Dimensions and homogeneity in mathematical structures

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To assess the *expressive power of a formal language* study the structures which can be described in full (up to isomorphism) by means of the language.

This property of the structure is called (absolute) **categoricity**.

We restrict the language by choosing a set of names Σ for relations and operations which have meaning in a structure

$$\mathbf{M} = (M, \Sigma),$$

where M is the universum of the structure. $\mathrm{Th}(\mathbf{M})$, the **theory of $\mathbf{M}$**, is the set of all sentences formulated in the language Σ which are true in $\mathbf{M}$.
Any structure satisfying the theory $\mathrm{Th}(\mathbf{M})$ is called **a model** of the theory.

Theorem (Lowenheim, Skolem) *The first order description* Th($\mathbf{M}$) *of* $\mathbf{M}$ *is absolutely categorical iff* M *is finite.*

Uninteresting once the description is of the same size as the structure it describes.

Definition A structure $\mathbf{M}$ is said to be **categorical in cardinality** λ if there is exactly one, up to isomorphism, structure $\mathbf{M}'$ of cardinality λ satisfying $\mathrm{Th}(\mathbf{M})$.

Of special interest is the case of *uncountable cardinality* λ.

Theorem (M.Morley 1964). *If a first-order countable theory is categorical in one uncountable* λ, *it is categorical in all uncountable cardinalities.*

We have then a large structure which has a concise (countable) categorical description.

This could be viewed as a formalization of what we would like to see as a **canonical structure**.

Definition A subset $S \subseteq M^n$ in structure $\mathbf{M}$ is said to be a **definable (using parameters from $X \subseteq M$) subset** if there is a first order property (formula) $\varphi(x_1, \ldots, x_n)$ in language Σ with $x_1, \ldots, x_n$ free variables such that

$$S = \{\langle a_1, \ldots, a_n \rangle \in M^n : \varphi(a_1, \ldots, a_n) \text{ holds in } \mathbf{M}\}.$$

If $S \subseteq M^n$ is a definable subset and E an equivalence relation on S which is definable too, then the quotient-set S/E is said to be a **definable set in M**.

Suppose sets S and relations $R_1 \subseteq S^{m_1}$, ...$R_k \subseteq S^{m_k}$ all are definable in $\mathbf{M}$. Then the structure $\mathbf{S}$ on the set S given by the relations $R_1, \ldots, R_k$ is said to be definable in $\mathbf{M}$.

Basic examples of uncountably categorical structures in countable languages:

(1) Trivial structures (Σ contains only the equality);
Graph of a countable group, uncountably many copies.

(2) Abelian divisible torsion-free groups;
Abelian groups of prime exponent;
Vector spaces over a countable division ring.

(3) Algebraically closed fields.

Given these structures one can construct in a **definable** way more complicated uncountably categorical structures.

Let $\mathbf{K}$ be an algebraically closed field and $G(\mathbf{K})$ an algebraic group over $\mathbf{K}$, (the $\mathbf{K}$-points of the group).
$G(\mathbf{K})$ *is uncountably categorical in a language witnessing the way* $G(\mathbf{K})$ *is definable in* $\mathbf{K}$.

Theorem *Assume G is a simple algebraic group. Then $G(\mathbf{K})$, in the pure language of groups, is uncountably categorical.*

No structural theory of algebraic groups is needed!

For G simple algebraic, the field $\mathbf{K}$ *is definable in* $G(\mathbf{K})$.

Need to know that maximal solvable connected (Borel) subgroups are non-abelian.

Corollary(Special case of Borel-Tits Thm) *Any (abstract) automorphism of* $G(\mathbf{K})$ *is a composition of an automorphism induced by a field automorphism* $\mathbf{K} \to \mathbf{K}$ *and a biregular automorphism* $G \to G$.

Conjecture(G.Cherlin 1979, B.Zilber 1977) *Any simple uncountably categorical group is isomorphic to an algebraic group* $G(\mathbf{K})$, *with* $\mathbf{K}$ *algebraically closed.*

Status: open.

Main logical problem

What properties of **M** make it categorical?

Why do certain structures allow an uncountably categorical description?

The answer (M.Morley, S.Shelah and others):

The key factor is the measurability by a *dimension* in the presence of *homogeneity.* The dimension notion satisfies certain geometric properties.

This is given by **Geometric Stability Theory**. When applied to fields, Stability Theory approach in many respects is very close to Algebraic Geometry.

Dimension notions for finite $X \subset \mathbf{M}$:

(1) Trivial structures: **size of** X

(2) Abelian divisible torsion-free groups; Abelian groups of prime exponent; Vector spaces over a division ring:

linear dimension of $\mathrm{span} X$

(3) Algebraically closed fields:
transcendense degree tr.d.(X).

Dually, the *dimension of an algebraic variety* $V \subseteq \mathbf{K}^n$:

$$\dim V = \max\{\mathrm{tr.d.}(\bar{x}) \mid \bar{x} \in V\}.$$

Atlas of Stability

Geometries

A closure operator $\mathrm{acl}(X)$ on $\mathbf{M}$:

$y \in \mathrm{acl}(X)$ iff there is a finite subset $Y \subseteq M$ which is definable using parameters in X such that $y \in Y$.

For any $\mathbf{M}$ the following property of the algebraic closure hold for any finite $X \subseteq M$, $y, z \in M$:
(i) $X \subseteq \mathrm{acl}(X)$;
(ii) $\mathrm{acl}(\mathrm{acl}(X)) = \mathrm{acl}(X)$;

A structure **M** is said to be a **combinatorial pregeometry** if the *exchange principle* holds:
(iii) $z \in \mathrm{acl}(X, y)$ and $z \notin \mathrm{acl}(X) \Rightarrow y \in \mathrm{acl}(X, z)$.

M is said to be a **combinatorial geometry** if $\mathrm{acl}(\{x\}) = \{x\}$ for any singleton.

M is said to be a **homogeneous** pregeometry if for any finite $X \subseteq M$ and $y, y' \in M \setminus \mathrm{acl}(X)$ there is an automorphism fixing $\mathrm{acl}(X)$ and sending $y \to y'$.

$\mathrm{acl}(X) =$ the **subspace** generated by X.

In combinatorial pregeometries the notions of **independence, basis, dimension** are well defined.

Theorem (Baldwin-Lachlan, Zilber, Harrington-Cherlin-Lachlan and others) *An uncountably categorical structure is* **coordinatizable** *by a* **unique** *homogeneous combinatorial geometry.*
Any ω-stable structure is coordinatizable by a collection of combinatorial geometries.

Weak Trichotomy Theorem (Zilber 1980)
Given an uncountably categorical **M**, *one and only one of the following holds:*

(i) the geometry of **M** *is trivial;*

(ii) the geometry of **M** *is isomorphic to the geometry of a* **projective** *or* **affine** *space over a division ring;*

(iii) there is a **pseudoplane** *definable in* **M**.

Definition (Lachlan) A **pseudoplane** is a two-sorted structure (P, L), P is said to be the set of points and L the set of lines, with a binary relation $I \subseteq P \times L$ ($I(x, y)$ is read as "point x belongs to line y") and satisfying the following:
any line contains infinitely many points;
through any point pass infinitely many lines;
any two distinct points belong to at most finitely many lines;
any two distinct lines intersect in at most finitely many points.

Theorem *If* $\mathbf{M}$ *is locally finite (*$\mathrm{acl}(X)$ *is finite for finite* X*) then there is no pseudoplane in* M*.*

Proof 1 (Zilber 84) Imitate algebraic geometry on the 'surface' P and develop intersection theory. 'Bezout Theorem' leads to a contradiction.$\square$

Proof 2 (Cherlin, Mills, P.Newmann 84) Use the classification of finite doubly transitive permutation groups, based on the *classification of finite simple groups.* $\square$

Corollaries
(1) *An infinite locally finite homogeneous combinatorial geometry is either* **trivial** *or an* **affine** *or* **projective** *geometry over a finite field.*
(2) *Ultrahomogeneous graphs are coordinatizable by trivial geometries.*

Both can be finitized.

In (1) dim ≥ 7 suffices.

(2) leads to a full classification of families of ultrahomogeneous finite structures (Lachlan, Cherlin, Shelah)

(1)&(2) is at the background of the classification of **Finite Strictures with few Types** (oncoming book by G.Cherlin and E.Hrushovski)
http://www.math.rutgers.edu/cherlin/

Trichotomy Conjecture *Given an uncountably categorical structure* $\mathbf{M}$ *one and only one of the following holds:*

(i) the combinatorial geometry of $\mathbf{M}$ *is* **trivial;**

(ii) the combinatorial geometry of $\mathbf{M}$ *is isomorphic to an* **affine or projective** *geometry over a countable division ring;*

(iii) $\mathbf{M}$ *is a structure of* **algebraic geometry** *over an algebraically closed field* $\mathbf{K}$.
($\mathbf{M}$ is definably equivalent to the field $\mathbf{K}$.)

The Trichotomy conjecture

was refuted in general (E.Hrushovski, 1989)

proved under extra **Zariski assumptions** (E.Hrushovski and B.Zilber, 1994)

Also, proved in the **o-minimality – real algebraic geometry context** (Y.Peterzil, S.Starchenko, 1996).

II. The fundamental trichotomy of Geometric Stability Theory, Zariski structures and Diophantine geometry

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The definition of Zariski structure M assumes:
- Weak **topology** on a structure M, with all *basic relations assumed closed* and satisfying the **descending chain condition** for closed subsets of M^n.
- The projection $\mathrm{pr} : M^{n+1} \to M^n$ maps irreducible closed sets to closed-minus-closed sets.
- **Dimension**, a number is assigned to any closed set, with natural geometric properties, e.g. for any closed irreducible $S \subseteq M^{n+1}$

$$\dim \mathrm{pr}(S) = \dim S - \dim S_a,$$

S_a a fiber of minimal dimension.

- **Pre-smoothness**. Given closed irreducible $S_1, S_2 \subseteq M^n$ and a point $a \in S_1 \cap S_2$, the dimension of an irreducible component containing a satisfies

$$\dim_a S_1 \cap S_2 \geq \dim S_1 + \dim S_2 - \dim M^n.$$

Theorem 2.1 *Zariski structures are of finite Morley rank. More precisely* $\mathrm{Mdim}\, A \leq \dim A$ *for any definable* A.

Examples.

1. Algebraic structure on a smooth algebraic variety M over an algebraically closed field is Zariski

n-ary basic relations P = algebraic subvarieties $P \subseteq M^n$.

2. Compact complex manifods are Zariski

n-ary basic relations P = analytic subsets $P \subseteq M^n$.

3. Proper analytic varieties in rigid analytic geometry are Zariski.

4. Finite Morley rank substructures of differentially closed fields $\mathbf{F} = (F, +, \cdot, \mathrm{D})$ of characteristic zero.

(Hrushovski, Sokolovich) *A definable substructure A of $\mathbf{F}$ of Morley rank one in an induced language is up to finite number of points a Zariski structure.*

Examples

4.1 $$C = \{x \in \mathbf{F} : \mathrm{D}x = 0\}$$
the field of constants of $\mathbf{F}$.

4.2 (Yu.Manin)
$A = \{(x, y) \in \mathbf{F}^2$ satisfying:
$$y^2 = x(x-1)(x-t); \qquad \mathrm{D}t = 1;$$
$$y^3 + 2(2t-1)(x-t^2)y \cdot \mathrm{D}x -$$
$$-2t(t-1)(y\mathrm{D}^2x - 2\mathrm{D}x \cdot \mathrm{D}^2y) = 0\},$$
a group containing $C(t)$-rational points on the elliptic curve $y^2 = x(x-1)(x-t)$.

Theorem (Hrushovski, Zilber 1994) *Zariski structures satisfy the Trichotomy conjecture.*

Proof Let **M** be a Zariski structure,

$$\dim M = 1.$$

Imitate algebraic geometry in **M**.

1. Develope an intersection theory (multiplicity of intersection in a point, tangency e.c.). Prove *Bezout Theorem.*

2. Assuming $M \times M$ is a pseudoplane, use tangency to define *jets of curves* through $(0,0)$. This forms a *group* under the composition operation. This group gives rise to a multiplicative and then to the additive group of a field **K**.
K *coordinatizes* M.

3. The proof that the coordinatization is given by *algebraic* functions amounts to proving the *Chow Theorem*: *any closed subset of* $\mathbb{P}^n(\mathbf{K})$ *is algebraic.*

4. $\mathbf{M}$ is an algebraic curve $C(\mathbf{K})$ with relations on it given by polynomial equations over $\mathbf{K}$. $\Box$

Diophantine Geometry context

Hrushovski, Manin:
The solutions of Manin's equations form a subgroup $A \subseteq \mathbb{A}(\mathbf{F})$ *of the abelian variety, and*

(1) $A \supseteq \Gamma$, *the group of rational points over a function field,*

(2) $\mathrm{Mdim}\, A$ *is finite,* A *satisfies the Trichotomy, thus either*

(i) A *does not have non-linear relations induced from the whole* $\mathbf{F}$, *or*

(ii) $A = G(C')$, *the* C'*-points of an algebraic group over a field* $C' \subseteq \mathbf{F}^n$,
$\mathrm{Mdim}\, C' = 1$. *In fact* $C = C'$.
Equivalently, $\mathbb{A}$ is *isogeneous* over the function field to an abelian variety over C.

Corollary (Holds in general once the linear case (i) has been identified).
If $V \subseteq \mathbb{A}$ *is an algebraic subvariety then*

$$\Gamma \cap V = \Gamma \cap \mathbb{B}_1 \cup \ldots \cup \Gamma \cap \mathbb{B}_k$$

for some abelian subvarieties $\mathbb{B}_i \subseteq \mathbb{A}$.

This is the statement of **Mordell-Lang conjecture for function fields of characteristic 0** for $\mathbb{A}$ *not isogeneous* to an abelian variety over the field of constants.

Similar scheme works for **the function field case in positive charactersitics**. The model theory of differentially closed fields is replaced by the model theory of *separably closed fields of positive characteristic.*

Similar scheme works for solving **the Manin-Mumford conjecture** (Γ the torsion subgroup of a semiabelian variety) the general case. The model theory of *difference fields* is used by Hrushovski.

Similar scheme works to prove a strong **uniform version of Manin-Mumford conjecture**.
T.Scanlon. Based on the structure of *Witt vectors with an automorphism* (A.Macintyre, L.Belair, T.Scanlon).

1.Model Theory puts the diophantine problems into a new geometric context:

Geometric Stability Theory - Zariski Geometries - the Trichotomy.

2. Model Theory works here by *synthesizing new structures with prescribed properties* to solve (or give a geometrical meaning to) mathematical problems:
Tame structures in which the problem is definable and analyzable.

More on compact complex manifolds(CCM).

A.Pillay, R.Moosa, T.Scanlon and others:

Model theory of CCM $\leftrightsquigarrow$
Classification theory of CCM.
(K.Ueno, A.Fujiki, F.Kampana...)

Zariski Geometries links

Compact complex manifolds	$\overset{?}{\leftrightsquigarrow}$	Finite-Morley-rank substructures of a diff. closed field

Useful informal analogies (A.Pillay)

Problem Find formal links.

Schanuel conjecure in differential fields

Using the equation

$$(*) \qquad \mathrm{D}y = y\mathrm{D}x$$

in differential fields J.Ax (1972) proved a *function field case* of Schanuel Conjecture:

If $x_1, y_1, \ldots x_n, y_n$ *satisfy the equation* (*) *and* $\mathrm{D}x_1, \ldots, \mathrm{D}x_n$ *are* $\mathbb{Q}$*-lineraly independent, then*

$$\mathrm{tr.d.}_C(x_1, y_1, \ldots x_n, y_n) > n.$$

W.D.Brownawell and K.K.Kubota (1977) extended this result to Weierstrass functions and Abelian integrals.

Problem Find a general model theoretic principle behind these results.

One more example

A powered algebraically closed field (with operations raising to non-standard powers)

$$(F,+,\cdot,x^{\eta})_{\eta\in\hat{\mathbb{N}}} \qquad \hat{\mathbb{N}} \succ \mathbb{N}.$$

Theorem Powered a.c.fields of characteristic 0 have superstable theories if the following is true:

Conjecture *Let $W \subseteq \mathbb{C}^n$ be an irreducible algebraic variety over $\mathbb{Q}$. There is finite family $\tau(W)$ of proper tori in $\mathbb{C}^n$ such that for any torus $T \subseteq \mathbb{C}^n$, given an irreducible component $S \subseteq W \cap T$ with $\dim S > \dim W + \dim T - n$, there is $A \in \tau(W)$ such that $S \subseteq A$.*

This can be generalized to semiabelian varieties. In this general form

The conjecture implies the Mordell-Lang conjecture over number and over function fields of characteristic 0.

III. Pseudo-analytic structures and Transcendental Number Theory

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Hrushovski construction of new structures

Given a class of structures $\mathbf{M}$ with a dimension notion ∂ we want to consider a new function $\mathbf{f}$ on $\mathbf{M}$.
Then on $(\mathbf{M}, \mathbf{f})$ we can calculate a *predimension*

$$\delta(X) = \partial(X \cup \mathbf{f}(X)) - \text{size}(X).$$

Consider now subclass of structures $(\mathbf{M}, \mathbf{f})$ which satisfy (Hrushovski inequality):

$$\delta(X) \geq 0 \text{ for any finite } X \subset \mathbf{M}.$$

Amalgamate all such structures to get a universal and homogeneous structure in the class. The resulting structure $(\mathbf{M}, \mathbf{f})$ will have a good dimension notion $\Delta(X)$ and a nice geometry.

Are Hrushovski structures mathematical pathologies?

Observation: If $\mathbf{M}$ is a field and we want $\mathbf{f} = \mathrm{ex}$ to be a group homomorphism

$$\mathrm{ex}(x_1 + x_2) = \mathrm{ex}(x_1) \cdot \mathrm{ex}(x_2)$$

then the corresponding predimension must be

$$\delta(X) = \mathrm{tr.d.}(X \cup \mathrm{ex}(X)) - \mathrm{l.d.}(X) \geq 0.$$

The Hrushovski inequality, under assumption that X is $\mathbb{Q}$-linearly independent, is equivalent to

$$\mathrm{tr.d.}(X \cup \mathrm{ex}(X)) \geq \mathrm{size}(X),$$

which in case $\mathbf{M} = (\mathbb{C}, +, \cdot)$ and $\mathrm{ex} = \exp$ is known as the **Schanuel conjecture.**

for linearly independent $x_1, \ldots, x_n \in \mathbb{C}$

$$\mathrm{tr.d.}(x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n}) \geq n.$$

Pseudo-exponentiation

Consider class $\mathcal{F}_{\mathrm{ex}}$ of fields of characteristic 0 with a function ex $\quad K_{\mathrm{ex}} = (K, +, \cdot, \mathrm{ex})$ satisfying

EXP1: $\quad \mathrm{ex}(x_1 + x_2) = \mathrm{ex}(x_1) \cdot \mathrm{ex}(x_2)$;

EXP2: $\quad \ker \mathrm{ex} = \pi\mathbb{Z}$;

and the Hrushovski inequality

SCH: $\quad$ for any finite $X \subset F$

$$\delta(X) = \mathrm{tr.d.}(X \cup \mathrm{ex}(X)) - \mathrm{l.d.}(X) \geq 0$$

Applying Fraisse amalgamation to $\mathcal{F}_{\mathrm{ex}}$ one gets **algebraically closed fields with pseudo-exponentiation $\mathbf{K}_{\mathrm{ex}}$.**

$\mathbf{K}_{\mathrm{ex}}$ satisfy **algebraic-exponential closedness** assumption.

EC: For any *free* and *non-overdetermined* irreducible system of polynomial equations

$$P(x_1, \ldots, x_n, y_1, \ldots, y_n) = 0$$

there exists a generic solution satisfying

$$y_i = \mathrm{ex}(x_i) \quad i = 1, \ldots, n.$$

And

Countable closure property

CC: 'Analytic' subsets of $\mathbf{K}^n$ of dimension 0 are countable.

Theorem *Given an uncountable cardinal* λ, *there is a unique model of axioms* $\mathrm{ACF}_0 + \mathrm{EXP} + \mathrm{SCH} + \mathrm{EC} + \mathrm{CC}$ *of cardinality* λ.

This is a consequence of Theorems A and B:

Theorem A *The* $L_{\omega_1,\omega}(Q)$*-sentence* $\mathrm{ACF}_0 + \mathrm{EXP} + \mathrm{SCH} + \mathrm{EC} + \mathrm{CC}$ *is axiomatizing a* **quasi-minimal excellent class**.

Theorem B (S.Shelah 1983) *A quasi-minimal excellent class has models and is categorical in any uncountable cardinality.*

The proof of Theorem A is essentially the proof of the following fact of *field arithmetic* (plus Galois (Kummer) theory)

Theorem C *Given algebraically closed subfields $L_1, \ldots, L_n \subseteq \mathbb{C}$ and elements $a_1, \ldots, a_m \in \mathbb{C}^*$ the multiplicative group of the field*

$$k = \mathbb{Q}(L_1, \ldots, L_n, a_1, \ldots, a_m)$$

can be represented as

$$A \cdot T \cdot L_1^* \cdot \ldots \cdot L_n^*,$$

where A is free *abelian, T the torsion subgroup (when $n = 0$) and L_i^* the multiplicative groups of L_i.*

In case of Abelian variety $\mathbb{A}(\mathbb{C})$ in place of $\mathbb{C}^*$, this corresponds to the similar properties of the group of k-rational points $\mathbb{A}(k)$, the impossibility of an *infinite descent* in the group.

The proof of Theorem C uses

1. Theory of fractional ideals of number fields.

2. Theory of Weil divisors and the normalization theorem.

Theorem B2 (S.Shelah) *The assumptions of excellency are* **necessary** *for an $L_{\omega_1,\omega}$-axiomatizable class to be categorical in all uncountable cardinalities.*

Corollary *The statement of Theorem C is essentially a form of model theoretic properties of categorical classes.*

Conjecture. The field of complex numbers with the genuine exponentiation $\mathbb{C}_{\exp}$ is isomorphic to the canonical field with exponentiation $\mathbf{K}_{\mathrm{ex}}$ of cardinality $2^{\aleph_0}$.

Two crucial properties of $\mathbf{K}_{\mathrm{ex}}$.

SCH: The Schanuel condition

EC: The exponential-algebraic closedness: *Any non-obviously-contradictory system of equations in* $+, \cdot, \mathrm{ex}$ *has a solution in* $\mathbf{K}_{\mathrm{ex}}$.

A test for the main conjecture

Consider the class of fields with a function $\mathbf{f}$ on them satisfying 'the absolute Schanuel condition'

$$\delta(X) = \mathrm{tr.d.}(X \cup \mathbf{f}(X)) - \mathrm{size}(X) \geq 0.$$

Construct, using Hrushovski-Fraisse method, the canonical structure $\mathbf{K_f} = (K, +, \cdot, \mathbf{f})$ of cardinality $2^{\aleph_0}$.

Problem Is there an analytic function $\mathbf{f}$ on $\mathbb{C}$ such that $\mathbb{C}_\mathbf{f}$ is isomorphic to $\mathbf{K_f}$?

It follows from the general theory that the first-order theory of $\mathbf{K_f}$ is *ω-stable* and is determined by two sets of axioms:
(SCH) stating the absolute Schanuel condition
(EC) stating the condition of $\mathbf{f}$-algebraic closedness.

Any such $\mathbb{C}_\mathbf{f}$ would also satisfy CC and thus will be quasi-minimal excellent.

Theorem (A.Wilkie + P.Koiran, 2002) *There is an entyre function* $\mathbf{f}$ *on* $\mathbb{C}$ *satisfying* SCH + EC