y_m\rangle\rangle \mapsto \langle x_1, \ldots, x_k, y_1, \ldots, y_m\rangle.$$

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A *projection*

$$\mathrm{pr}_{i_1,\ldots,i_m} : \langle x_1, \ldots, x_n\rangle \mapsto \langle x_{i_1}, \ldots, x_{i_m}\rangle, \quad i_1, \ldots, i_m \in \{1, \ldots, n\}.$$

is a continuous map, by L6:

the inverse image of a closed set S is closed. Indeed,

$$\mathrm{pr}^{-1}_{i_1,\ldots,i_m} S = S \times M^{n-m}$$

up to the order of coordinates.

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A subset of M^n will be called **projective** if it is a finite union of sets of the form $\mathrm{pr}\, S_i$, for some $S_i \subseteq_{\rm cl} U_i \subseteq_{\rm op} M^{n+k_i}$ and projections $\mathrm{pr}^{(i)} : M^{n+k_i} \to M^n$.
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A topological structure is said to be **complete** if
(P) **Properness** of projections condition holds:
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A topological structure M will be called **quasi-compact** (or just **compact**) if it is complete and satisfies
(QC) For any finitely consistent family $\{C_t : t \in T\}$ of closed subsets

$$\bigcap_{t \in T} C_t \text{ is non-empty.}$$

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A topological structure is called **Noetherian** if it also satisfies:
(DCC) **Descending chain condition** for closed subsets: for any closed

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A definable set S is called **irreducible** if there are no relatively closed subsets $S_1 \subseteq_{\mathrm{cl}} S$ and $S_2 \subseteq_{\mathrm{cl}} S$ such that $S_1 \subsetneq S_2$, $S_2 \subsetneq S_1$ and $S = S_1 \cup S_2$.

Noetherian Zariski structures: Definition and Axioms

Good dimension

We assume that to any non-empty projective S a non-negative integer called **the dimension of** S, $\dim S$, is attached.

We postulate the following properties of a good dimension notion:

(DP) **Dim of a point** is 0;

(DU) **Dim of unions:** $\dim(S_1 \cup S_2) = \max\{\dim S_1, \dim S_2\}$;

(SI) **Strong irreducibility:** For any irreducible $S \subseteq_{\mathrm{cl}} U \subseteq_{\mathrm{op}} M^n$ and its closed subset $S_1 \subseteq_{\mathrm{cl}} S$, if $S_1 \neq S$ then $\dim S_1 < \dim S$;

(AF) **Addition formula:** For any irreducible $S \subseteq_{\mathrm{cl}} U \subseteq_{\mathrm{op}} M^n$ and a projection map $\mathrm{pr} : M^n \to M^m$,

$$\dim S = \dim \mathrm{pr}\,(S) + \min_{a \in \mathrm{pr}\,(S)} \dim(\mathrm{pr}^{-1}(a) \cap S).$$

(FC) **Fibre condition:** For any irreducible $S \subseteq_{\mathrm{cl}} U \subseteq_{\mathrm{op}} M^n$ and a projection map $\mathrm{pr} : M^n \to M^m$ there exists $V \subseteq_{\mathrm{op}} \mathrm{pr}\, S$ (relatively open) such that

$$\min_{a \in \mathrm{pr}\,(S)} \dim(\mathrm{pr}^{-1}(a) \cap S) = \dim(\mathrm{pr}^{-1}(v) \cap S), \text{ for any } v \in V \cap \mathrm{pr}\,(S).$$

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More generally we replace (P) by

(SP) **semi-Properness** of projection mappings: given a closed irreducible subset $S \subseteq_{\mathrm{cl}} M^n$ and the projection map $\mathrm{pr} : M^n \to M^k$, there is a proper closed subset $F \subset \overline{\mathrm{pr}\, S}$ such that $\overline{\mathrm{pr}\, S} \setminus F \subseteq \mathrm{pr}\, S$.

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Noetherian topological structures with good dimension and satisfying (SP) will be called **(Noetherian) Zariski structures.**

Noetherian Zariski structures: Definition and Axioms

In many cases we assume that a Zariski structure satisfies also (EU) **Essential uncountability:** If a closed $S \subseteq M^n$ is a union of countably many closed subsets, then there are finitely many among the subsets, the union of which is S.

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The following is an extra condition crucial for developing a rich theory for Zariski structures.

(PS) **Presmoothness:** For any closed irreducible $S_1, S_2 \subseteq M^n$, for any irreducible component S_0 of $S_1 \cap S_2$,

$$\dim S_0 \geq \dim S_1 + \dim S_2 - \dim M^n.$$

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This can be generalised to a definition of a **(n-dim Noetherian) Zariski geometry.**

Noetherian Zariski geometries: Examples

1. Smooth algebraic varieties over an uncountable algebraically closed field, in the natural language (1990).

"Uncountable" needed to satisfy (EU).
Natural language: $\mathcal{C}$ consists of Zariski-closed subsets of M^n.

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2. Compact complex manifolds, in the natural language (1993).

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1. Smooth algebraic varieties over an uncountable algebraically closed field, in the natural language (1990).

2. Compact complex manifolds, in the natural language (1993).

3. Definable substructures of $\mathrm{DCF}_0(n)$ of finite Morley rank. (2001)

More precisely: every definable substructure of finite Morley rank can be made Zariski in a natural language by removing a subset of smaller rank.

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4. "Quantum geometries".

Model theory of Noetherian Zariski structures

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Proof. Use Theorem 1 to show by induction on dim Q, constructible Q, that $\mathrm{Mrk}\, Q \leq \dim Q$.
(EU) provides $\aleph_0$-saturation for countable fragments of the language.

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Theorem 3 Assume **M** satisfies (EU). Given $\mathbf{M}' \succeq \mathbf{M}$ one can naturally extend the topology to M' so that $\mathbf{M}'$ becomes a Noetherian Zariski structure satisfying (EU).

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Again, (EU) is essential in providing a saturation.

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Proposition. Suppose $\mathbf{M}$ is a quasi-compact structure, $\mathbf{M}' \succeq \mathbf{M}$. Then there is a total specialisation $\pi : \mathbf{M}' \rightarrow \mathbf{M}$. Moreover, any partial specialisation can be extended to a total one.

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The inverse also holds for a right choice of topology on **M**.

Specialisations and infinitesimal calculus

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for any $\mathbf{M}' \succeq {}^*\mathbf{M} \succeq \mathbf{M}$, any finite subset $A \subset M'$ and a specialisation $\pi' : A \cup {}^*M \to M$ extending π, there is an elementary embedding $\alpha : A \to {}^*M$, over $A \cap {}^*M$, such that

$$\pi' = \pi \circ \alpha \text{ on } A.$$

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This depends strongly on $\mathbf{M}'$ and π.

A specialisation $\pi : {}^*\mathbf{M} \to \mathbf{M}$, for ${}^*\mathbf{M} \succeq \mathbf{M}$, is said to be **universal** if:
for any $\mathbf{M}' \succeq {}^*\mathbf{M} \succeq \mathbf{M}$, any finite subset $A \subset M'$ and a specialisation $\pi' : A \cup {}^*M \to M$ extending π, there is an elementary embedding $\alpha : A \to {}^*M$, over $A \cap {}^*M$, such that

$$\pi' = \pi \circ \alpha \text{ on } A.$$

Proposition. Every specialisation $\pi^0 : \mathbf{M}^0 \to \mathbf{M}$ can be extended to a universal one $\pi : {}^*\mathbf{M} \to \mathbf{M}$.

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Assuming π is universal, the geometric properties of $\mathcal{V}_a$ are independent on π and ${}^*\mathbf{M}$.

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Corollary. For $a \in \mathrm{Reg}\, F/D$ and $\langle a, b\rangle \in F$ the set $F \cap (\mathcal{V}_a \times \mathcal{V}_b)$ is the graph of a function $\varphi : \mathcal{V}_a \to \mathcal{V}_b$ (local function).

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Let L_1, L_2 and P be constructible irreducible presmooth sets and $I_i \subseteq_{\text{cl}} L_i \times P$, $i = 1, 2$, irreducible. We will call a **curve** coded by $\ell \in L_i$ the set

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Lemma. Given a family of curves L on P as above, the set of jets $[L]_p$ through p is definable (interpretable) and under certain assumptions can be identified with a Zariski constructible set.

Classification of 1-dim non-locally modular Noetherian Zariski geometries

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Proposition. Non-local modularity implies: some irreducible $P \subseteq_{\mathrm{op}} M \times M$, some Zariski irreducible presmooth set L in $\mathbf{M}$ and $I \subseteq_{\mathrm{cl}} L \times P$ define a 2-dimensional family of curves on P.

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At a generic point $\langle a, b\rangle \in M^2$ a generic curve ℓ_1 locally (i.e. in infinitesimal neighbourhood) is the graph of a local function

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Given ℓ_1 and ℓ_2 the local function

$$\lambda_1^{-1} \circ \lambda_2 : \mathcal{V}_a \to \mathcal{V}_a$$

corresponds to a new curve through $\langle a, a \rangle$ (rather a **branch of a curve**).

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Corollary. There is a group structure $(G, \circ)$ definable by Zariski-closed predicates on a 1-dim irreducible Zariski set G. (Copy the proof of Weil's group chunk theorem).

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With more work one obtains

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- The theory of multiplicities can be applied to get an intersection theory in projective spaces. In particular, the following generalisation of Bezout's theorem holds: given in $\mathbf{P}^2(K)$ a curve ℓ and an algebraic curve ℓ_{alg}

$$\#_{\mathrm{mult}}(\ell \cap \ell_{\mathrm{alg}}) = \deg \ell \cdot \deg \ell_{\mathrm{alg}},$$

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  where $\deg \ell$ is defined as a number of point in the intersection of a generic straight line in $\mathbf{P}^2(K)$ with ℓ.
- The latter implies that any $S \subseteq_{\mathrm{cl}} \mathbf{P}^n(K)$ must be algebraic (generalisation of Chow's theorem).

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- In general, such functions can be seen as co-ordinate functions and given $\mathbf{f} = \langle \bar{f}_1, \ldots, \bar{f}_n \rangle$ we obtain a map

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$\mathbf{f}(M)$ is a *quasi-projective* curve $C \subseteq \mathbf{P}^N(K)$ and

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- The latter classifies $\mathbf{M}$ up to the finite fibres $\mathbf{f}^{-1}(a)$, $a \in C$.

Lecture III

Generalities:

The classification of 1-dimensional Zariski geometries found its application in e.g. Diophantine Geometry.

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The classification of 1-dimensional Zariski geometries found its application in e.g. Diophantine Geometry.

But even more interesting is that it lead to the discovery of a class of **new geometric objects**.

New geometric objects

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There exists 1-dimensional $\mathbf{M}$ such that no covering $\mathbf{f} : \mathbf{M} \to C$ is bijective (C an algebraic curve). In other words, 1-dimensional Zariski geometry can be different from an algebraic curve.

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Example. Let M be the set

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So, the set $K = M/E$ is definable and we have all polynomially defined relations on K, lifted to relations on M, in our language. Let $R \subseteq K \setminus \{0\}$ be a subset with the property:

$$y \in R \text{ iff } -y \notin R.$$

Introduce a new ternary relation $A \in \mathcal{C}$, $A \subseteq M \times M \times K$:

$$A(\langle x_1, \epsilon_1\rangle, \langle x_2, \epsilon_2\rangle, y) \text{ iff } x_2 = x_1 + 1 \ \& y^2 = x_1^2 \ \&$$

$$\&\ ((y \in R \ \&\ \epsilon_1 = \epsilon_2) \vee (y \notin R \ \& y \neq 0 \ \&\ \epsilon_1 \neq \epsilon_2) \vee y = 0)$$

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- Consider *Galois theory* of $(K(\langle x, \epsilon\rangle) : K)$ and prove that one can not interprete $\langle x, \epsilon\rangle$ as a tuple in a field extension of K.

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We have two linear operators $\mathbf{a}$ and $\mathbf{a}^\dagger$ acting in the linear space generated by the e_x which satisfy

$$(\mathbf{a}^\dagger\mathbf{a} - \mathbf{a}\mathbf{a}^\dagger)e_x = e_x.$$

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5. $(\mathcal{A}(\mathbf{M}), *)$ *does not depend on* $\mathcal{H}(\mathbf{M})$, *only on* $\mathbf{M}$. *One recovers the whole of structure* $\mathbf{M}$ *from* $\mathcal{A}(\mathbf{M})$.

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is well-established only for special class of algebras $\mathcal{A}$ and structures $\mathbf{M}$.

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Examples

- The algebra T_q^2 generated by U and V with defining relation

$$UV = qVU, \text{ in case } q^N = 1.$$

- Many other algebras, e.g. quantum groups $\mathrm{SL}(2, K)_q$, $Usl_q(2, K)$.

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Theorem. There is a canonical procedure that puts in correspondence to any K-algebra $\mathcal{A}$ at root of unity, K algebraically closed, a Zariski geometry $\mathbf{M}$, so that $\mathcal{A}$ can be canonically recovered from $\mathbf{M}$.

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Construction. Consider the affine variety $V = V(\mathcal{A})$ corresponding to the affine commutative algebra $Z(\mathcal{A})$. To each point of V corresponds a unique, up to isomorphism, N-dimensional $\mathcal{A}$-module. The bundle of such modules over V is $\mathbf{M}(\mathcal{A})$.

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Question. What to do for a general value of q?

Lecture IV

Trichotomy conjecture and Hrushovski counterexamples

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Classical first-order λ-categorical structures for **uncountable** λ :

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Almost true for Zariski geometries (HZ,1993).

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- *The resulting structure* $(\tilde{\mathbf{M}}, f)$ *will have a good dimension notion and a nice geometry.*

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This is **the Schanuel conjecture.**

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Amalgamation process produces an *algebraically-exponentially closed* **field with pseudo-exponentiation**, $K_{\mathrm{ex}}(\lambda)$, of cardinality λ.

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Algebraic-exponential closedness (**existential closedness**) takes the form:

EC: For any *non-overdetermined* irreducible system of polynomial equations

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ACF_0 : Axioms for algebraically closed fields of characteristic 0.

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Main Theorem *Given an uncountable cardinal* λ, *there is a unique, up to isomorphism, structure* K_{ex} *of cardinality* λ *satisfying*

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Equivalently: $\mathbb{C}_{\mathrm{exp}}$ *satisfies* $\mathrm{SCH} + \mathrm{EC}$.

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3. The new fact (with M.Bays): Let $L_1, \ldots, L_n$ be algebraically closed fields *mutually linearly disjoint over their intersections*. Then, for the multiplicative group of their composite, $(L_1 \cdot \ldots \cdot L_n)^* \cong L_1^* \cdot \ldots \cdot L_n^* \times A$, for a free abelian group A.

Conclusion

Hrushovski's counter-examples are not pathologies.

Lecture V

Generalities:

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- Noetherian Zariski Geometry is an extension of Algebraic Geometry (into a non-commutative domain).
- Some interesting mathematics may lie outside the narrow context of Noetherian Zariski geometries.

Analytic Zariski geometries

Definition. We say that $\mathbf{M} = (M, \mathcal{C})$ is a **pre-analytic** Zariski structure if:

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- (case $\dim M = 1$) given $F \subseteq_{\mathrm{cl}} V \subseteq_{\mathrm{op}} M^{n+k}$ with the projection $\mathrm{pr} : M^{n+k} \to M^n$ such that $\dim \mathrm{pr}\, F = n$, there exists $D \subseteq_{\mathrm{op}} M^n$ such that $D \subseteq \mathrm{pr}\, F$.

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- For every $S \subseteq_{\mathrm{cl}} U \subseteq_{\mathrm{op}} M^n$ there are at most countably many constructible irreducible sets $S_i \subseteq M^n$, $I \in \mathbb{N}$, with

$$S = \bigcup S_i.$$

Analytic Zariski geometries

Definition (continued) A pre-analytic Zariski **M** is said to be **analytic** if

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- Given a subset $S \subseteq_{\text{cl}} U \subseteq_{\text{op}} M^n$ the natural number ${}_U(S)$, (**analytic rank**) is well-defined by:
  1. ${}_U(S) = 0$ iff $S = \emptyset$;
  2. ${}_U(S) \leq k + 1$ iff there is a set $S' \subseteq_{\text{cl}} S$ such that ${}_U(S') \leq k$, and the set $S^0 = S \setminus S'$ is a countable union of irreducible closed subsets.

A subset $S \subseteq_{\text{cl}} U \subseteq_{\text{op}} M^n$ is said to be **analytic** if ${}_U(S) = 1$.

Model theory of pre-analytic Zariski structures

Let **M** be an analytic Zariski structure of dimension 1.
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For finite $X \subseteq M$ we define the $\mathcal{C}_0$-predimension

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For $X \subseteq M$ finite, we say that X is **self-sufficient** and write $X \leq M$, if $\mathrm{d}(X) = \delta(X)$.

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Lemma 2. Given X, X', XY all finite self-sufficient, suppose $X \equiv_{\text{qftp}} X'$. Then there is Y' such that $XY \equiv_{\text{qftp}} X'Y'$.

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5. $Y \equiv^{\exists}_{(X)} Y' \Rightarrow$ exists an elementary monomorphism over $(X), \; (XY) \to (XY')$.

In other words, **M** *is quasi-minimal ω-homogeneous over submodels.*

Is M excellent?

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Fact. For all *natural* analytic Zariski **M**, when the answer is known: **yes.**

Theorem 4 Suppose **M** is excellent. Then for every $\kappa > \mathrm{card} M$ there is a (pre)analytic Zariski **M**′ of cardinality κ,

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This **M**′ is unique up to isomorphism.

Examples of analytic Zariski geometries

1. Abstract covers of the algebraic torus K^*, for an uncountable algebraically closed field K, any characteristic.

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4. Pseudo-exponentiation, as a pre-analytic structure (?)