Covers of modular curves, categoricity and Drinfeld's GT

B.Z. joint with C.Daw

University of Oxford

June, 2020

Some history

- *The $L_{\omega_1,\omega}$-theory of the universal cover* $\exp : \mathbb{C} \to \mathbb{G}_m(\mathbb{C})$ *is categorical.* (B.Z. 2004, B.Z. and M.Bays 2011)

Required Kummer theory and Dedekind theory of Galois action on μ_∞.

Some history

- *The $L_{\omega_1,\omega}$-theory of the universal cover* $\exp : \mathbb{C} \to \mathbb{G}_m(\mathbb{C})$ *is categorical.* (B.Z. 2004, B.Z. and M.Bays 2011)

Required Kummer theory and Dedekind theory of Galois action on μ_∞.

- *The $L_{\omega_1,\omega}$-theory of the universal cover of an elliptic curve is categorical.* (M.Bays 2012 using M.Gavrilovich 2007)

Required Kummer-Bashmakov theory and Serre's open image theorem (on Galois action on torsion of the elliptic curve).

Some history

- *The $L_{\omega_1,\omega}$-theory of the universal cover of an abelian variety is categorical conditional on extension of Serre's open image theorem on torsion(= $\hat{\pi}_1^{top}$) of the abelian variety*
(Bays-Hart-Pillay 2015 based on Bays 2013)

Requires Kummer-Ribet-Larsen theory.

Some history

- *The $L_{\omega_1,\omega}$-theory of the universal cover of an abelian variety is categorical* *conditional* *on extension of Serre's open image theorem on torsion(= $\hat{\pi}_1^{top}$) of the abelian variety* (Bays-Hart-Pillay 2015 based on Bays 2013)

Requires Kummer-Ribet-Larsen theory.

- *The $L_{\omega_1,\omega}$-theory of the cover $j_\Gamma : \mathbb{H} \to \mathbb{Y}(\Gamma)$ with fixed* $\mathrm{GL}^+(2, \mathbb{Q})$- *action is categorical* (A.Harris 2013, A.Harris and C.Daw 2014)

Requires Serre's open image theorem for tuples of elliptic curves without CM.

Problems

Formulate a geometrically natural $L_{\omega_1,\omega}$-theory of $\tilde{\mathbb{X}}^{an}(\mathbb{C})$ for hyperbolic curves (and more general), in particular, for $\mathbb{X} = \mathbf{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$.

Formulate necessary and sufficient conditions for categoricity of this theory.

Problems

Formulate a geometrically natural $L_{\omega_1,\omega}$-theory of $\tilde{\mathbb{X}}^{an}(\mathbb{C})$ for hyperbolic curves (and more general), in particular, for $\mathbb{X} = \mathbf{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$.

Formulate necessary and sufficient conditions for categoricity of this theory.

Why: A categorical theory is a **complete formal invariant** of a geometric structure.

Motivating Claims

Claim 1. The necessary condition for categoricity of $\tilde{\mathbb{X}}^{an}$ $\mathbb{X} = \mathbf{P}^1 \setminus \{0, 1, \infty\}$, is to identify GT, the Grothendieck - Teichmuller group: the Galois action on the $\hat{\mathrm{Fr}}_2$ (Esquisse de un programme)

Drinfeld's conjecture (1990) identifies GT.

Motivating Claims

Claim 1. The necessary condition for categoricity of $\tilde{\mathbb{X}}^{an}$ $\mathbb{X} = \mathbf{P}^1 \setminus \{0, 1, \infty\}$, is to identify GT, the Grothendieck - Teichmuller group: the Galois action on the $\hat{\mathrm{Fr}}_2$ (Esquisse de un programme)

Drinfeld's conjecture (1990) identifies GT.

Claim 2. The categoricity of the structure $\tilde{\mathbb{Y}}^{an}_{mod}$ is a weaker version of the case above:

$\mathbb{Y} = \mathbb{Y}(1) = \Gamma \backslash \mathbb{H}$, the modular curve, $\Gamma = \mathrm{SL}(2, \mathbb{Z})/(-\mathbf{I})$, and $\tilde{\mathbb{Y}}^{an}_{mod}$ includes finite covers **by modular curves only**

$$\mathbb{Y}(n) \to \mathbb{Y}(1).$$

Note: $\mathbb{Y}(2) = \mathbf{P}^1 \setminus \{0, 1, \infty\}$.

Motivating Claims

Claim 1. The necessary condition for categoricity of $\tilde{\mathbb{X}}^{an}$ $\mathbb{X} = \mathbf{P}^1 \setminus \{0, 1, \infty\}$, is to identify GT, the Grothendieck - Teichmuller group: the Galois action on the $\hat{\mathrm{Fr}}_2$ (Esquisse de un programme)

Drinfeld's conjecture (1990) identifies GT.

Claim 2. The categoricity of the structure $\tilde{\mathbb{Y}}^{an}_{mod}$ is a weaker version of the case above:

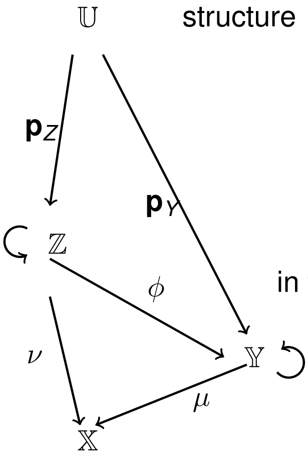
$\mathbb{Y} = \mathbb{Y}(1) = \Gamma \backslash \mathbb{H}$, the modular curve, $\Gamma = \mathrm{SL}(2, \mathbb{Z})/(-\mathbf{I})$, and $\tilde{\mathbb{Y}}^{an}_{mod}$ includes finite covers **by modular curves only**

$$\mathbb{Y}(n) \to \mathbb{Y}(1).$$

Note: $\mathbb{Y}(2) = \mathbf{P}^1 \setminus \{0, 1, \infty\}$.

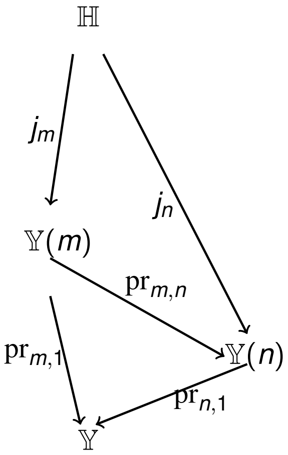
The universal cover $\tilde{\mathbb{X}}^{an}(\mathbb{C})$ with given $\mathrm{Gal}_k \hookrightarrow \pi_1^{et}$

analytic structure



finite étale covers in the field sort $\mathbb{C}$

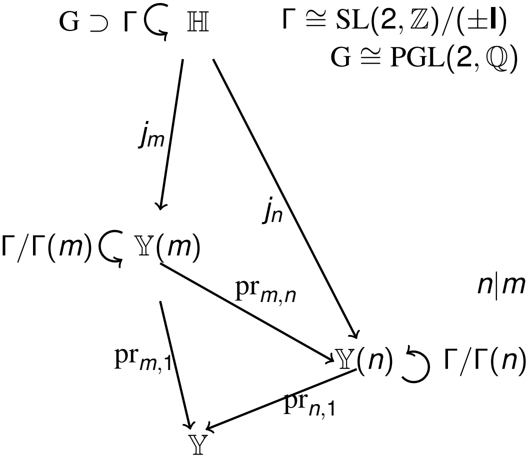
The cover of $\mathbb{Y} = \mathbb{Y}(1) = A^1$ by modular curves over $\mathbb{Q}$



$n|m$

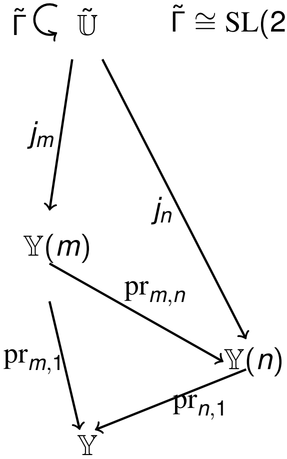
$\mathbb{Y}(n) \subset A^3(\mathbb{C})$

The cover of $\mathbb{Y} = \mathbb{Y}(1) = A^1$ by modular curves over $\mathbb{Q}$



$\mathbb{Y}(n) \subset A^3(\mathbb{C})$

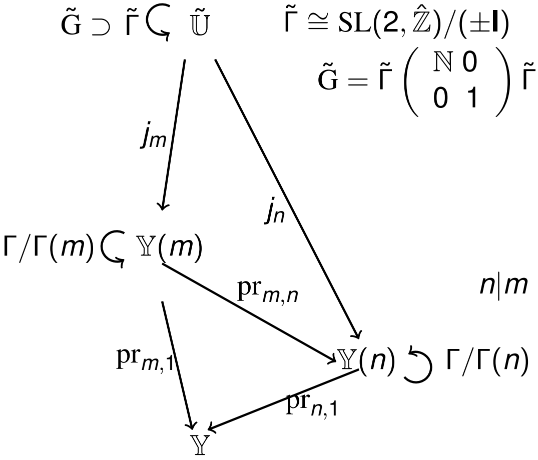
An adelic version of the modular cover



$$\tilde{\Gamma} \cong \mathrm{SL}(2, \hat{\mathbb{Z}})/(\pm \mathbf{I})$$

$n|m$

An adelic version of the modular cover



Two multisorted structures

Let $\mathbb{X}(n) = \bar{\mathbb{Y}}(n) = j_n(\mathbb{H}^*)$.
$\infty_n = j_n(\infty) \in \mathbb{X}(n) \setminus \mathbb{Y}(n)$. $\mathrm{UT} = g \in \Gamma : g \cdot \infty = \infty$.

Proposition.

$$(\tilde{\mathbb{U}}, \tilde{\mathrm{G}}, \tilde{\mathrm{UT}}, \tilde{\Gamma}(n))_{n\in\mathbb{N}}$$

and

$$(\mathbb{X}(n), \mathrm{pr}_{n,n/d}, \infty_n, \Psi_{m,n})_{n\in\mathbb{N},\ d|n}$$

are bi-interpretable (in L_{ω_1,ω_1}).

where $\Psi_{m,n}(x_1, x_2, y_1, y_2) \subset \mathbb{X}(n)^4$:

$$\exists g \in \mathrm{G}_{\det=m}\ \exists u, v \in \mathbb{H}^* :$$

$$\langle x_1, x_2\rangle = \langle j_n(u), j_n(gu)\rangle\ \&\ \langle y_1, y_2\rangle = \langle j_n(v), j_n(gv)\rangle$$

The CM-substructure

$$(\tilde{\mathbb{U}}, \tilde{\mathrm{G}}, \tilde{\mathrm{UT}}, \tilde{\Gamma}(n))_{n\in\mathbb{N}}$$

is determined by

$$(\tilde{\mathrm{G}}, \tilde{\mathrm{UT}}, \tilde{\Gamma}(n))_{n\in\mathbb{N}} \ \textit{ and } \tilde{\mathrm{E}} = \{e \in \tilde{\mathrm{G}} : \exists u_e \ e \cdot u_e = u_e\}$$

the **elliptic** *transformation.*

The CM-substructure

$$(\tilde{\mathbb{U}}, \tilde{\mathrm{G}}, \tilde{\mathrm{UT}}, \tilde{\Gamma}(n))_{n\in\mathbb{N}}$$

is determined by

$$(\tilde{\mathrm{G}}, \tilde{\mathrm{UT}}, \tilde{\Gamma}(n))_{n\in\mathbb{N}} \ \textit{and}\ \tilde{\mathrm{E}} = \{e \in \tilde{\mathrm{G}} : \exists u_e\ e \cdot u_e = u_e\}$$

the **elliptic** *transformation.*

$$(\tilde{\mathrm{G}}, \tilde{\mathrm{UT}}, \tilde{\mathrm{E}}, \tilde{\Gamma}(n))_{n\in\mathbb{N}} \cong_{\mathrm{bi-int}} (\mathrm{CM}_*, \Psi_*)$$

The CM-substructure

$$(\tilde{\mathbb{U}}, \tilde{\mathrm{G}}, \tilde{\mathrm{UT}}, \tilde{\Gamma}(n))_{n\in\mathbb{N}}$$

is determined by

$$(\tilde{\mathrm{G}}, \tilde{\mathrm{UT}}, \tilde{\Gamma}(n))_{n\in\mathbb{N}} \text{ and } \tilde{\mathrm{E}} = \{e \in \tilde{\mathrm{G}} : \exists u_e \; e \cdot u_e = u_e\}$$

the **elliptic** *transformation.*

$$(\tilde{\mathrm{G}}, \tilde{\mathrm{UT}}, \tilde{\mathrm{E}}, \tilde{\Gamma}(n))_{n\in\mathbb{N}} \cong_{\mathrm{bi-int}} (\mathrm{CM}_*, \Psi_*)$$

$$\mathrm{Aut}(\tilde{\mathrm{G}}, \tilde{\mathrm{UT}}, \tilde{\mathrm{E}}, \tilde{\Gamma}(n))_{n\in\mathbb{N}} = \mathrm{Aut}(\mathrm{CM}_*, \Psi_*)$$

Galois action

Theorem. There is a subgroup

$$\mathrm{Out}_*\tilde{\Gamma} \subseteq \mathrm{Out}\,\tilde{\Gamma}$$

and an isomorphism

$$\mathrm{Aut}(\tilde{\mathrm{G}}, \tilde{\mathrm{UT}}, \tilde{\mathrm{E}}, \tilde{\Gamma}(n))_{n\in\mathbb{N}} \cong \mathrm{Out}_*\tilde{\Gamma}$$

Moreover, there is an embedding

$$\mathfrak{h} : \mathrm{Gal}(\mathbb{Q}(\mathrm{CM}_*) : \mathbb{Q}) \hookrightarrow \mathrm{Out}_*\tilde{\Gamma}$$

Galois action

Theorem. There is a subgroup

$$\mathrm{Out}_*\tilde{\Gamma} \subseteq \mathrm{Out}\,\tilde{\Gamma}$$

and an isomorphism

$$\mathrm{Aut}(\tilde{\mathrm{G}}, \tilde{\mathrm{UT}}, \tilde{\mathrm{E}}, \tilde{\Gamma}(n))_{n\in\mathbb{N}} \cong \mathrm{Out}_*\tilde{\Gamma}$$

Moreover, there is an embedding

$$\mathfrak{h} : \mathrm{Gal}(\mathbb{Q}(\mathrm{CM}_*) : \mathbb{Q}) \hookrightarrow \mathrm{Out}_*\tilde{\Gamma}$$

Conjecture (GT/CM). $\mathfrak{h}$ *is an isomorphism.*

The group $\mathrm{Out}_*\tilde{\Gamma}$

Theorem. Let

$$\mathbf{t} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \ \mathbf{t}_- = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \ \mathbf{t}^\lambda = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, \ \mathbf{s} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}.$$

Any automorphism $\phi \in \mathrm{Out}_*\tilde{\Gamma}$ has the form

$$\phi_{\lambda,x} : \mathbf{t} \mapsto \mathbf{t}^\lambda, \ \ \mathbf{s} \mapsto \mathbf{sf}, \ \ \mathbf{t}_- \mapsto \mathbf{f}^{-1}\mathbf{t}_-^\lambda \mathbf{f}$$

where

$$\mathbf{f} = \mathbf{f}_{\lambda,x} = \begin{pmatrix} \lambda^{-1} & x \\ x & (1+x^2)\lambda \end{pmatrix}, \ \lambda \in \hat{\mathbb{Z}}^*, \ x, \lambda - 1 \in 2\hat{\mathbb{Z}}.$$

The group $\mathrm{Out}_*\tilde{\Gamma}$

Moreover, in terms of the canonical free generators X, Y of $\tilde{\Gamma}(2)$, the (limit) word $\mathbf{f} = \mathbf{f}(X, Y)$ satisfies

$$\mathbf{f}(Y, X) = \mathbf{f}(X, Y)^{-1}$$

and, for Z such that $XYZ = \mathbf{I}$, $\mu = \frac{\lambda - 1}{2}$, we the identity holds:

$$\mathbf{f}(Z, X)\, Z^{\mu}\, \mathbf{f}(Y, Z)\, Y^{\mu}\, \mathbf{f}(X, Y)\, X^{\mu} = \mathbf{I}.$$

Drinfeld's associator $\mathfrak{f}$

Drinfeld defined a subgroup of the automorphism group of $\hat{\mathrm{Fr}}_2$ on generators X, Y :

$$\mathrm{GT} := \{\phi \in \mathrm{Aut}\,\hat{\mathrm{Fr}}_2 : \phi(X) = X^\lambda,\ \phi(Y) = \mathfrak{f}^{-1} Y^\lambda \mathfrak{f}\}$$

where $\lambda \in \hat{\mathbb{Z}}^*$, $\lambda - 1 \in 2\hat{\mathbb{Z}}$,

$$\mathfrak{f} = \mathfrak{f}(X, Y) = \mathfrak{f}(Y, X)^{-1} \in \hat{\mathrm{Fr}}_2$$

and, for $\mu = (\lambda - 1)/2$, once $XYZ = 1$, it is assumed to hold:

$$\mathfrak{f}(Z, X) Z^\mu \mathfrak{f}(Y, Z) Y^\mu \mathfrak{f}(X, Y) X^\mu = 1.$$

Drinfeld's associator $\mathfrak{f}$

Drinfeld defined a subgroup of the automorphism group of $\hat{\mathrm{Fr}}_2$ on generators X, Y :

$$\mathrm{GT} := \{\phi \in \mathrm{Aut}\,\hat{\mathrm{Fr}}_2 : \phi(X) = X^\lambda,\ \phi(Y) = \mathfrak{f}^{-1} Y^\lambda \mathfrak{f}\}$$

where $\lambda \in \hat{\mathbb{Z}}^*$, $\lambda - 1 \in 2\hat{\mathbb{Z}}$,

$$\mathfrak{f} = \mathfrak{f}(X, Y) = \mathfrak{f}(Y, X)^{-1} \in \hat{\mathrm{Fr}}_2$$

and, for $\mu = (\lambda - 1)/2$, once $XYZ = 1$, it is assumed to hold:

$$\mathfrak{f}(Z, X)Z^\mu \mathfrak{f}(Y, Z)Y^\mu \mathfrak{f}(X, Y)X^\mu = 1.$$

Drinfeld's Theorem *There is an embedding*

$$\mathfrak{h}_{\mathrm{Dr}} : \mathrm{Gal}_{\mathbb{Q}} \hookrightarrow \mathrm{GT}.$$

Drinfeld's associator $\mathfrak{f}$

Drinfeld defined a subgroup of the automorphism group of $\hat{\mathrm{Fr}}_2$ on generators X, Y :

$$\mathrm{GT} := \{\phi \in \mathrm{Aut}\,\hat{\mathrm{Fr}}_2 : \phi(X) = X^\lambda,\ \phi(Y) = \mathfrak{f}^{-1} Y^\lambda \mathfrak{f}\}$$

where $\lambda \in \hat{\mathbb{Z}}^*$, $\lambda - 1 \in 2\hat{\mathbb{Z}}$,

$$\mathfrak{f} = \mathfrak{f}(X, Y) = \mathfrak{f}(Y, X)^{-1} \in \hat{\mathrm{Fr}}_2$$

and, for $\mu = (\lambda - 1)/2$, once $XYZ = 1$, it is assumed to hold:

$$\mathfrak{f}(Z, X) Z^\mu \mathfrak{f}(Y, Z) Y^\mu \mathfrak{f}(X, Y) X^\mu = 1.$$

Drinfeld's Theorem *There is an embedding*

$$\mathfrak{h}_{\mathrm{Dr}} : \mathrm{Gal}_{\mathbb{Q}} \hookrightarrow \mathrm{GT}.$$

Drinfeld's conjecture: $\mathfrak{h}_{\mathrm{Dr}}$ is an isomorphism.

The comparison statement

Proposition. There are natural surjective homomorphisms γ and g :

$$\begin{array}{ccc} \mathrm{Gal}_{\mathbb{Q}} & \hookrightarrow & \mathrm{GT} \\ \gamma \downarrow & & \downarrow g \\ \mathrm{Gal}(\mathbb{Q}(\mathrm{CM}_*) : \mathbb{Q}) & \hookrightarrow & \mathrm{Out}_* \tilde{\Gamma} \end{array}$$

such that the diagram commutes.

The comparison statement

Proposition. There are natural surjective homomorphisms γ and g :

$$\begin{array}{ccc} \mathrm{Gal}_{\mathbb{Q}} & \hookrightarrow & \mathrm{GT} \\ \gamma \downarrow & & \downarrow g \\ \mathrm{Gal}(\mathbb{Q}(\mathrm{CM}_*) : \mathbb{Q}) & \hookrightarrow & \mathrm{Out}_* \tilde{\Gamma} \end{array}$$

such that the diagram commutes.

Drinfeld's Conjecture **implies** *the* GT/CM *Conjecture.*

Axioms

Group axioms:

$$(\mathrm{G}, \Gamma) \cong (\mathrm{PGL}^+(2, \mathbb{Q}), \mathrm{PSL}^+(2, \mathbb{Z})).$$

Action axiom:

$$\mathrm{G} \text{ acts on } \mathbb{U}; \quad \forall e \in \mathrm{G}\,(\exists u\; e \cdot u = u \leftrightarrow e \in \mathrm{E})$$

Algebraically closed field of characteristic 0 and sorts $\mathbb{Y}(n)$:

$$\mathrm{F} \models \mathrm{ACF}_0$$

and

$$\mathbb{Y}(n) \subset \mathrm{F}^M; \quad \mathrm{pr}_{n.m} : \mathbb{Y}(n) \to \mathbb{Y}(m); \quad \Psi_{m,n} \subset \mathbb{Y}(n)^4$$

are given by specific equations over $\mathbb{Q}$.

Axioms (cont.)

Functional equations:

$$\Psi_{m,n}(x_1, x_2, y_1, y_2) \Leftrightarrow \exists g \in \mathrm{G}_{\det=m}\ \langle x_1, x_2\rangle, \langle y_1, y_2\rangle \in C_{g,n} \subset \mathbb{Y}(n)^2$$

where $C_{g,n} = j_n(\mathrm{graph}\, g)$.

Fibre formula:

$$\forall u, v \in \mathbb{U}\ j_n(u) = j_n(v) \leftrightarrow \exists \gamma \in \Gamma(n)\ v = \gamma \cdot u$$

The caterocity statement

Theorem *Assuming* GT/CM*-conjecture, the* $L_{\omega_1,\omega}$*-sentence determines an uncountably categorical class.*