

# Axiomatic Set Theory

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## 1 Introduction

One of our main aims in this course is to prove the following: <sup>1 2 3</sup>

**Theorem 1.1** (*Gödel 1938*) *If set theory without the Axiom of Choice (ZF) is consistent (i.e. does not lead to a contradiction), then set theory with the axiom of choice (ZFC) is consistent.*

*Importance of this result:* Set theory is the axiomatization of mathematics, and without AC no-one seriously doubts its truth, or at least consistency. However, much of mathematics requires AC (eg. every vector space has a basis, every ideal can be extended to a maximal ideal). Probably most mathematicians don't doubt the truth, or at least consistency, of set theory with AC, but it does lead to some bizarre, seemingly paradoxical results—eg. the Banach-Tarski paradox. Hence it is comforting to have Gödel's theorem.

I formulate the axioms of set theory below. For the moment we have:

**(AC.)** *Axiom of Choice* (Zermelo) *If  $X$  is a set of non-empty pairwise disjoint sets, then there is a set  $Y$  which has exactly one element in common with each element of  $X$ .*

To complement Gödel's theorem, there is also the following result which is beyond this course:

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<sup>1</sup>These lecture were originally written by Peter Koepke many years ago and subsequently modified and taught at Oxford for a number of years by Alex Wilkie. In 2006 the course was given by R.Knight, who edited and typed them up. I have introduced a few more editorial changes.

<sup>2</sup>Prerequisite for this course is B1.

<sup>3</sup>See Andreas Blass, "On the inadequacy of inner models", JSL 37 no. 3 (Sept 72) 569–571.

**Proposition 1.2** (Cohen 1963) *If ZF is consistent, so is ZF with  $\neg AC$ .*

We shall also discuss Cantor's continuum problem which is the following.

Cantor defined the cardinality, or size, of an arbitrary set. The cardinality of  $A$  is denoted  $\text{card } A$ . He showed that  $\text{card } \mathbb{R} > \text{card } \mathbb{N}$ , but could not find any set  $S$  such that  $\text{card } \mathbb{R} > \text{card } S > \text{card } \mathbb{N}$ , so conjectured:

**(CH.)** *Cantor's Continuum Hypothesis* For any set  $S$ , either  $\text{card } S \leq \text{card } \mathbb{N}$ , or  $\text{card } S \geq \text{card } \mathbb{R}$ .

Again Gödel (1938) showed:

**Theorem 1.3** *If ZF is consistent, so is ZF+AC+CH,*

and Cohen (1963) showed:

**Proposition 1.4** *If ZF is consistent, so is ZF+AC+ $\neg$ CH.*

We shall prove Gödel's theorem but not Cohen's.

Of course Gödel's theorem on CH was perhaps not so mathematically pressing as his theorem on AC since mathematicians rarely want to assume CH, and if they do, then they say so.

We first make Gödel's theorem precise, by defining set theory and its language.

## 2 Basics

See D. Goldrei *Classic Set Theory*, Chapman and Hall 1996, or H.B. Enderton *Elements of Set Theory*, Academic Press, 1977.

The *language of set theory*, LST, is first-order predicate calculus with equality having the membership relation  $\in$  (which is binary) as its only non-logical symbol.

Thus the basic symbols of LST are:  $=$ ,  $\in$ ,  $\forall$ ,  $\neg$ ,  $\exists$ , ( and ), and an infinite list  $v_0, v_1, \dots, v_n, \dots$  of variables (although for clarity we shall often use  $x, y, z, t, \dots, u, v, \dots$  etc. as variables).

The *well-formed formulas*, or just *formulas*, of LST are those expressions that can be built up from the *atomic formulas*:  $v_i = v_j$ ,  $v_i \in v_j$ , using the rules: (1) if  $\phi$  is a formula, so is  $\neg\phi$ , (2) if  $\phi$  and  $\psi$  are formulas, so is  $(\phi \vee \psi)$ , and (3) if  $\phi$  is a formula, so is  $\forall v_i \phi$ .

## 2.1 Some standard abbreviations

We write  $(\phi \wedge \psi)$  for  $\neg(\neg\phi \vee \neg\psi)$ ;  $(\phi \rightarrow \psi)$  for  $(\neg\phi \vee \psi)$ ;  $(\phi \leftrightarrow \psi)$  for  $((\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi))$ ;  $\exists x\phi$  for  $\neg\forall x\neg\phi$ ;  $\exists!x\phi$  for  $\exists x(\phi \wedge \forall y(\phi(y) \leftrightarrow x = y))$ ;  $\exists x \in y \phi$  for  $\exists x(x \in y \wedge \phi)$ ;  $\forall x \in y \phi$  for  $\forall x(x \in y \rightarrow \phi)$ ;  $\forall x, y \phi$  (etc.) for  $\forall x \forall y \phi$ ;  $x \notin y$  for  $\neg x \in y$ .

We shall also often write  $\phi$  as  $\phi(x)$  to indicate free occurrences of a variable  $x$  in  $\phi$ . The formula  $\phi(z)$  (say) then denotes the result of substituting every free occurrence of  $x$  in  $\phi$  by  $z$ . Similarly for  $\phi(x, y)$ ,  $\phi(x, y, z), \dots$ , etc.

## 2.2 The Axioms

(A1.) *Extensionality*

$$\forall x, y(x = y \leftrightarrow \forall t(t \in x \leftrightarrow t \in y))$$

Two sets are equal iff they have the same members.

(A2.) *Empty set*

$$\exists x \forall y y \notin x$$

There is a set with no members, *the empty set*, denoted  $\emptyset$ .

(A3.) *Pairing*

$$\forall x, y \exists z \forall t(t \in z \leftrightarrow (t = x \vee t = y))$$

For any sets  $x, y$  there is a set, denoted  $\{x, y\}$ , whose only elements are  $x$  and  $y$ .

(A4.) *Union*

$$\forall x \exists y \forall t(t \in y \leftrightarrow \exists w(w \in x \wedge t \in w))$$

For any set  $x$ , there is a set, denoted  $\bigcup x$ , whose members are the members of the members of  $x$ .

(A5.) *Separation Scheme* If  $\phi(\mathbf{x}, y)$  is a formula of LST, the following is an axiom:

$$\forall \mathbf{x} \forall u \exists z \forall y(y \in z \leftrightarrow (y \in u \wedge \phi(\mathbf{x}, y)))$$

For given sets  $\mathbf{x}, u$  there is a set, denoted  $\{y \in u : \phi(\mathbf{x}, y)\}$ , whose elements are those elements  $y$  of  $u$  which satisfy the formula  $\phi(\mathbf{x}, y)$ .

**(A6.) Replacement Scheme** If  $\phi(x, y)$  is a formula of LST (possibly with other free variables  $\mathbf{u}$ , say) then the following is an axiom:

$$\forall \mathbf{u}[\forall x, y, y'((\phi(x, y) \wedge \phi(x, y')) \rightarrow y = y') \rightarrow \forall s \exists z \forall y(y \in z \leftrightarrow \exists x \in s \phi(x, y))]$$

The set  $z$  is denoted  $\{y : \exists x \phi(x, y) \wedge x \in s\}$ . “The image of a set under a function is a set.”

**(A7.) Power Set**

$$\forall x \exists y \forall t(t \in y \leftrightarrow \forall z(z \in t \rightarrow z \in x))$$

For any set  $x$  there is a set, denoted  $\mathbb{P}(x)$ , whose members are exactly the subsets of  $x$ .

**(A8.) Infinity**

$$\exists x[\exists y(y \in x \wedge \forall z(z \notin y) \wedge \forall y(y \in x \rightarrow \exists z(z \in x \wedge \forall t(t \in z \leftrightarrow (t \in y \vee t = y)))))]$$

There is a set  $x$  such that  $\emptyset \in x$  and whenever  $y \in x$ , then  $y \cup \{y\} \in x$ . (Such a set is called a *successor set*. The set  $\omega$  of natural numbers is a successor set.

**(A9.) Foundation**

$$\forall x(\exists z z \in x \rightarrow \exists z(z \in x \wedge \forall y \in zy \notin x))$$

If the set  $x$  is non-empty, then for some  $z \in x$ ,  $z$  has no members in common with  $x$ .

**(A10.) Axiom of Choice**

$$\begin{aligned} \forall u[[\forall x \in u \exists y y \in x \wedge \forall x, y((x \in u \wedge y \in u \wedge x \neq y) \rightarrow \forall z(z \notin x \vee z \notin y))] \rightarrow \\ \rightarrow \exists v \forall x \in u \exists! y(y \in x \wedge y \in v)] \end{aligned}$$

We write  $\text{ZF}^*$  for the collection of axioms A1–A8;  $\text{ZF}$  for A1–A9;  $\text{ZFC}$  for A1–A10.

### 2.3 Proofs in principle and proofs in practice

Suppose that  $T$  is one of the above collections of axioms. If  $\sigma$  is a sentence of LST (ie. a formula without free variables), we say that  $\sigma$  is a theorem of  $T$ , or that  $\sigma$  can be proved from  $T$ , and write  $T \vdash \sigma$ , if there is a finite sequence  $\sigma_1, \dots, \sigma_n$  of LST formulas such that  $\sigma_n$  is  $\sigma$ , and each  $\sigma_i$  is either in  $T$  or else follows from earlier formulas in the sequence by a rule of logic. Clearly every theorem of ZF is a theorem of ZFC and every theorem of ZF\* is a theorem of ZF. To say that  $T$  is consistent means that for no sentence  $\phi$  of LST is  $(\phi \wedge \neg\phi)$  a theorem of  $T$  (which is in fact equivalent to saying that there is some sentence which is not provable from  $T$ ). This now makes theorem 1.1 precise: we must show that if ZF is consistent, then so is ZFC.

Now in proving this theorem we shall need to build up a large stock of theorems of ZF (and we shall discuss some theorems of ZFC as well) but to give formal proofs of these would not only be tedious but also infeasible. We shall therefore employ the standard short-cut of *adopting a Platonic viewpoint*. That is, we shall think of the collection of all sets as being a clearly defined notion and whenever we want to show a sentence,  $\sigma$ , say, of LST has a formal proof (from ZF say) we shall simply give an informal argument that the proposition asserted by  $\sigma$  about this collection is true. Indeed, we shall often not bother to write out  $\sigma$  as a formula of LST at all; we shall simply write down (in English plus a few logical and mathematical symbols) “what it is saying”. Of course we shall take care that, in our informal argument, we only use those propositions about the collection of all sets asserted by the axioms of ZF. Thus, for example, if I write:

**Theorem 2.1** (ZF\*) *There is no set containing every set.*

then I mean that from the axioms of ZF\* there is a formal proof of the LST sentence

$$\forall x \exists y y \notin x.$$

Actually, it probably wouldn't be too difficult to give a formal proof of this, but we shall supply the following as a proof:

*Proof.* Suppose  $A$  were a set containing every set. By A5  $\{x \in A : x \notin x\}$  is a set, call it  $B$ . Then  $B \in B$  iff  $B \in A$  and  $B \notin B$ . But  $B \in A$  is true (as  $A$  contains every set), so  $B \in B$  iff  $B \notin B$ —a contradiction.  $\square$

Of course in all such cases, the reader should convince him- or herself that (a) the informal statement we are proving can be written as a sentence

of LST, and (b) the given proof can be converted, at least in principle, to a formal proof from the specified collection of axioms.

## 2.4 Interpretations

The Completeness Theorem for first-order predicate calculus (also due to Gödel) states that a sentence  $\sigma$  (of any first-order language) is provable from a collection of sentences  $S$  (in the same language) if and only if every model of  $S$  is a model of  $\sigma$ . Equivalently,  $S$  is consistent if and only if  $S$  has a model. Let us examine this in our present context. Firstly, a *structure for LST* is specified by a domain of discourse  $M$  over which the quantifiers  $\forall x \dots$  and  $\exists x \dots$  range, and a binary relation  $E$  on  $M$  to interpret the membership relation  $\in$ . If  $\sigma$  is a sentence of LST which is true under this interpretation we say that  $\sigma$  is true in  $\langle M, E \rangle$  or  $\langle M, E \rangle$  is a model of  $\sigma$ , and write  $\langle M, E \rangle \models \sigma$ . If  $T$  is a collection of sentences of LST we also write  $\langle M, E \rangle \models T$  iff  $\langle M, E \rangle \models \sigma$  for each sentence  $\sigma$  in  $T$ . (If  $\phi(x_1, \dots, x_n)$  is a formula of LST with free variables among  $x_1, \dots, x_n$  and  $a_1, \dots, a_n$  are in the domain  $M$ , we also write  $\langle M, E \rangle \models \phi(a_1, \dots, a_n)$  to mean  $\phi(x_1, \dots, x_n)$  is true of  $a_1, \dots, a_n$  in the interpretation  $\langle M, E \rangle$ .)

For example, suppose  $M$  contains just the two distinct elements  $a$  and  $b$ , and  $E$  is specified by  $a \rightarrow b$ , ie.  $E(a, b)$ , not  $E(b, a)$ , not  $E(a, a)$ , not  $E(b, b)$ . Then  $\langle M, E \rangle \models A2$ , ie.  $M \models \exists x \forall y y \notin x$ , since it is true that there is an  $x$  in  $M$  (namely  $a$ ) such that for all  $y \in M$ , not  $E(y, x)$ . It is also easy to see that  $\langle M, E \rangle \models A1$  and  $\langle M, E \rangle \models \neg A3$ . Notice that, by the completeness theorem, this implies that  $A3$  is not provable from the axioms  $A1, A2$  since we have found a model of the latter two axioms which is not a model of the former.

**Exercise 2.2** Let  $\mathbb{Q}$  be the set of rational numbers and  $\in$  the usual ordering of  $\mathbb{Q}$ . Which axioms of ZF are true in  $\langle \mathbb{Q}, \in \rangle$ ?

Note that the Platonic viewpoint adopted here amounts to regarding a sentence,  $\sigma$ , say, of LST as true, if and only if  $\langle V^*, \in \rangle \models \sigma$ , where  $V^*$  is the collection of all sets, and  $\in$  is the usual membership relation.

The completeness theorem provides a method for establishing theorem 1.1. For we can rephrase that theorem as: If ZF has a model then so does ZFC. Indeed we shall construct a subcollection  $L$  of  $V^*$  such that if we assume  $\langle V^*, \in \rangle \models ZF$ , then  $\langle L, \in \rangle \models ZFC$ . (Actually our proof will yield somewhat more which ought to be enough to satisfy any purist. Namely, it will

produce an effective procedure for converting any proof of a contradiction (ie. a sentence of the form  $(\phi \wedge \neg\phi)$ ) from ZFC to a proof of a contradiction from ZF.)

We now turn to the development of some basic set theory from the axioms ZF\*.

## 2.5 New sets from old

The axioms of ZF are of three types: (a) those that assert that all sets have a certain property (A1, A9), (b) those that sets with certain properties exist (A2, A8), and (c) those that tell us how we may construct new sets out of given sets (A3–A7). Our aim here is to combine the operations implicit in the axioms of type (c) to obtain more ways of constructing sets and to introduce notations for these constructions (just as, for example, we introduced the notation  $\bigcup x$  for the set  $y$  given by A4). It will be convenient to use the class notation  $\{x : \phi(x)\}$  for the collection (or *class*) of sets  $x$  satisfying the LST formula  $\phi(x)$ .<sup>4</sup> As we have seen, such a class need not be a set. However, in the following definitions it can be shown (from the axioms ZF\*) that we always do get a set. This amounts to showing that for some set  $a$ , if  $b$  is any set such that  $\phi(b)$  holds (ie.  $V^* \models \phi(b)$ ) then  $b \in a$ , so that  $\{x : \phi(x)\} = \{x \in a : \phi(x)\}$  which is a set by A5. I leave all the required proofs as exercises—they can also be found in the books.

In the following,  $A, B, \dots, a, b, c, \dots, f, g, a_1, a_2, \dots, a_n, \dots$  etc. all denote sets.

1.  $\{a_1, \dots, a_n\} := \{x : x = a_1 \vee \dots \vee x = a_n\}$ .
2.  $a \cup b := \bigcup\{a, b\} = \{x : x \in a \vee x \in b\}$ .
3.  $a \cap b := \{x : x \in a \wedge x \in b\}$ .
4.  $a \setminus b := \{x : x \in a \wedge x \notin b\}$ .
5.  $\bigcap a := \begin{cases} \{x : \forall y \in a, x \in y\} & \text{if } a \neq \emptyset \\ \text{undefined} & \text{if } a = \emptyset \end{cases}$ .

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<sup>4</sup>Actually,  $\phi(x)$  will be allowed to have parameters (ie. names for given sets), so is not strictly a formula of LST. Notice, however, that parameters are allowed in A5 and A6 (the “**x**” and “**u**”).

6.  $\langle a, b \rangle := \{\{a\}, \{a, b\}\}$ .  
**(Lemma.**  $\langle a, b \rangle = \langle c, d \rangle \leftrightarrow (a = c \wedge b = d)$ .)
7.  $a \times b := \{x : \exists c \in a \exists d \in b \langle c, d \rangle = x\}$ .
8.  $a \times b \times c := a \times (b \times c), \dots$ , etc.
9.  $a^2 := a \times a$ ,  $a^3 := a \times a \times a, \dots$ , etc.
10. We write  $a \subseteq b$  for  $\forall x \in a (x \in b)$ .
11.  $c$  is a *binary relation* on  $a$  we take to mean  $c \subseteq a^2$ . (Similarly for ternary,  $\dots$ ,  $n$ -ary,  $\dots$  relations.)
12. If  $A$  is a binary relation on  $a$  we usually write  $xAy$  for  $\langle x, y \rangle \in A$ .  
 $A$  is called a (*strict*) *partial order* on  $a$  iff
- (a)  $\forall x, y \in a (xAy \rightarrow \neg yAx)$ ,
  - (b)  $\forall x, y, z \in a ((xAy \wedge yAx) \rightarrow xAz)$ .
- If in addition we have (3)  $\forall x, y \in a (x = y \vee xAy \vee yAx)$ , then  $A$  is called a (*strict*) *total* (or *linear*) *order* of  $a$ .
13. Write  $f : a \rightarrow b$  ( $f$  is a *function* with *domain*  $a$  and *codomain*  $b$ , or simply  $f$  is a *function from  $a$  to  $b$* ) if  $f \subseteq a \times b$  and  $\forall c \in a \exists! d \in b \langle c, d \rangle \in f$ . Write  $f(c)$  for this unique  $d$ .
14. If  $f : a \rightarrow b$ ,  $f$  is called *injective* (or *one-to-one*) if  $\forall c, d \in a (c \neq d \rightarrow f(c) \neq f(d))$ , *surjective* (or *onto*) if  $\forall d \in b \exists c \in a f(c) = d$ , and *bijective* if it is both injective and surjective.
15. We write  $a \sim b$  if  $\exists f (f : a \rightarrow b \wedge f \text{ bijective})$ .
16.  ${}^a b := \{f : f : a \rightarrow b\}$ .
17. A set  $a$  is called a *successor set* if
- (a)  $\emptyset \in a$  and
  - (b)  $\forall b (b \in a \rightarrow b \cup \{b\} \in a)$ .

Axiom A8 implies a successor set exists and it can be further shown that a unique such set, denoted  $\omega$ , exists with the property that  $\omega \subseteq a$  for every successor set  $a$ . The set  $\omega$  is called the set of natural numbers. If  $n, m \in \omega$  we often write  $n + 1$  for  $n \cup \{n\}$  and  $n < m$  for  $n \in m$  and  $0$  for  $\emptyset$  (in this context). The relation  $\in$  (ie.  $<$ ) is a total order of  $\omega$  (more precisely  $\{\langle x, y \rangle : x \in \omega, y \in \omega \wedge x \in y\}$  is a total order of  $\omega$ ).

18. The set  $\omega$  satisfies the *principle of mathematical induction*, ie. if  $\psi(x)$  is any formula of LST such that  $\psi(0) \wedge \forall n \in \omega (\psi(n) \rightarrow \psi(n+1))$  holds, then  $\forall n \in \omega \psi(n)$  holds.
19. The set  $\omega$  also satisfies the *well-ordering principle*, ie. for any set  $a$ , if  $a \subseteq \omega$  and  $a \neq \emptyset$  then  $\exists b \in a \forall c \in a (c > b \vee c = b)$ .
20. *Definition by recursion*

Suppose that  $f : A \rightarrow A$  is a function and  $a \in A$ . Then there is a unique function  $g : \omega \rightarrow A$  such that:

- (a)  $g(0) = a$ , and
- (b)  $\forall n \in \omega g(n+1) = f(g(n))$ .

(Thus,  $g(n) = \underbrace{f(f \cdots (f a))}_{n \text{ times}} \cdots$ .)

More generally, if  $f : B \times \omega \times A \rightarrow A$  and  $h : B \rightarrow A$  are functions, then there is a unique function  $g : B \times \omega \rightarrow A$  such that

- (a)  $\forall b \in B g(b, 0) = h(b)$ , and
- (b)  $\forall b \in B \forall n \in \omega g(b, n+1) = f(b, n, g(b, n))$ .

Using this result one can define the addition, multiplication and exponentiation functions.

**(Remark** I have adopted here the usual convention of writing  $g(b, n+1)$  for  $g(\langle b, n+1 \rangle)$ . Similarly for  $f$ .)

21. A set  $a$  is called *finite* iff  $\exists n \in \omega a \sim n$ .
22. A set  $a$  is called *countably infinite* iff  $a \sim \omega$ .

23. A set  $a$  is called *countable* iff  $a$  is finite or countably infinite. (Equivalently: iff  $\exists f(f : a \rightarrow \omega \wedge f$  injective).)

(**Theorem**  $\mathbb{P}\omega$  is not countable. In fact, for no set  $A$  do we have  $A \sim \mathbb{P}A$ . (Cantor))

### 3 Classes, class terms and recursion

$V^*$  = the collection of all sets (assuming only ZF\*).

We call collections of the form  $\{x : \phi(x)\}$ , where  $\phi$  is a formula of LST, *classes*.

Every set is a class,  $a = \{x : x \in a\}$ . (so  $\phi(x)$  is  $x \in a$  here).

We must be careful in their use—we cannot quantify over them but some operations will still apply, eg. if  $U_1 = \{x : \phi(x)\}$  and  $U_2 = \{x : \psi(x)\}$ , then

$$\begin{aligned} U_1 \cap U_2 &= \{x : \phi(x) \wedge \psi(x)\} \\ U_1 \cup U_2 &= \{x : \phi(x) \vee \psi(x)\} \\ U_1 \times U_2 &= \{x : \exists y(y = \langle s, t \rangle \wedge \phi(s) \wedge \psi(t))\} \end{aligned} \tag{1}$$

and so on.  $x \in U_1$  means  $\phi(x)$  and  $U_1 \subseteq U_2$  means  $\forall x(\phi(x) \rightarrow \psi(x))$ .

Classes are only a notation—we can always eliminate their use.

Note that  $V^*$  is a class— $V^* = \{x : x = x\}$ .

If  $F, U_1, U_2$  are classes with the properties that  $F \subseteq U_1 \times U_2$  and  $\forall x \in U_1 \exists! y \in U_2 \langle x, y \rangle \in F$ , then  $F$  is called a *class term*, or just a *term*, and we write  $F(x) = y$  instead of  $\langle x, y \rangle \in F$ . We also write  $F : U_1 \rightarrow U_2$ , although  $F$  may not be a function, as  $U_1$  may not be a set. So if  $F = \{x : \exists y_1, y_2(x = \langle y_1, y_2 \rangle \wedge y_2 = \cup y_1)\}$ , so for all sets  $F(x) = \cup x$ , then  $F$  is a class term. We need class terms for *higher* recursion.

#### 3.1 The recursion theorem for $\omega$

(We assume only ZF\* throughout.)

**Theorem 3.1** *Suppose  $G : U \rightarrow U$  is a class term and  $a \in U$ . Then there is a term  $F : \omega \rightarrow U$  (which is therefore a function) such that*

1.  $F(0) = a$  and

2.  $\forall n \in \omega \ F(n+1) = G(F(n))$ .

*Proof.*

**Lemma 3.2** *Suppose that  $n \in \omega$ . Then there is a unique function  $f$ , with domain  $n+1$ , such that*

1.  $f(0) = a$  and
2.  $\forall m \in n \ f(m+1) = G(f(m))$ .

(Recall  $n = \{m : m < n\}$ .)

*Proof. Existence:* By induction on  $n$ .

For  $n = 0$ : Let  $f = \{\langle 0, a \rangle\}$ . Then  $f$  is a function with domain  $\{0\} = 1 = 0+1$ , such that  $f(0) = a$  and  $\forall m \in 0 \ f(m+1) = G(f(m))$ . (trivially)

Suppose true for  $n$ . Let  $f$  have domain  $n$  and satisfy (1) and (2). Let  $b = f(n)$ . Let  $f' = f \cup \{\langle n+1, G(b) \rangle\}$ . Then  $f'$  is a function with domain  $n+1 \cup \{n+1\} = (n+1)+1$ . Further  $f'(0) = f(0) = a$  (since  $0 \in n+1 = \text{dom } f$ ) (using (1)) and if  $m \in n+1 = n \cup \{n\}$ , then either  $m \in n$ , in which case  $m+1 \in n+1 = \text{dom } f$ , so  $f'(m+1) = f(m+1) = G(f(m))$  (using (2)) (by properties of  $f$ ), or  $m = n$ , so  $f'(m+1) = f'(n+1) = G(b) = G(f(n)) = G(f(m))$ , as required. So the proposition is true for  $n+1$ .

The uniqueness is also by induction.  $\square$

We now define  $F$  by

$$F = \{z : \exists x \in \omega \exists y \in U \ z = \langle x, y \rangle \wedge \exists f (f \text{ is a function with domain } x+1 \\ \text{ such that } f(0) = a \wedge \forall w \in x \ f(w+1) = G(f(w)) \wedge f(x) = y)\}$$

—the stuff after the colon is a formula of LST.

It is easy to show that  $\forall x \in \omega \exists! y \in U \ \langle x, y \rangle \in F$ , and that  $F$  satisfies (1) and (2) of Theorem 3.1.  $\square$

Some applications:

**Definition 3.3** *A set  $a$  is called transitive if  $\forall x \in a \forall y \in x \ y \in a$ . (ie.  $x \in a \rightarrow x \subseteq a$ , or  $\bigcup a \subseteq a$ .)*

**Lemma 3.4**  *$\omega$  is transitive; and if  $n \in \omega$ , then  $n$  is transitive.*

*Proof.* See the books.  $\square$

**Theorem 3.5** For any set  $a$ , there is a unique set  $b$ , denoted  $TC(a)$ , and called the transitive closure of  $a$ , such that

1.  $a \subseteq b$ ,
2.  $b$  is transitive,
3. whenever  $a \subseteq c$  and  $c$  is transitive, then  $b \subseteq c$ .

*Proof.* Uniqueness is clear since if  $a \subseteq b_1$  and  $a \subseteq b_2$ ,  $b_1$  and  $b_2$  transitive and both satisfying (3), then  $b_1 \subseteq b_2$  and  $b_2 \subseteq b_1$ , so  $b_1 = b_2$ .

For existence (idea:  $b = a \cup \cup a \cup \cup \cup a \cup \dots$ ).

Let  $G$  be the class term given by  $G(x) = \cup x$  (for  $x \in V^*$ ). Apply 3.1, to get a term  $F$  such that

1.  $F(0) = a$ , and
2.  $\forall n \in \omega F(n+1) = G(F(n)) = \cup F(n)$ .

By replacement, there is a set  $B$  such that  $B = \{y : \exists x \in \omega F(x) = y\}$ .

Let  $b = \cup B = \cup \{F(n) : n \in \omega\}$ . Then

1. Since  $a = F(0)$  and  $F(0) \in B$ , we have  $a \in B$ , so  $a \subseteq \cup B = b$ .
2. Suppose  $x \in b$  and  $y \in x$ . We must show  $y \in b$ . But  $x \in b$  implies  $x \in \cup B$  implies  $x \in F(n)$  for some  $n \in \omega$  implies  $x \subseteq \cup F(n)$ , so  $y \in \cup F(n)$ , so  $y \in F(n+1)$ , so  $y \in \cup B$ , so  $y \in b$ .
3. Suppose  $a \subseteq c$ ,  $c$  transitive.

We prove by induction on  $n$  that  $F(n) \subseteq c$ .

$F(0) = a \subseteq c$ .

Suppose  $F(n) \subseteq c$ .

We want to show that  $F(n+1) \subseteq c$ , so suppose  $x \in F(n+1)$ , ie  $x \in \cup F(n)$ . Then for some  $y \in F(n)$ ,  $x \in y$ . Thus  $x \in y \in F(n) \subseteq c$ , so  $x \in y \in c$ , so  $x \in c$ , since  $c$  is transitive, as required.

Thus, by induction,  $\forall n \in \omega F(n) \subseteq c$ , so  $\cup \{F(n) : n \in \omega\} \subseteq c$ , ie.  $b \subseteq c$ , as required.

□

**Recursion on  $\in$ .**

**Theorem 3.6** (Requires Foundation—ie. assume ZF) For  $\psi(x)$  any formula of LST (with parameters) if  $\forall x(\forall y \in x \psi(y) \rightarrow \psi(x))$ , then  $\forall x\psi(x)$ . (The hypothesis trivially implies  $\psi(\emptyset)$ .)

*Proof.* Suppose  $\forall x(\forall y \in x \psi(y) \rightarrow \psi(x))$ , but that there is some set  $a$  such that  $\neg\psi(a)$ . Then  $a \neq \emptyset$ . Let  $b = TC(a)$ , so  $a \subseteq b$ , and hence  $b \neq \emptyset$ . Let  $C = \{x \in b : \neg\psi(x)\}$ . Then  $C \neq \emptyset$ , since otherwise we would have  $\forall x \in b \psi(x)$ , hence  $\forall x \in a \psi(x)$  (since  $a \subseteq b$ ), and hence  $\psi(a)$ , contradiction.

By foundation there is some  $d \in C$  such that  $d \cap C = \emptyset$ , ie.  $d \in b$ ,  $\neg\psi(d)$ , but  $\forall x \in d x \in b$  (since  $b$  is transitive) and  $x \notin C$ . But this means  $\forall x \in d \psi(x)$ , so  $\psi(d)$ —contradiction.  $\square$

Our present aim is to prove that if ZF\* is consistent then so is ZF—so we won't use 3.6. Instead we find another generalization of induction.

**Definition 3.7** Suppose that  $a$  is a set and  $R$  is a binary relation on  $a$ . Then  $R$  is called a well-ordering of  $a$  if

1.  $R$  is a total ordering of  $a$ .
2. If  $b$  is a non-empty subset of  $a$ , then  $b$  contains an  $R$ -least element.  
ie.  $\exists x \in b \forall y \in b (y = x \vee xRy)$ .

**Remark.** AC iff every set is well-orderable.

**Definition 3.8** Suppose that  $R_1$  is a total order of  $a$ , and  $R_2$  is a total order of  $b$ . Then we say that  $\langle a, R_1 \rangle$  is order-isomorphic to  $\langle b, R_2 \rangle$ , written  $\langle a, R_1 \rangle \sim \langle b, R_2 \rangle$ , if there is a bijective function  $f : a \rightarrow b$  such that  $\forall x, y \in a (x < y \leftrightarrow f(x) < f(y))$ .

**Definition 3.9** We say  $x$  is an ordinal,  $On(x)$ , or  $x \in On$ , if

1.  $x$  is transitive, and
2.  $\in$  is a well-ordering of  $x$ .

We usually use  $\alpha, \beta$ , etc., for ordinals.  
 $On$  is a class.

**Theorem 3.10** (Enderton)

1. If  $R$  is a well-order of the set  $a$ , then there is a unique ordinal  $\alpha$  such that  $\langle a, R \rangle \sim \langle \alpha, \in \rangle$ .
2.  $\emptyset \in On$ . (Write  $\emptyset = 0$ .)
3.  $\alpha \in On \rightarrow \alpha + 1 \in On$  (so all natural numbers are ordinals, by induction).
4. If  $a$  is a set and  $a \subseteq On$ , then  $\bigcup a \in On$ . (Hence  $\omega \in On$ .)
5. If  $\alpha, \beta \in On$ , either  $\alpha = \beta$ ,  $\alpha \in \beta$ , or  $\beta \in \alpha$ , and exactly one occurs.
6. If  $\alpha, \beta, \gamma \in On$ , and  $\alpha \in \beta$  and  $\beta \in \gamma$ , then  $\alpha \in \gamma$ .
7. If  $\alpha, \beta \in On$ ,  $\alpha \subseteq \beta$  iff  $\alpha \in \beta$  or  $\alpha = \beta$ .
8. If  $\alpha \in On$  and  $a \in \alpha$ , then  $a \in On$ .

(Note that (4) implies that  $On$  is not a set.)

**Theorem 3.11** (Which is required to prove the above.) Suppose that  $\phi(x)$  is a formula of LST, such that  $\forall \alpha \in On (\forall \beta \in \alpha \phi(\beta) \rightarrow \phi(\alpha))$ . Then  $\forall \alpha \in On \phi(\alpha)$ .

*Proof.* Exercise  $\square$

**Theorem 3.12** (Well-ordering of the class of ordinals) Suppose  $U$  is a class and  $U \subseteq On$ ,  $U \neq \emptyset$ . Then there is an ordinal  $\alpha \in U$  such that  $\forall \beta \in U (\beta = \alpha \vee \alpha \in \beta)$ .

**Definition 3.13** (1) An ordinal  $\alpha$  is called a successor ordinal if  $\alpha = \beta \cup \{\beta\}$  for some (necessarily unique) ordinal  $\beta$ . (Write  $\alpha = \beta + 1$ .)

(2) An ordinal  $\alpha$  is called a limit ordinal if  $\alpha \neq \emptyset$  and  $\alpha$  is not a successor ordinal.

Theorem 3.11 is often applied in the following way:

To prove  $\forall \alpha \in On \phi(\alpha)$ :

1. Show  $\phi(0)$
2. Show  $\forall \alpha (\phi(\alpha) \rightarrow \phi(\alpha + 1))$

3. Show  $\forall \alpha < \delta \phi(\alpha) \rightarrow \phi(\delta)$ , for limit  $\delta$

**Theorem 3.14** (*Definition by recursion on On*) Suppose  $F : V^* \rightarrow V^*$  is a class term, and  $a \in V^*$ . Then there is a unique class term  $G : On \rightarrow V^*$  such that

1.  $G(0) = a$
2.  $G(\alpha + 1) = F(G(\alpha))$
3.  $G(\delta) = \bigcup_{\alpha \in \delta} G(\alpha)$  for  $\delta$  a limit.

*Proof.* Let  $\phi(g, \alpha)$  be the formula of LST expressing:

" $g$  is a function with domain  $\alpha + 1$  such that  $\forall \beta < \alpha \ g(\beta + 1) = F(g(\beta))$  and if  $\beta$  is a limit  $g(\beta) = \bigcup \{g(\alpha) : \alpha < \beta\}$  and  $g(0) = a$ ".

((\*) Note that if  $\phi(g, \alpha)$  and  $\beta \leq \alpha$ , then  $\phi(g \upharpoonright (\beta + 1), \beta)$ .)

**Lemma 3.15**  $\forall \alpha \in On \exists! g \phi(g, \alpha)$ .

*Proof.* Induction on  $\alpha$ .

$\alpha = 0$ : Clearly  $g = \{\langle 0, a \rangle\}$  is the only set satisfying  $\phi(g, 0)$ .

Suppose true for  $\alpha$ . Let  $g$  be the unique set satisfying  $\phi(g, \alpha)$ . (Note  $g : \alpha + 1 \rightarrow V^*$ .) Certainly  $g^* = g \cup \{\langle \alpha + 1, F(g(\alpha)) \rangle\}$  satisfies  $\phi(g^*, \alpha + 1)$ . If  $g'$  also satisfied  $\phi(g', \alpha + 1)$ , then  $\phi(g' \upharpoonright \alpha + 1, \alpha)$  holds, so by the inductive hypothesis  $g = g' \upharpoonright \alpha + 1$ . But  $\phi(g', \alpha + 1)$  implies  $g'(\alpha + 1) = F(g'(\alpha)) = F(g(\alpha))$ . So  $g' = g \cup \{\langle \alpha + 1, F(g(\alpha)) \rangle\} = g^*$ , as required.

Suppose  $\delta$  is a limit and  $\forall \alpha < \delta \exists! g \phi(g, \alpha)$ . For given  $\alpha < \delta$  let the unique  $g$  be  $g_\alpha$ . Notice that  $S = \{g_\alpha : \alpha < \delta\}$  is a set by Replacement. But  $\alpha_1 < \alpha_2$  implies  $g_{\alpha_1} = g_{\alpha_2} \upharpoonright \alpha_1 + 1$ . Let  $g^* = \bigcup S$ . Then  $g^*$  is a function with domain  $\{\alpha : \alpha < \delta\} = \delta$ , and  $\forall \alpha < \delta \ g^*(\alpha + 1) = F(g^*(\alpha))$  and if  $\beta$  is a limit  $< \delta$ , then  $g^*(\beta) = \bigcup \{g^*(\alpha) : \alpha < \beta\}$  and  $g^*(0) = a$ . (Since for any  $\alpha < \delta$ ,  $g^*$  coincides with  $g_\alpha$  on  $\alpha + 1$ , and the  $g_\alpha$ 's satisfy the condition by the inductive hypothesis.) Further  $g^*$  is the only such function by (\*).

Now define  $g = g^* \cup \{\langle \delta, \bigcup \{g^*(\alpha) : \alpha < \delta\} \rangle\}$ . Then  $g$  is unique such that  $\phi(g, \delta)$ .

Now set  $G = \{\langle x, \alpha \rangle : \exists g (\phi(g, \alpha) \wedge g(\alpha) = x)\}$ .

Then  $G$  satisfies the required conditions since by the lemma for each  $\alpha \in On$ ,  $G \upharpoonright \alpha + 1$  is the unique  $g$  such that  $\phi(g, \alpha)$ .

We get uniqueness of  $G$  by induction.  $\square$

The following can be easily proved using the above.

**Theorem 3.16** Suppose  $F : V^* \times On \rightarrow V^*$  and  $H : V^* \rightarrow V^*$  are class terms. Then there is a unique class term  $G : V^* \times On \rightarrow V^*$  such that

1.  $G(x, 0) = H(x)$
2.  $G(x, \alpha + 1) = F(x, G(x, \alpha))$
3.  $G(x, \delta) = \bigcup_{\alpha < \delta} G(x, \alpha)$  for  $\delta$  a limit.

Some applications:

**Definition 3.17** Ordinal addition: Set  $F(x, y) = y \cup \{y\}$ ,  $H(x) = x$ . We get  $G$  such that

1.  $G(x, 0) = x$
2.  $G(x, \alpha + 1) = G(x, \alpha) \cup \{G(x, \alpha)\}$
3.  $G(x, \delta) = \bigcup_{\alpha < \delta} G(x, \alpha)$ .

Suppose  $\alpha, \beta \in On$ . Write  $\alpha + \beta$  for  $G(\alpha, \beta)$ . Then:

1.  $\alpha + 0 = \alpha$
2.  $\alpha + (\beta + 1) = (\alpha + \beta) + 1$
3.  $\alpha + \delta = \bigcup_{\beta < \delta} \alpha + \beta$ .

**Definition 3.18** Ordinal multiplication:

1.  $\alpha \cdot 0 = 0$  (So  $H(x) = 0$ )
2.  $\alpha \cdot (\beta + 1) = \alpha \cdot \beta + \alpha$  (So  $F(x, y) = y + x$ )
3.  $\alpha \cdot \delta = \bigcup_{\beta < \delta} \alpha \cdot \beta$ .

## 4 The Cumulative Hierarchy and the consistency of the Axiom of Foundation

We apply Theorem 3.14 with  $a = \emptyset$  and  $F(x) = \mathbb{P}x$ , to get  $V : On \rightarrow V^*$  defined by

1.  $V(0) = \emptyset$
2.  $V(\alpha + 1) = \mathbb{P}V(\alpha)$ , and
3.  $V(\delta) = \bigcup_{\alpha < \delta} V(\alpha)$  for  $\delta$  a limit.

We write  $V_\alpha$  for  $V(\alpha)$ . Each  $V_\alpha$  is a set and we also write  $V$  for the class  $\{x : \exists \alpha \in On \ x \in V_\alpha\}$  “=”  $\bigcup_{\alpha \in On} V_\alpha$ .

**Theorem 4.1** *For each  $\alpha \in On$ ,*

1.  $V_\alpha$  is transitive,
2.  $V_\alpha \subseteq V_{\alpha+1}$ ,
3.  $\alpha \in V_{\alpha+1}$ .

*Proof.* Simultaneous induction on  $\alpha$ .

$\alpha = 0$   $V_0 = \emptyset$ , which is transitive.  $V_0 \subseteq V_1$ , and  $0 = \emptyset \in \{\emptyset\} = V_1$ .

*Suppose true for  $\alpha$ .*

(1) Suppose  $x \in y \in V_{\alpha+1}$ .  $V_{\alpha+1} = \mathbb{P}V_\alpha$ , so  $x \in y \subseteq V_\alpha$ , so  $x \in V_\alpha$ . Since  $V_\alpha \subseteq V_{\alpha+1}$  by the inductive hypothesis, we get  $x \in V_{\alpha+1}$  as required.

(2) Suppose  $x \in V_{\alpha+1}$ . Then  $x \subseteq V_\alpha$ . But  $V_\alpha \subseteq V_{\alpha+1}$  by the inductive hypothesis, so  $x \subseteq V_{\alpha+1}$ . Hence  $x \in V_{(\alpha+1)+1}$ , as required.

(3)  $\alpha \in V_{\alpha+1}$  by hypothesis. So  $\alpha \subseteq V_{\alpha+1}$ , since  $V_{\alpha+1}$  is transitive. Thus  $\alpha \cup \{\alpha\} \subseteq V_{\alpha+1}$ . Hence  $\alpha + 1 = \alpha \cup \{\alpha\} \in V_{(\alpha+1)+1}$ , as required.

—Hence the result is true for  $\alpha + 1$ .

*Suppose  $\delta$  a limit and (1), (2) and (3) are true for all  $\alpha < \delta$ .*

(1) Suppose  $x \in y \in V_\delta = \bigcup_{\alpha < \delta} V_\alpha$ . Then  $x \in y \in V_\alpha$  for some  $\alpha < \delta$ . So  $x \in V_\alpha$  by ind hyp. But  $V_\alpha \subseteq V_\delta$ , so  $x \in V_\delta$ .

(2) Suppose  $x \in V_\delta$ . Since  $y \in x \in V_\delta \rightarrow y \in V_\delta$ , we have  $x \subseteq V_\delta$ , so  $x \in V_{\delta+1}$ . Thus  $V_\delta \subseteq V_{\delta+1}$ .

(3) Now for all  $\alpha < \delta$ ,  $\alpha \in V_{\alpha+1}$ , by the inductive hypothesis. So  $\forall \alpha < \delta$   $\alpha \in V_\delta$  (since  $V_{\alpha+1} \subseteq V_\delta$ ). Thus  $\delta \subseteq V_\delta$  (note  $\delta = \{\alpha : \alpha < \delta\}$ ) and so  $\delta \in \mathbb{P}V_\delta = V_{\delta+1}$ , as required.  $\square$

**Corollary 4.2** (1)  $V$  is a transitive class (ie.  $x \in y \in V \rightarrow x \in V$ ) containing all the ordinals.

(2)  $\forall \alpha < \beta \ V_\alpha \subseteq V_\beta$ .

**Theorem 4.3**  $\langle V, \in \rangle \models ZF$ .

*Proof.* (Note that  $\langle V, \in \rangle$  is a substructure of  $\langle V^*, \in \rangle$ , so for  $a, b \in V$ ,  $\langle V, \in \rangle \models a \in b$  iff  $a \in b$ , and  $\langle V, \in \rangle \models a = b$  iff  $a = b$ .)

**A1.** Suppose  $x, y \in V$ , and  $\langle V, \in \rangle \models \forall t(t \in x \leftrightarrow t \in y)$  (\*). We must show  $\langle V, \in \rangle \models x = y$ , ie  $x = y$ . Suppose  $x \neq y$ . Say  $a \in x$ ,  $a \notin y$ . Since  $a \in x \in V$  we have  $a \in V$  (by Corollary 4.2). But by (\*),  $\forall t \in V, t \in x \leftrightarrow t \in y$ . In particular  $a \in x \leftrightarrow a \in y$ —contradiction.

So  $x = y$ .

**A2.** We must show  $\langle V, \in \rangle \models \exists x \forall y \ y \notin x$ . Since  $\emptyset \in V$ , we have  $\emptyset \in V$ , and clearly  $\forall y \in V, \emptyset \notin y$ .

**A3.** Suppose  $a, b \in V$ . We must show  $\langle V, \in \rangle \models \exists z \forall t(t \in z \leftrightarrow (t = a \vee t = b))$ . Let  $c = \{a, b\}$ . Now by 4.2 (ii), there is some  $\alpha$  such that  $a, b \in V_\alpha$ . So  $c \subseteq V_\alpha$ , so  $c \in V_{\alpha+1}$ , so  $c \in V$ . It remains to show  $\forall t \in V(t \in c \leftrightarrow (t = a \vee t = b))$ , which is clear since this is true  $\forall t \in V^*$ .

**A4.**  $\langle V, \in \rangle \models$  Unions—exercise.

**A7.** *Power Set* Suppose  $a \in V$ . We must show  $\langle V, \in \rangle \models \exists y \forall t(t \in y \leftrightarrow \forall z(z \in t \rightarrow z \in a))$ .

Now suppose  $a \in V_\alpha$ .

*Exercise:*  $\forall \alpha \in On$ , if  $b \in a \in V_\alpha$ , then  $b \in V_\alpha$ .

It follows that  $\forall b \in \mathbb{P}(a), b \in V_\alpha$ . Thus  $\mathbb{P}(a) \subseteq V_\alpha$ , so  $\mathbb{P}(a) \in V_{\alpha+1}$ . So  $\mathbb{P}(a) \in V$ . Let  $c = \mathbb{P}(a)$ .

We show  $\langle V, \in \rangle \models \forall t(t \in c \leftrightarrow \forall z(z \in t \rightarrow z \in a))$ .

So suppose  $t \in V$ .

$\Rightarrow$ ): If  $\langle V, \in \rangle \models t \in c$ , then  $t \in c$ , so  $t \subseteq a$ , ie.  $\forall z \in V^*(z \in t \rightarrow z \in a)$ , thus  $\forall z \in V(z \in t \rightarrow z \in a)$ .

$\Leftarrow$ ): Suppose  $\langle V, \in \rangle \models \forall z(z \in t \rightarrow z \in a)$  (\*) (ie.  $\langle V, \in \rangle \models t \subseteq a$ ). We show that really,  $t \subseteq a$ . Suppose  $d \in t$ . Since  $t \in V$ , we have  $d \in V$  (by 4.2 (i)). Hence, by (\*),  $d \in a$ . Thus  $t \subseteq a$ , so  $t \in c$ , so  $\langle V, \in \rangle \models t \in c$  as required.

**A8.** *Infinity* Exercise (Note:  $\omega \in V_{\omega+1}$ , so  $\omega \in V$ ).

**A9.** *Foundation* Suppose  $a \in V, a \neq \emptyset$ . We must find  $b \in a$  such that  $b \cap a = \emptyset$ .

[Since then  $b \in V$ , by transitivity, and  $\langle V, \in \rangle \models \forall y \in by \ y \notin a$ .]

Let  $x \in a$ . Then  $x \in V$ , so  $x \in V_\alpha$  for some  $\alpha$ . This shows  $\exists \alpha \in On, a \cap V_\alpha \neq \emptyset$ . Choose  $\beta$  minimal such that  $a \cap V_\beta \neq \emptyset$ . Then  $\beta$  is a successor ordinal since, for  $\delta$  a limit,  $a \cap V_\delta = a \cap \bigcup_{\alpha < \delta} V_\alpha = \bigcup_{\alpha < \delta} (a \cap V_\alpha)$ , so if  $a \cap V_\delta \neq \emptyset$ , then  $a \cap V_\alpha \neq \emptyset$  for some  $\alpha < \delta$ .

Say  $\beta = \gamma + 1$ . Now choose  $b \in a \cap V_\beta$ .

We claim that  $b \cap a = \emptyset$ . Suppose  $x \in a \cap b$ . Now  $b \in V_\beta$ , so  $b \subseteq V_\gamma$ , so  $x \in V_\gamma$ . But  $x \in a$ , so  $a \cap V_\gamma \neq \emptyset$ —a contradiction to the minimality of  $\beta$ .

**A5. Separation** Suppose  $\phi(x_1, \dots, x_n, y)$  is a formula of LST and  $a_1, \dots, a_n \in V$ , and  $u \in V$ . We want  $b \in V$  such that

$$\langle V, \in \rangle \models \forall y (y \in b \leftrightarrow (y \in u \wedge \phi(a_1, \dots, a_n, y))).$$

**Definition 4.4** Relativization of formulas Suppose  $U$  is a class, say  $U = \{x : \Phi(x)\}$ , and  $\phi(v_1, \dots, v_k)$  is a formula of LST. We define the formula  $\phi^U(v_1, \dots, v_k)$  (or  $\phi^\Phi(v_1, \dots, v_k)$ ), which has the same free variables as  $\phi$ , as follows (by recursion on  $\phi$ ):

1. If  $\phi$  is  $v_i = v_j$  or  $v_i \in v_j$ , then  $\phi^U$  is just  $\phi$ .
2. If  $\phi$  is  $\neg\psi$ , then  $\phi^U$  is  $\neg\psi^U$ .
3. If  $\phi$  is  $(\psi \vee \psi')$ , then  $\phi^U$  is  $(\psi^U \vee (\psi')^U)$ .
4. If  $\phi$  is  $\forall v_i \psi$ , then  $\phi^U$  is  $\forall v_i (\Phi(v_i) \rightarrow \psi^U)$ .

(We tacitly assume  $\phi$  and  $\Phi$  have no bound variables in common.)

**Lemma 4.5** For any  $\phi(v_1, \dots, v_k)$  and  $a_1, \dots, a_k \in U$ ,  $\langle U, \in \rangle \models \phi(a_1, \dots, a_k)$  iff  $\phi^U(a_1, \dots, a_k)$ .

*Proof.* Obvious.  $\square$

To return to the proof of A5 in  $\langle V, \in \rangle$ : Suppose  $u \in V_\alpha$ . Let  $b = \{y \in u : \phi^V(a_1, \dots, a_n, y)\}$ . Then  $b \subseteq u \in V_\alpha$ , so  $b \in V_\alpha$  (by an exercise), so  $b \in V$ .

Suppose  $y \in V$ .

We want to show  $\langle V, \in \rangle \models y \in b \leftrightarrow (y \in u \wedge \phi(a_1, \dots, a_n, y))$ .

$\Rightarrow$ ): Suppose  $y \in b$ . Then  $y \in u$ , and  $\phi^V(a_1, \dots, a_n, y)$ . Hence, by lemma 4.5,  $\langle V, \in \rangle \models y \in u \wedge \phi(a_1, \dots, a_n, y)$ .

$\Leftarrow$ ): Suppose  $\langle V, \in \rangle \models y \in u \wedge \phi(a_1, \dots, a_n, y)$ . Then  $y \in u$  and  $\phi^V(a_1, \dots, a_n, y)$  (by 4.5), so  $y \in b$ , as required.

**A6. Replacement** Suppose  $\phi(x, y)$  is a formula of LST (possibly involving parameters from  $V$ ).

Suppose  $\langle V, \in \rangle \models \forall x, y, y' ((\phi(x, y) \wedge \phi(x, y')) \rightarrow y = y')$ .

Let  $\psi(x, y)$  be  $\overbrace{x \in V}^{V(x)} \wedge \overbrace{y \in V}^{V(y)} \wedge \phi^V(x, y)$ . [Note  $V(x)$  has no parameters.]

Then we have (in  $V^*$ )  $\forall x, y, y' ((\psi(x, y) \wedge \psi(x, y')) \rightarrow y = y')$ , by lemma 4.5.

Let  $s \in V$ .

Hence there is a set  $z$  such that

$$\forall y (y \in z \leftrightarrow \exists x \in s \psi(x, y)) \quad (*)$$

(by replacement in  $V^*$ ). We want to show  $z \in V$ .

Now by (\*), if  $y \in z$ , then  $\exists x \in s \psi(x, y)$ , so  $\exists x \in s (x \in V \wedge y \in V \wedge \phi^V(x, y))$ , so  $y \in V$ .

Thus for each  $y \in z$ ,  $\exists \alpha \in On, y \in V_\alpha$ .

Let  $\chi(u, v)$  be “ $u \in z \wedge v$  is the least ordinal such that  $u \in V_v$ ”.

Then by replacement in  $V^*$ , there is a set  $S$  such that

$$\forall v (\exists u \in z (\chi(u, v)) \leftrightarrow v \in S).$$

Clearly  $S$  is a set of ordinals, so  $\bigcup S$  is an ordinal,  $\beta$  say.

Clearly  $\forall y \in z, y \in V_\beta$ . Hence  $z \subseteq V_\beta$ , so  $z \in V_{\beta+1}$ , so  $z \in V$ .

We must show  $\langle V, \in \rangle \models \forall y (y \in z \leftrightarrow \exists x \in s \phi(x, y))$ .

$\Rightarrow$ ): So suppose  $y \in V$  and  $y \in z$ .

By (\*),  $\exists x \in s \psi(x, y)$ , ie.  $\exists x \in s (x \in V \wedge y \in V \wedge \phi^V(x, y))$ , so  $\langle V, \in \rangle \models \exists x \in s \phi(x, y)$ .

$\Leftarrow$ ): Conversely, if  $y \in V$ , and  $\langle V, \in \rangle \models \exists x \in s \phi(x, y)$ , then  $\exists x \in S (x \in V \wedge \phi^V(x, y))$ , so  $\exists x \in s (x \in V \wedge y \in V \wedge \phi^V(x, y))$ , ie  $\exists x \in s \psi(x, y)$ , so by (\*),  $y \in z$ .  $\square$

**Corollary 4.6** *If  $ZF^*$  is consistent, then so is  $ZF$ .*

*Proof.* If  $\sigma$  is an axiom of  $ZF$ , we have shown that  $ZF^* \vdash \sigma^V$ . Hence if  $\sigma_1, \sigma_2, \dots, \sigma_k$  were a proof of a contradiction from  $ZF$ , then (roughly)  $\sigma_1^V, \dots, \sigma_k^V$  could be converted into one from  $ZF^*$ .  $\square$

From now on we assume Foundation, and hence may assume (exercise) that  $V^* = V$ .

## 5 Lévy's Reflection Principle

### 5.1

**Theorem 5.1** (LRP) (ZF—for each individual  $\chi$ )

Suppose  $\tilde{W} : On \rightarrow V$  is a class term, and write  $W_\alpha$  for  $\tilde{W}(\alpha)$ . Suppose  $\tilde{W}$  satisfies:

1.  $\alpha < \beta \rightarrow W_\alpha \subseteq W_\beta$  ( $\forall \alpha, \beta \in On$ )
2.  $W_\delta = \bigcup_{\alpha \in \delta} W_\alpha$  for all limit ordinals  $\delta$ .

Let  $W = \bigcup_{\alpha \in On} W_\alpha$  ( $= \{x : \exists \alpha \in On, x \in W_\alpha\}$ , so  $W$  is a class; each  $W_\alpha$  is a set.)

Suppose  $\chi(v_1, \dots, v_n)$  is a formula of LST (without parameters). Then, for any  $\alpha \in On$ , there is  $\beta \in On$  such that  $\beta \geq \alpha$ , and such that  $\forall a_1, \dots, a_n \in W_\beta$ ,  $\langle W, \in \rangle \models \chi(a_1, \dots, a_n)$  iff  $\langle W_\beta, \in \rangle \models \chi(a_1, \dots, a_n)$ ; ie. for all  $a_1, \dots, a_n \in W_\beta$ ,  $\chi^W(a_1, \dots, a_n) \leftrightarrow \chi^{W_\beta}(a_1, \dots, a_n)$ .

*Proof.* For any formula  $\phi$  of LST, by the collection of subformulas of  $\phi$ ,  $SF(\phi)$ , we mean all those formulas that go into the building up of  $\phi$ . Formally

1.  $SF(\phi) = \{\phi\}$  if  $\phi$  is of the form  $x = y$  or  $x \in y$ ;
2.  $SF(\neg\phi) = \{\neg\phi\} \cup SF(\phi)$ ;
3.  $SF(\phi \vee \psi) = \{\phi \vee \psi\} \cup SF(\phi) \cup SF(\psi)$ ;
4.  $SF(\forall x\phi) = \{\forall x\phi\} \cup SF(\phi)$ .

Clearly  $SF(\phi)$  is a finite collection for any formula  $\phi$ , and  $\phi \in SF(\phi)$ .

Suppose now that  $S$  is any finite collection of formulas, which is closed under taking subformulas—ie. if  $\phi \in S$ , then  $SF(\phi) \subseteq S$ .

Define  $T_S = \{\beta \in On : \forall \chi \in S \forall \mathbf{a} \in W_\beta (\chi^{W_\beta}(\mathbf{a}) \leftrightarrow \chi^W(\mathbf{a}))\}$ . (Abuse of notation here.) ( $T_S$  is a class since  $S$  is finite.)

We must show that  $T_S$  is unbounded in the ordinals. (LRP follows by taking  $S = SF(\chi)$ .)

We first show.

**Lemma 5.2** For any  $S$  as above,  $T_S$  is a closed class of ordinals, ie. it contains all its limits, ie. when  $X$  is a subset of  $T_S$ , then  $\sup X \in T_S$ .

*Proof.* We prove this by induction on the total number  $n$  of occurrences of connectives in formulas of  $S$ . We write this  $n$  as  $\#S$ .

If  $n = 0$ , then all formulas of  $S$  are of the form  $x = y$  or  $x \in y$  (for variables  $x$  and  $y$ ), so  $T_S = On$ , so  $T_S$  is definitely closed.

Now suppose that  $\#S = n + 1$ . Let  $\chi$  be a formula in  $S$  with maximal number of connectives.

Let  $S' = S \setminus \{\chi\}$ . Clearly  $S'$  is also closed under taking subformulas and  $\#S' \leq n$ . Also since  $S' \subseteq S$ , we have  $T_{S'} \subseteq T_S$ .

Let  $X \subseteq T_S$ , a subset, and suppose  $X$  has no greatest element. Note that  $X \subseteq T_{S'}$ , so  $\sup X \in T_{S'}$  by the inductive hypothesis.

We want to show that  $\sup X \in T_S$ .

*Case 1.*  $\chi$  is  $\neg\psi$ . Note  $\psi \in S'$ , so  $T_S = T_{S'}$ . So  $\sup X \in T_S$ .

*Case 2.*  $\chi$  is  $\psi_1 \vee \psi_2$ . Then again  $\psi_1, \psi_2 \in S'$ , so we can easily check  $T_S = T_{S'}$ , and the result follows by the inductive hypothesis.

*Case 3.*  $\chi$  is  $\forall v_{n+1}\psi(v_1, \dots, v_n, v_{n+1})$ .

Then  $\psi(v_1, \dots, v_n, v_{n+1}) \in S'$ . Let  $\eta = \sup X$ . Now since  $X$  has no greatest element  $\eta$  is a limit ordinal, so  $W_\eta = \bigcup_{\alpha < \eta} W_\alpha = \bigcup_{\alpha \in X} W_\alpha$ .

But by the inductive hypothesis we have for all  $\phi \in S'$ , for all  $\mathbf{a} \in W_\eta$

$$\phi^{W_\eta}(\mathbf{a}) \leftrightarrow \phi^W(\mathbf{a}) \quad (*)$$

We clearly only have to show:

$$\forall \mathbf{a} \in W_\eta (\chi^{W_\eta}(\mathbf{a}) \leftrightarrow \chi^W(\mathbf{a})). \quad (\dagger)$$

Now since  $X \subseteq T_S$  we have

$$\forall \beta \in X \forall \mathbf{a} \in W_\beta (\chi^{W_\beta}(\mathbf{a}) \leftrightarrow \chi^W(\mathbf{a})). \quad (**)$$

*Proof of  $\leftarrow$  in  $(\dagger)$*

Suppose  $\mathbf{a} \in W_\eta$  and  $\chi^W(\mathbf{a})$ . Thus

$$(\forall v_{n+1}\psi(\mathbf{a}, v_{n+1}))^W, \text{ ie. } \forall v_{n+1} \in W \psi^W(\mathbf{a}, v_{n+1}).$$

But  $W_\eta \subseteq W$ , so  $\forall v_{n+1} \in W_\eta \psi^W(\mathbf{a}, v_{n+1})$ . Let  $a_{n+1} \in W_\eta$ . Then  $\psi^W(\mathbf{a}, a_{n+1})$ . But  $\psi \in S'$  (since  $\psi$  is a subformula of  $\chi$  different from  $\chi$ ), so by  $(*)$   $\psi^{W_\eta}(\mathbf{a}, a_{n+1})$ . Since this holds for any  $a_{n+1} \in W_\eta$  we have  $\forall v_{n+1} \in W_\eta \psi^{W_\eta}(\mathbf{a}, v_{n+1})$ , ie.  $\chi^{W_\eta}$  as required.

*Proof of  $\rightarrow$  in  $(\dagger)$*

Suppose  $\mathbf{a} \in W_\eta$  and  $\chi^{W_\eta}(\mathbf{a})$ . Since  $W_\eta = \bigcup_{\alpha \in X} W_\alpha$  we have  $\mathbf{a} \in W_\beta$  for some  $\beta \in X$ . Now  $\forall v_{n+1} \in W_\eta \psi^{W_\eta}(\mathbf{a}, v_{n+1})$ . Since  $W_\beta \subseteq W_\eta$ , we have  $\forall v_{n+1} \in W_\beta \psi^{W_\eta}(\mathbf{a}, v_{n+1})$ . Now let  $a_{n+1} \in W_\beta$ . Then  $\psi^{W_\eta}(\mathbf{a}, a_{n+1})$ . Hence by (\*),  $\psi^W(\mathbf{a}, a_{n+1})$ . But  $\beta \in X \subseteq T_{S'}$  (and  $\psi \in S'$ ), so  $\psi^{W_\beta}(\mathbf{a}, a_{n+1})$ . Since  $a_{n+1} \in W_\beta$  was arbitrary, we have  $\forall v_{n+1} \in W_\beta \psi^{W_\beta}(\mathbf{a}, v_{n+1})$ , ie.  $\chi^{W_\beta}(\mathbf{a})$ . Hence by (\*\*),  $\chi^W(\mathbf{a})$  as required.  $\square$

To complete the proof of the theorem we now show that

$$\forall \alpha \in On \exists \beta \in On (\beta > \alpha \wedge \beta \in T_S).$$

The proof is again by induction on  $\#S$ , and the only difficult case is when  $\chi$  is  $\forall v_{n+1} \psi(\mathbf{v}, v_{n+1})$  and  $S' = S \setminus \{\chi\}$ ,  $S'$  closed under taking subformulas.

By our inductive hypothesis we have

$$\forall \alpha \exists \beta > \alpha \beta \in T_{S'}. \quad (***)$$

It remains to show that given any  $\alpha \in On, \exists \beta > \alpha \beta \in T_{S'}$ , such that  $\forall \mathbf{a} \in W_\beta (\chi^{W_\beta}(\mathbf{a}) \leftrightarrow \chi^W(\mathbf{a}))$ . (For then such a  $\beta$  will be in  $T_S$ .)

Let  $\alpha \in On$  be given.

Now  $\chi(v)$  is  $\forall v_{n+1} \psi(v_1, \dots, v_n, v_{n+1})$ .

Define the term  $f : On \times V^n \rightarrow On$  so that  $\forall \gamma \in On \forall a_1, \dots, a_n \in V$   $f(\gamma, a_1, \dots, a_n)$  is the least  $\theta \in On$  such that  $\theta > \gamma$  and  $\exists a_{n+1} \in W_\theta$  such that  $\neg \psi^W(a_1, \dots, a_n, a_{n+1})$ , if such a  $\theta$  exists.

Now define the term  $F : On \rightarrow On$  so that  $\forall \gamma \in On$   $F(\gamma)$  is the least  $\theta \in T_{S'}$  such that  $\theta > \sup\{f(\gamma, a_1, \dots, a_n) : \langle a_1, \dots, a_n \rangle \in W_\gamma^n\}$ . (This last thing is a set by replacement since  $W_\gamma^n$  is.  $\theta$  exists using (\*\*\*) .)

Notice that for all  $\gamma$ ,  $F(\gamma) > \gamma$ ,  $F(\gamma) \in T_{S'}$ , and if  $a_1, \dots, a_n \in W_\gamma$ ,

$$\forall v_{n+1} \in W_{F(\gamma)} \psi^W(a_1, \dots, a_n, v_{n+1}) \Rightarrow \forall v_{n+1} \in W \psi^W(a_1, \dots, a_n, v_{n+1}) \quad (\dagger\dagger)$$

(For otherwise,  $\exists a_{n+1} \in W \neg \psi^W(a_1, \dots, a_n, a_{n+1})$ , so for some minimal  $\eta$ ,  $\exists a_{n+1} \in W^\eta \neg \psi^W(a_1, \dots, a_n, a_{n+1})$  (since  $W = \bigcup_{\eta \in On} W_\eta$ ), so  $F(\gamma) \geq f(\gamma, a_1, \dots, a_n) \geq \eta$ , so  $\exists a_{n+1} \in W_{F(\gamma)} \neg \psi^W(a_1, \dots, a_n, a_{n+1})$  since  $W_{F(\gamma)} \supseteq W_\eta$ —contradiction.)

Now by the recursion theorem on  $\omega$  define the function  $g : \omega \rightarrow On$  by

1.  $g(0) = F(\alpha)$ ,
2.  $g(n+1) = F(g(n))$ ;

let  $X = \text{rang } g$ . Clearly  $X$  has no greatest element and  $X \subseteq T_{S'}$ . Let  $\beta = \sup X$ . Since  $T_{S'}$  is closed (Lemma above), we have  $\beta \in T_{S'}$ . We also have  $\beta > \alpha$ , and:

For all  $a_1, \dots, a_n \in W_\beta$ ,  
if  $\forall v_{n+1} \in W_\beta \psi^W(a_1, \dots, a_n, v_{n+1})$ , then  $\forall v_{n+1} \in W \psi^W(a_1, \dots, a_n, v_{n+1})$ .  
(\*\*\*)

*Proof.* Suppose  $a_1, \dots, a_n \in W_\beta$ . Since  $W_\beta = \bigcup_{\gamma \in X} W_\gamma$ , we have  $a_1, \dots, a_n \in W_\gamma$ , for some  $\gamma \in X$ . Suppose  $\forall v_{n+1} \in W_\beta \psi^W(a_1, \dots, a_n, v_{n+1})$ .

Since  $F(\gamma) \in X$ , and hence  $W_{F(\gamma)} \subseteq W_\beta$ , we have  $\forall v_{n+1} \in W_{F(\gamma)} \psi^W(a_1, \dots, a_n, v_{n+1})$ . Hence by ( $\dagger\dagger$ ) we have  $\forall v_{n+1} \in W \psi^W(a_1, \dots, a_n, v_{n+1})$ , as required.  $\square$

Now show that (\*\*\*) implies  $\beta \in T_S$  as required (exercise, Problem sheet 4).  $\square$

## 6 Gödel's Constructible Universe

### 6.1

For any set  $a$  and  $n \in \omega$  we define  ${}^n a$  to be  $\{f : f : n \rightarrow a\}$ , and  ${}^{<\omega} a = \bigcup_{n \in \omega} {}^n a$ . (Exercise: this is a set.)

We shall define the class term  $Def : V \rightarrow V$  so that

$$Def(A) = \{X \subseteq A : X \text{ is definable from } A\},$$

where  $X$  is definable from  $A$  if there is formula  $\phi(x_1, \dots, x_n, x)$  of LST and there are elements  $a_1, \dots, a_n$  of  $A$  such that  $X = \{a \in A : \langle A, \in \rangle \models \phi(a_1, \dots, a_n, a)\}$ .

WARNING: it is difficult to prove that  $Def(A)$  is a class. We postpone this till section 8.

In order to construct  $Def$  we shall construct a class term  $G : \omega \times V \times V \rightarrow V$  such that

$$\forall m \in \omega \forall a, s \in V \ G(m, a, s) \subseteq a.$$

Further to each formula  $\psi(v_0, \dots, v_{n-1}, v_n)$  of LST with free variables amongst  $v_0, \dots, v_n$  (with  $n \geq 1$ ), there will be assigned a number  $m \in \omega$  ( $m = \lceil \psi(v_0, \dots, v_n) \rceil$ ) with the property that for all  $a, s \in V$ ,

$G(m, a, s) = \{b \in a : \langle a, \in \rangle \models \psi(s(0), \dots, s(n-1), b)\}$  if  $s \in {}^{<\omega} a$  and  $\text{dom } s \geq n$  and  $\emptyset$  otherwise.

We then define the class term  $Def : V \rightarrow V$  by

$$Def(a) = \{G(m, a, s) : m \in \omega, s \in {}^{<\omega} a\}.$$

Thus  $Def(a)$  consists of all the definable (with parameters) subsets of the structure  $\langle a, \in \rangle$ .

**Definition 6.1** (*The constructible hierarchy*)

We define the class term  $L : On \rightarrow V$  (writing  $L_\alpha$  for  $L(\alpha)$ ) by recursion on  $On$  as follows:

1.  $L_0 = \emptyset$ ;
2.  $L_{\alpha+1} = Def(L_\alpha)$ ;
3.  $L_\delta = \bigcup_{\alpha < \delta} L_\alpha$  for limit  $\delta$ .

$L$  is called the Constructible Universe.  
Throughout we assume ZF holds in  $V$ .

**Lemma 6.2** *For all  $\alpha, \beta \in On$ :*

1.  $\alpha < \beta \rightarrow L_\alpha \subseteq L_\beta$ ;
2.  $\alpha < \beta \rightarrow L_\alpha \in L_\beta$ ;
3.  $L_\beta$  is transitive;
4.  $L_\beta \subseteq V_\beta$ ;
5.  $On \cap L_\beta = \beta$ .

*Proof.* Fix  $\alpha$ . We prove (1)–(5) (simultaneously) by induction on  $\beta$ .

$\beta = 0$ : trivial.

*The successor case:* Suppose (1)–(5) true for  $\beta$ .

(1) Suffices to show  $L_\beta \subseteq L_{\beta+1}$ . Suppose  $x \in L_\beta$ . Then  $x \subseteq L_\beta$  (by IH(3)). Let  $s = \{\langle 0, x \rangle\}$ ; then  $s \in {}^{<\omega}L_\beta$  and  $\text{doms} = 1$ . Then  $A = G(\ulcorner v_1 \in v_0 \urcorner, L_\beta, s) \in Def(L_\beta) = L_{\beta+1}$ .

Also  $A = \{b \in L_\beta : \langle L_\beta, \in \rangle \models b \in s(0)\} = \{b \in L_\beta : b \in x\} = x$  (since  $x \subseteq L_\beta$ ).

Thus  $x \in L_{\beta+1}$  as required.

(2) Suffices to show (by (1)) that  $L_\beta \in L_{\beta+1}$ . (Since if  $\alpha < \beta$  then  $L_\alpha \in L_\beta$  (by IH) and  $L_\beta \subseteq L_{\beta+1}$  (by (1)).

Must show that  $L_\beta \in Def(L_\beta)$ .

Let  $s = \emptyset$ . Then  $G(\ulcorner v_1 = v_0 \urcorner, L_\beta, s) = \{b \in L_\beta : \langle L_\beta, \in \rangle \models b = b\} = L_\beta$ , so  $L_\beta \in Def(L_\beta)$ , as required.

(3) If  $x \in L_{\beta+1}$ , then  $x \subseteq L_\beta$ . But  $L_\beta \subseteq L_{\beta+1}$ , by (1), so  $x \subseteq L_{\beta+1}$ . Thus  $L_{\beta+1}$  is transitive.

(4) By IH  $L_\beta \subseteq V_\beta$ .

Also  $x \in L_{\beta+1} \rightarrow x \subseteq L_\beta \rightarrow x \subseteq V_\beta \rightarrow x \in \mathbb{P}V_\beta = V_{\beta+1}$ .

Thus  $L_{\beta+1} \subseteq V_{\beta+1}$ .

(5) By IH  $On \cap L_\beta = \beta$ .

Suppose  $x \in On \cap L_{\beta+1}$ . Then  $x \in On$  and  $x \subseteq L_\beta$ .

But every member of  $x$  is an ordinal, so  $x \subseteq L_\beta \cap On$ , so  $x \subseteq \beta$ . Thus either  $x \in \beta$  or  $x = \beta$ . In either case  $x \in \beta \cup \{\beta\} = \beta + 1$ . Thus  $On \cap L_{\beta+1} \subseteq \beta + 1$ .

Suppose  $x \in \beta + 1$ . Then either  $x \in \beta$ , in which case  $x \in On \cap L_\beta \subseteq On \cap L_{\beta+1}$  (by (1)), or  $x = \beta$ . So it remains to show  $\beta \in L_{\beta+1}$ .

Let  $s = \emptyset$ .

Then  $A = G([On(v_0)], L_\beta, s) = \{b \in L_\beta : \langle L_\beta, \in \rangle \models On(b)\}$ , and  $A \in Def(L_\beta) = L_{\beta+1}$ . We show  $A = \beta$ .

But  $On(v_0)$  is an absolute formula, that is has the same meaning in any transitive class (exercise).

Thus  $\forall b \in L_\beta, \langle L_\beta, \in \rangle \models On(b)$  iff  $b \in On$ .

Thus  $A = L_\beta \cap On = \beta$  by IH, as required.

*The Limit Step* Suppose  $\delta > 0$  is a limit ordinal and (1)–(5) hold for all  $\beta < \delta$ . Since  $L_\delta = \bigcup_{\beta < \delta} L_\beta$ , (1)–(5) for  $\delta$  are all easy.  $\square$

**Lemma 6.3** For all  $n \in \omega$ ,  $L_n = V_n$ .

*Proof.* By induction on  $n$ .

For  $n = 0$ , this is clear.

Suppose now that  $L_n = V_n$ .

Now  $L_{n+1} \subseteq V_{n+1}$  by 6.2.

Suppose  $x \in V_{n+1}$ . Then  $x \subseteq V_n$ , so  $x$  is finite. Also  $x \subseteq L_n$  by IH. Say  $x = \{a_0, \dots, a_{k-1}\}$  ( $k \in \omega$ ), so that  $a_0, \dots, a_{k-1} \in L_n$ .

Let  $s = \{\langle 0, a_0 \rangle, \dots, \langle k-1, a_{k-1} \rangle\}$ , so  $s \in {}^k L_n$ .

Let  $A = G([\{v_k = v_0 \vee \dots \vee v_k = v_{k-1}\}], L_n, s) = \{b \in L_n : \langle L_n, \in \rangle \models (b = a_0 \vee \dots \vee b = a_{k-1})\} = \{a_0, \dots, a_{k-1}\} = x$ .

Thus  $x \in Def(L_n) = L_{n+1}$ .

Thus  $V_{n+1} \subseteq L_{n+1}$ .

So  $V_{n+1} = L_{n+1}$ .  $\square$

**Lemma 6.4** Suppose  $a, c \in L$ . Then

1.  $\{a, c\} \in L$ .

2.  $\cup a \in L$ .

3.  $\mathbb{P}a \cap L \in L$ .

4.  $\omega \in L$ .

*Proof.* (1) Suppose  $a, c \in L_\alpha$ . Define  $s = \{\langle 0, a \rangle, \langle 1, c \rangle\}$ , so  $s \in {}^{<\omega}L_\alpha$ .

Then  $L_{\alpha+1} \ni G(\lceil v_2 = v_0 \vee v_2 = v_1 \rceil, L_\alpha, s) = \{b \in L_\alpha : \langle L_\alpha, \exists \rangle \models b = a \vee b = c\} = L_\alpha \cap \{a, c\} = \{a, c\}$ .

So  $\{a, c\} \in L_{\alpha+1} \subseteq L$ .

(2) Suppose  $a \in L_\alpha$ . Let  $s = \{\langle 0, a \rangle\}$ . Then  $L_{\alpha+1} \ni G(\lceil \exists v_2 \in v_0 (v_1 \in v_2) \rceil, L_\alpha, s) = \{b \in L_\alpha : \langle L_\alpha, \in \rangle \models \exists v_2 \in a (b \in v_2)\} = A$ , say.

We claim that  $A = \cup a$ .

Suppose that  $b \in A$ .

Then  $\langle L_\alpha, \in \rangle \models \exists v_2 \in a (b \in v_2)$ .

Say  $d \in L_\alpha$  is such that  $\langle L_\alpha, \in \rangle \models d \in a \wedge b \in d$ .

Then  $d \in a \wedge b \in d$ , so  $b \in \cup a$ .

Conversely, suppose  $b \in \cup a$ . Then for some  $d \in a$ ,  $b \in d$ . But  $L_\alpha$  is transitive, and  $a \in L_\alpha$ , so  $d \in L_\alpha$ , and hence  $b \in L_\alpha$ .

So  $\langle L_\alpha, \in \rangle \models d \in a \wedge b \in d$ . Hence  $\langle L_\alpha, \in \rangle \models \exists v_2 \in a (b \in v_2)$  (and  $b \in L_\alpha$ ) so  $b \in A$  as required.

Thus  $\cup a \in L_{\alpha+1} \in L$ .

(3) Let  $f : \mathbb{P}a \rightarrow On$  be defined so that  $f(x)$  is the least  $\alpha$  such that  $x \in L_\alpha$  if there is one,  $f(x) = 0$  otherwise.

Then by replacement  $\text{ran } f$  is a set, and hence  $\exists \beta \in On$  such that  $\beta > \alpha$  for all  $\alpha \in \text{ran } f$ .

Clearly  $\mathbb{P}a \cap L \subseteq L_\beta$  (using 6.2 (1)).

We may also suppose that  $a \in L_\beta$ .

Let  $s = \{\langle 0, a \rangle\}$ .

Then  $L_{\beta+1} \ni G(\lceil \forall v_2 \in v_1 (v_2 \in v_0) \rceil, L_\beta, s) = \{b \in L_\beta : \langle L_\beta, \in \rangle \models \forall v_2 \in b (v_2 \in a)\} = A$ , say.

Suffices to show  $A = \mathbb{P}a \cap L$ .

Suppose  $b \in A$ . Then  $b \in L_\beta$  (so  $b \in L$ ) and  $\langle L_\beta, \in \rangle \models \forall v_2 \in b (v_2 \in a)$ .

Now suppose  $d \in b$ . Then  $d \in L_\beta$  since  $L_\beta$  is transitive. Hence  $\langle L_\beta, \in \rangle \models d \in b \wedge d \in a$ , so  $d \in a$ .

Hence  $b \subseteq a$ , so  $b \in \mathbb{P}a \cap L$ . Thus  $A \subseteq \mathbb{P}a \cap L$ .

Conversely suppose  $b \in \mathbb{P}a \cap L$ . Then  $b \in L_\beta$ .

Also  $\forall v_2 \in b (v_2 \in a)$ . Hence  $\forall v_2 \in L_\beta (v_2 \in b \rightarrow v_2 \in a)$ , so  $\langle L_\beta, \in \rangle \models \forall v_2 \in b (v_2 \in a)$ .

So  $b \in A$ .

Hence  $\mathbb{P}a \cap L = A$ .  $\square$

It is now easy to check that

**Corollary 6.5** *Extensionality, empty-set, pairs, unions, power-set are all true in  $L$ .*

**Lemma 6.6**  $\langle L, \in \rangle \models$  *separation.*

*Proof.* Suppose  $u \in L$ , and  $a_0, \dots, a_n \in L$ . Say  $u, a_0, \dots, a_n \in L_\alpha$ . Let  $\phi(v_0, \dots, v_{n+1})$  be a formula of LST. By Lévy's Reflection Principle, there is some  $\beta \geq \alpha$  such that  $\forall c, c_1, \dots, c_{n+1} \in L_\beta$

$$\langle L_\beta, \in \rangle \models (c \in c_{n+1} \wedge \phi(c_0, \dots, c_n, c)) \Leftrightarrow \langle L, \in \rangle \models (c \in c_{n+1} \wedge \phi(c_0, \dots, c_n, c)). \quad (*)$$

Let  $\psi(v_0, \dots, v_{n+2}) = (v_{n+2} \in v_{n+1} \wedge \phi(v_0, \dots, v_n, v_{n+2}))$ .

Let  $s = \{\langle 0, a_0 \rangle, \dots, \langle n, a_n \rangle, \langle n+1, u \rangle\}$ .

Then  $L_{\beta+1} \ni G(\ulcorner \psi(v_0, \dots, v_{n+2}) \urcorner, L_\beta, s) = \{b \in L_\beta : \langle L_\beta, \in \rangle \models \psi(a_0, \dots, a_n, u, b)\} = \{b \in L_\beta : \langle L_\beta, \in \rangle \models (b \in u \wedge \phi(a_0, \dots, a_n, b))\} = A$ , say. (So  $A \in L$ .)

Sufficient to show  $\langle L, \in \rangle \models \forall x (x \in A \leftrightarrow (x \in u \wedge \phi(a_0, \dots, a_n, x)))$ .

$\Rightarrow$ : Suppose  $x \in L$  and  $x \in A$ . Then  $x \in L_\beta$ , and  $\langle L_\beta, \in \rangle \models x \in u \wedge \phi(a_0, \dots, a_n, x)$ .

By (\*),  $\langle L, \in \rangle \models x \in u \wedge \phi(a_0, \dots, a_n, x)$ , as required.

$\Leftarrow$ : Suppose  $x \in L$ , and  $x \in u \wedge \phi(a_0, \dots, a_n, x)$ . Then  $x \in L_\beta$ , since  $x \in L_\beta$  and  $L_\beta$  is transitive. Hence, using (\*),  $\langle L_\beta, \in \rangle \models x \in u \wedge \phi(a_0, \dots, a_n, x)$ , so  $x \in A$ , as required.  $\square$

**Lemma 6.7**  $\langle L, \in \rangle \models$  *replacement.*

*Proof.* Suppose  $a_0, \dots, a_n \in L$ ,  $\mathbf{a} = \langle a_0, \dots, a_n \rangle$ ,  $u \in L$ ,  $\phi(\mathbf{x}, y, z)$  a formula of LST, and  $\langle L, \in \rangle \models \underbrace{\forall z, y, y' ((\phi(\mathbf{a}, z, y) \wedge \phi(\mathbf{a}, z, y')) \rightarrow y = y')}_{\sigma}$ .

Now choose  $\beta$  so large that  $a_0, a_1, \dots, a_n, u \in L_\beta$ , and such that (by LRP) for all  $z \in L_\beta$   $\langle L, \in \rangle \models \sigma \wedge \exists y (\phi(\mathbf{a}, z, y) \wedge z \in u) \Leftrightarrow \langle L_\beta, \in \rangle \models \sigma \wedge \exists y (\phi(\mathbf{a}, z, y) \wedge z \in u)$ , and for all  $c, d \in L_\beta$ ,  $\langle L, \in \rangle \models \phi(\mathbf{a}, c, d)$  iff  $\langle L_\beta, \in \rangle \models \phi(\mathbf{a}, c, d)$ .

Now let  $A = \{b \in L_\beta : \langle L_\beta, \in \rangle \models \exists z \in u (\phi(\mathbf{a}, z, b))\}$ , so  $A \in L_{\beta+1}$ .

Then, as in the proof of separation,  $\langle L, \in \rangle \models \forall z \in u (\exists y \phi(\mathbf{a}, z, y) \leftrightarrow \exists y \in A (\phi(\mathbf{a}, z, y)))$ , as required.  $\square$

**Lemma 6.8**  $\langle L, \in \rangle \models \text{Foundation}$ .

*Proof.* Suppose  $a \in L$ . Choose  $b \in V$  such that  $b \in a \wedge b \cap a = \emptyset$ . Since  $L$  is transitive,  $b \in L$  and clearly  $\langle L, \in \rangle \models b \in a \wedge b \cap a = \emptyset$ .  $\square$

**Theorem 6.9**  $\langle L, \in \rangle \models ZF$ .

## 7 Absoluteness

### 7.1

**Definition 7.1** The  $\Sigma_0$ -formulas of LST are defined as follows:

1.  $x \in y$ ,  $x = y$ ,  $\neg x \in y$ ,  $\neg x = y$  are  $\Sigma_0$ -formulas for any variables  $x$  and  $y$ .
2. If  $\psi$ ,  $\phi$  are  $\Sigma_0$ -formulas, so are  $\psi \wedge \phi$ ,  $\psi \vee \phi$ ,  $\forall x \in y \phi$  and  $\exists x \in y \phi$  (where  $x$  and  $y$  are distinct variables).
3. Nothing else is a  $\Sigma_0$  formula.

**Lemma 7.2** If  $\phi$  is a  $\Sigma_0$  formula, then  $\neg\phi$  is logically equivalent to a  $\Sigma_0$  formula.

*Proof.* Easy induction on  $\phi$ . Note that  $\neg\forall x \in y \phi$  is logically equivalent to  $\exists x \in y \neg\phi$ .  $\square$

**Lemma 7.3** If  $\phi(x_1, \dots, x_n)$  is a  $\Sigma_0$ -formula and  $U_1$  and  $U_2$  are transitive classes such that  $U_1 \subseteq U_2$ , then for all  $a_1, \dots, a_n \in U_1$ ,

$$\langle U_1, \in \rangle \models \phi(a_1, \dots, a_n) \Leftrightarrow \langle U_2, \in \rangle \models \phi(a_1, \dots, a_n).$$

We say  $\phi$  is absolute between  $U_1$  and  $U_2$ .

*Proof.* Exercise—induction on  $\phi$ .  $\square$

**Definition 7.4** The  $\Sigma_1$ -formulas of LST are defined as follows:

1.  $x \in y$ ,  $x = y$ ,  $\neg x \in y$ ,  $\neg x = y$  are  $\Sigma_1$ -formulas for any variables  $x$  and  $y$ .

2. If  $\psi, \phi$  are  $\Sigma_1$ -formulas, so are  $\psi \wedge \phi, \psi \vee \phi, \forall x \in y \phi$  and  $\exists x \in y \phi$  (where  $x$  and  $y$  are distinct variables), and  $\exists x \phi$ .
3. Nothing else is a  $\Sigma_1$  formula.

**Remark 7.5** Note that every  $\Sigma_0$  formula is  $\Sigma_1$ .

**Lemma 7.6** If  $\phi(x_1, \dots, x_n)$  is a  $\Sigma_1$ -formula, and  $U_1$  and  $U_2$  are transitive classes with  $U_1 \subseteq U_2$ , then for all  $a_1, \dots, a_n \in U_1$

$$\langle U_1, \in \rangle \models \phi(a_1, \dots, a_n) \Rightarrow \langle U_2, \in \rangle \models \phi(a_1, \dots, a_n).$$

(ie.  $\phi$  is preserved up or is upward absolute between  $U_1$  and  $U_2$ .)

**Definition 7.7** (1) A formula  $\phi(\mathbf{x})$  is called  $\Sigma_0^{ZF}$  (respectively  $\Sigma_1^{ZF}$ ) if there is a  $\Sigma_0$  (or  $\Sigma_1$ ) formula  $\psi(\mathbf{x})$  such that  $ZF \vdash \forall \mathbf{x}(\phi(\mathbf{x}) \leftrightarrow \psi(\mathbf{x}))$ .

(2) A formula  $\phi$  is called  $\Delta_1^{ZF}$  if  $\phi$  and  $\neg\phi$  are  $\Sigma_1^{ZF}$ .

(3) Suppose  $n \in \omega$  and  $F : V^n \rightarrow V$  is a class term. Then  $F$  is called  $\Delta_1^{ZF}$  if the formula  $\phi(x_1, \dots, x_n, x_{n+1})$  defining  $F(x_1, \dots, x_n) = x_{n+1}$  is  $\Delta_1^{ZF}$ , and if  $ZF$  proves that  $F$  is a class term.

**Remark 7.8** We need only verify that  $\phi$  in part (3) is  $\Sigma_1^{ZF}$ , since  $\neg\phi$  is  $\Sigma_1^{ZF}$  thus:

$$ZF \vdash \forall x_1, \dots, x_n, x_{n+1} (\neg\phi(x_1, \dots, x_n, x_{n+1}) \leftrightarrow \exists y (\phi(x_1, \dots, x_n, y) \wedge \neg y = x_{n+1}))$$

—and the bit on the right is  $\Sigma_1^{ZF}$  if  $\phi$  is.

**Remark 7.9** Every  $\Sigma_0^{ZF}$  formula is  $\Delta_1^{ZF}$  by 7.2 and 7.5.

**Theorem 7.10** Suppose  $\phi(x_1, \dots, x_n)$  is  $\Delta_1^{ZF}$  and  $U_1$  and  $U_2$  are transitive classes such that  $U_1 \subseteq U_2$  and  $\langle U_i, \in \rangle \models ZF$  ( $i = 1, 2$ ). Then for all  $a_1, \dots, a_n \in U_1$ ,

$$\langle U, \in \rangle \models \phi(a_1, \dots, a_n) \Leftrightarrow \langle U_2, \in \rangle \models \phi(a_1, \dots, a_n).$$

(ie.  $\phi$  is ZF-absolute.)

*Proof.* Let  $\psi(x_1, \dots, x_n)$  be  $\Sigma_1$  such that  $\text{ZF} \vdash \forall \mathbf{x}(\phi(\mathbf{x}) \leftrightarrow \psi(\mathbf{x}))$  (\*).  
Then

$$\begin{aligned}
\langle U_1, \in \rangle \models \phi(\mathbf{a}) &\Rightarrow \langle U_1, \in \rangle \models \psi(\mathbf{a}) && (*) \text{ and } \langle U_1, \in \rangle \models \text{ZF} \\
&\Rightarrow \langle U_2, \in \rangle \models \psi(\mathbf{a}) && \text{by 7.6} \\
&\Rightarrow \langle U_2, \in \rangle \models \phi(\mathbf{a}) && (*) \text{ and } \langle U_1, \in \rangle \models \text{ZF}
\end{aligned} \tag{2}$$

Now let  $\chi(x_1, \dots, x_n)$  be  $\Sigma_1$  such that  $\text{ZF} \vdash \forall \mathbf{x}(\neg\phi(\mathbf{x}) \leftrightarrow \chi(\mathbf{x}))$  (\*).  
Then as above,

$$\begin{aligned}
\langle U_1, \in \rangle \models \neg\phi(\mathbf{a}) &\Rightarrow \langle U_1, \in \rangle \models \chi(\mathbf{a}) && (*) \text{ and } \langle U_1, \in \rangle \models \text{ZF} \\
&\Rightarrow \langle U_2, \in \rangle \models \chi(\mathbf{a}) && \text{by 7.6} \\
&\Rightarrow \langle U_2, \in \rangle \models \neg\phi(\mathbf{a}) && (*) \text{ and } \langle U_1, \in \rangle \models \text{ZF}
\end{aligned} \tag{3}$$

□

**Theorem 7.11** *The following formulas and class terms are all  $\Sigma_0^{\text{ZF}}$  (and hence  $\Delta_0^{\text{ZF}}$ ):*

1.  $x = y$
2.  $x \in y$
3.  $x \subseteq y$
4.  $F(x_1, \dots, x_n) = \{x_1, \dots, x_n\}$  (for each  $n$ )
5.  $F(x_1, \dots, x_n) = \langle x_1, \dots, x_n \rangle$  (for each  $n$ )
6. (where  $n \geq 1$  and  $0 \leq i \leq n - 1$ )  $F(x) = x_i$  if  $x$  is an  $n$ -tuple  $\langle x_0, \dots, x_{n-1} \rangle$ ,  $\emptyset$  otherwise.
7.  $F(x, y) = x \cup y$ .
8.  $F(x, y) = x \cap y$ .
9.  $F(x) = \bigcup x$ .
10.  $F(x) = \bigcap x$  if  $x \neq \emptyset$ ,  $F(x) = \emptyset$  otherwise.

11.  $F(x, y) = x \setminus y$ .
12.  $x$  is an  $n$ -tuple.
13.  $x$  is an  $n$ -ary relation on  $y$ .
14.  $x$  is a function.
15.  $F(x) = \text{dom } x$  if  $x$  is a function,  $\emptyset$  otherwise.
16.  $F(x) = \text{ran } x$  if  $x$  is a function,  $\emptyset$  otherwise.
17.  $F(x, y) = x[y]$  ( $= \{x(t) : t \in y\}$ ) if  $x$  is a function,  $\emptyset$  otherwise.
18.  $F(x, y) = x \upharpoonright y$  if  $x$  is a function,  $\emptyset$  otherwise.
19.  $F(x) = x^{-1}$  if  $x$  is a function,  $\emptyset$  otherwise.
20.  $F(x) = x \cup \{x\}$ .
21.  $x$  is transitive.
22.  $x$  is an ordinal.
23.  $x$  is a successor ordinal.
24.  $x$  is a limit ordinal.
25.  $x : y \rightarrow z$ .
26.  $x : y \sim z$ .
27.  $x$  is a natural number.
28.  $x = \omega$ .
29.  $x$  is a finite sequence of elements of  $y$ .

*Proof.* (Selections) (3)  $x \subseteq y \Leftrightarrow \forall z \in x (z \in y)$  which is  $\Sigma_0$ .

Note that all the class terms  $F$  above are in ZF provably class terms, so we only have to show that the statement  $F(\mathbf{x}) = y$  can be put in  $\Sigma_0$  form.

(4)  $F(x_1, \dots, x_n) = y \Leftrightarrow x_1 \in y \wedge x_2 \in y \wedge \dots \wedge x_n \in y \wedge \forall z \in y (z = x_1 \vee \dots \vee z = x_n)$ .

(5)  $F(x_1, x_2) = y \Leftrightarrow \exists z_1 \in y \exists z_2 \in y (z_1 = \{x_1\} \wedge z_2 = \{x_1, x_2\} \wedge \forall t \in y (t = z_1 \vee t = z_2))$ , which is  $\Sigma_0$  by (4).

(12)  $x$  is a 2-tuple iff  $\exists z_1 \in x \exists x_1 \in z_1 \exists x_2 \in z_1 (x = \langle x_1, x_2 \rangle)$ , which is  $\Sigma_0$  by (5).

(13)  $x$  is a 2-ary relation on  $y$  iff  $\forall z \in x \exists y_1 \in y \exists y_2 \in y (z = \langle y_1, y_2 \rangle)$ , which is  $\Sigma_0$  by (5).

(29)  $x$  is a natural number iff  $(x \text{ is an ordinal}) \wedge (x \text{ is not a limit ordinal}) \wedge (\forall y \in x \ y \text{ is not a limit ordinal})$ , which is  $\Sigma_0$  by (24), (26) and the fact that  $\Sigma_0^{ZF}$  formulas are closed under  $\neg$ .  $\square$

**Lemma 7.12** *Suppose  $F$  and  $G$  are  $\Delta_1^{ZF}$  class terms. Then “ $F(\mathbf{x}) = G(\mathbf{y})$ ” is  $\Delta_1^{ZF}$ .*

*Proof.* Let  $\psi(\mathbf{x}, z)$  and  $\chi(\mathbf{y}, t)$  be  $\Sigma_1$  formulas defining (in ZF)  $F(\mathbf{x}) = y$  and  $G(\mathbf{y}) = t$  respectively. Then

$$F(\mathbf{x}) = G(\mathbf{y}) \underset{ZF}{\Leftrightarrow} \exists z (\psi(\mathbf{x}, z) \wedge \chi(\mathbf{y}, z)),$$

which is  $\Sigma_1$ , and

$$F(\mathbf{x}) \neq G(\mathbf{y}) \underset{ZF}{\Leftrightarrow} \exists z \exists t (\psi(\mathbf{x}, z) \wedge \chi(\mathbf{y}, t) \wedge \neg z = t),$$

which is  $\Sigma_1$ .

Hence “ $F(\mathbf{x}) = G(\mathbf{y})$ ” is  $\Delta_1^{ZF}$ .  $\square$

**Theorem 7.13** *Suppose  $F : V \times V \rightarrow V$  is a  $\Delta_1^{ZF}$  class term. Then the class term  $G$  defined from  $F$  by recursion on  $On$ , ie:*

1.  $G(0, x) = x$
2.  $G(\alpha + 1, x) = F(G(\alpha, x), x)$  for all  $\alpha \in On$
3.  $G(\delta, x) = \bigcup_{\alpha < \delta} G(\alpha, x)$  for all limit  $\delta \in On$
4.  $G(y, x) = \emptyset$  for all  $y \notin On$

is  $\Delta_1^{ZF}$ .

*Proof.* As in the proof of 3.14 define  $\phi(g, \alpha, x)$  by

$$\begin{array}{ll}
& On(\alpha) & \chi_1 \\
\wedge & g \text{ is a function} & \chi_2 \\
\wedge & \text{dom } g = \alpha \cup \{\alpha\} & \chi_3 \\
\wedge & g(0) = x & \chi_4 \\
\wedge & \forall \beta \in \alpha \exists y_1 \exists y_2 (y_1 = \beta \cup \{\beta\} \wedge y_2 = g(\beta) \wedge g(y_1) = F(y_2)) & \chi_5 \\
\wedge & \forall \beta \in \alpha (\beta \text{ is a limit ordinal} \rightarrow g(\beta) = \bigcup \{g(\alpha) : \alpha \in \beta\}). & \chi_6
\end{array} \tag{4}$$

$\chi_1$  is  $\Sigma_0^{ZF}$  by 7.11 (24);  $\chi_2$  is  $\Sigma_0^{ZF}$  by (14);  $\chi_3$  is by (15), (22) and 7.12;  $\chi_4$  can be rewritten as  $\exists y((\forall z \in y (z \neq z) \wedge g(y) = x)$  so is  $\Sigma_1^{ZF}$  by (17);  $\chi_5$  is  $\Sigma_1^{ZF}$  by (22), (17) and the fact that  $F$  is  $\Sigma_1^{ZF}$ , and using 7.12;  $\chi_6$  is  $\Sigma_1^{ZF}$  by (26) and the fact that “ $g(\beta) = \bigcup \{g(\alpha) : \alpha \in \beta\}$ ” is equivalent to  $\exists y \exists z (y = g[\beta] \wedge z = \bigcup y \wedge g(\beta) = z)$ , which is  $\Sigma_1^{ZF}$  by (18), (9) and (17).

Hence  $\phi(g, \alpha, x)$  is  $\Sigma_1^{ZF}$ .

Now recall from the proof of 3.14 that  $G$  can be defined by:

$$G(\alpha, x) = y \Leftrightarrow \exists g(\phi(g, \alpha, x) \wedge g(\alpha) = y) \vee (\neg On(\alpha) \wedge y = \emptyset).$$

This shows  $G$  is  $\Sigma_1^{ZF}$ , and hence  $\Delta_1^{ZF}$  by 7.8.  $\square$

**Corollary 7.14** *Assuming the class term  $G$  (from the beginning of subsection 6) is  $\Delta_1^{ZF}$ , then so is the class term  $\bar{L} : On \rightarrow V$ . (Strictly  $\bar{L} : V \rightarrow V$ , where  $\bar{L}(x) = \emptyset$  if  $x \notin On$ .)*

*Proof.* By 7.13 it is sufficient to show  $Def$  is  $\Delta_1^{ZF}$ . Recall that  $Def : V \rightarrow V$  is defined by

$$Def(a) = \{G(m, a, s) : m \in \omega, s \in {}^{<\omega}a\}.$$

Hence  $Def(a) = y$  iff  $\exists w \exists x (w = \omega \wedge x = {}^{<\omega}a \wedge \forall m \in w \forall s \in x \exists t \in y t = G(m, a, s) \wedge \forall t \in y \exists m \in w \exists s \in x t = G(m, a, s))$ .

Now  $x = {}^{<\omega}a$  is  $\Delta_1^{ZF}$ , so  $Def$  is  $\Sigma_1^{ZF}$  by 7.11 (29), (30), (31), and because  $G$  is.

Hence  $Def$  is  $\Delta_1^{ZF}$  by 7.8.  $\square$

**Definition 7.15**  $V=L$  is the sentence of LST:  $\forall x \exists \alpha (On(\alpha) \wedge x \in \bar{L}(\alpha))$  (writing  $L_\alpha$  for  $\bar{L}(\alpha)$ ).

**Theorem 7.16**  $\langle L, \in \rangle \models V=L$ .

*Proof.* Suppose  $a \in L$ . We must show  $\langle L, \in \rangle \models \exists \alpha (On(\alpha) \wedge a \in \bar{L}(\alpha))$ . Now choose  $\alpha$  such that  $a \in L_\alpha$ , ie.  $\langle V, \in \rangle \models a \in \bar{L}(\alpha)$ .

Let  $X$  be the set  $\bar{L}(\alpha)$  (ie.  $L_\alpha$ ). Then  $X \in L_{\alpha+1}$  by 6.2 (2). Hence  $X \in L$ . Since  $\langle V, \in \rangle \models a \in X$  we have  $\langle L, \in \rangle \models a \in X$ . Now  $\langle V, \in \rangle \models On(\alpha) \wedge X = \bar{L}(\alpha)$ . But the formula “ $x = \bar{L}(y)$ ” is  $\Delta_1^{ZF}$ , and  $On(\alpha)$  is  $\Delta_1^{ZF}$ , so by 7.10 (since  $\alpha, X \in L$ ),

$$\langle L, \in \rangle \models On(\alpha) \wedge X = \bar{L}(\alpha) \wedge a \in X.$$

Hence  $\langle L, \in \rangle \models \exists \alpha \exists x (On(\alpha) \wedge x = \bar{L}(\alpha) \wedge a \in x)$ , so  $\langle L, \in \rangle \models \exists \alpha (On(\alpha) \wedge a \in \bar{L}(\alpha))$ , as required.  $\square$

**Corollary 7.17** *If ZF is consistent, so is ZF+V=L.*

(Same argument as for Foundation.)

Later we'll show  $ZF + V=L \vdash AC, GCH$ .

## 8 Gödel numbering and the construction of *Def*

### 8.1

(Throughout, if we say “ $F : U_1 \times \dots \times U_n \rightarrow V$  is a  $\Delta_1^{ZF}$  term” we mean that the classes  $U_1, \dots, U_n$  are  $\Delta_1^{ZF}$  (ie. defined by  $\Delta_1^{ZF}$  formulas) and that “ $F(x_1, \dots, x_n) = y$ ” can be expressed by a  $\Sigma_1$  formula. By earlier sections this guarantees that the extension  $F' : V^n \rightarrow V$  of  $F$  defined by  $F'(x_1, \dots, x_n) = F(x_1, \dots, x_n)$  if  $x_1 \in U_1, \dots, x_n \in U_n$  and  $= \emptyset$  otherwise, is  $\Delta_1^{ZF}$  in the sense given.)

To give numbers to formulas we first define  $F : \omega^3 \rightarrow \omega$  by  $F(n, m, l) = 2^n 3^m 5^l$ . Then  $F$  is injective and easily seen to be  $\Delta_1^{ZF}$ . Write  $[n, m, l]$  for  $F(n, m, l)$ . We now define  $[\phi]$  by induction on  $\phi$ :

$$\begin{aligned} [v_i = v_j] &= [0, i, j]; \\ [v_i \in v_j] &= [1, i, j]; \\ [\phi \vee \psi] &= [2, [\phi], [\psi]]; \end{aligned}$$

$$\begin{aligned}
[\neg\phi] &= [3, [\phi], [\phi]]; \\
[\forall v_i\phi] &= [4, i, [\phi]].
\end{aligned}
\tag{5}$$

Of course this definition does not take place in ZF and is not actually used in the following definition of *Def*. However it should be borne in mind in order to see what's going on.

Now defined the class term  $Sub : V^4 \rightarrow V$  by  $Sub(a, f, i, c) = f(c/i)$  if  $f \in {}^{<\omega}a$ ,  $c \in a$  and  $i \in \omega$  and  $= \emptyset$  otherwise; where if  $f \in {}^{<\omega}a$ ,  $c \in a$  and  $i \in \omega$ ,  $f(c/i) \in {}^{<\omega}a$  is defined by  $\text{dom}(f(c/i)) = \text{dom } f$ , and for  $j \in \text{dom } f$ ,  $f(c/i)(j) = f(j)$  if  $j \neq i$ , and  $c$  if  $j = i$ .

It's easy to check that  $Sub$  is  $\Delta_1^{ZF}$ .

We now define a class term  $Sat : \omega \times V \rightarrow V$ . The idea is that if  $m \in \omega$  and  $m = [\phi(v_0, \dots, x_{n_1})]$ , for some formula  $\phi$  of LST, and  $a \in V$ , then

$$(*) \quad Sat(m, a) = \{f \in {}^{<\omega}a : \text{dom } f \geq n \wedge \langle a, \in \rangle \models \phi(f(0), \dots, f(n-1))\}.$$

We simply mimic the definition of satisfaction from predicate logic. (This definition uses a version of the recursion theorem which is slightly different from the usual one, see 8.2.)

**Definition 8.1** *Firstly if  $a \in V$ ,  $m \in \omega$  but  $m$  is not of the form  $[i, j, k]$ , for any  $i, j, k \in \omega$  with  $i < 5$ , then  $Sat(m, a) = \emptyset$ . Otherwise, if  $a \in V$  and  $m = [i, j, k]$  with  $i < 5$ , then*

$$\begin{aligned}
Sat([0, j, k], a) &= \{f \in {}^{<\omega}a : j, k \in \text{dom } f \wedge f(j) = f(k)\}. \\
Sat([1, j, k], a) &= \{f \in {}^{<\omega}a : j, k \in \text{dom } f \wedge f(j) \in f(k)\}. \\
Sat([2, j, k], a) &= Sat(j, a) \cup Sat(k, a). \\
Sat([3, j, k], a) &= ({}^{<\omega}a \setminus Sat(j, a)) \cap \{g \in {}^{<\omega}a : \exists f \in Sat(j, a), \text{dom } f \leq \text{dom } g\}. \\
Sat([4, j, k], a) &= \{f \in {}^{<\omega}a : j \in \text{dom } f \wedge \forall x \in a, Sub(a, f, j, x) \in Sat(k, a)\}.
\end{aligned}
\tag{6}$$

The generalized version of the recursion theorem (on  $\omega$ ) required here is:

**Lemma 8.2** *Suppose that  $\pi_1, \pi_2, \pi_3 : \omega \rightarrow \omega$  are  $\Delta_1^{ZF}$  class terms and  $H : V^4 \times \omega \rightarrow V$  is a  $\Delta_1^{ZF}$  class term. Suppose further that  $\forall n \in \omega \setminus \{0\} \pi_i(n) < n$  for  $i = 1, 2, 3$ . Then there is a  $\Delta_1^{ZF}$  class term  $F : \omega \times V \rightarrow V$  such that*

1.  $F(0, a) = 0$
2. and  $\forall n \in \omega \setminus \{0\}$

$$F(n, a) = H(F(\pi_1(n), (a)), F(\pi_2(n), (a)), F(\pi_3(n), (a)), a, n).$$

(Thus instead of defining  $F(n, a)$  in terms of  $F(n - 1, a)$ , we are defining  $F(n, a)$  in terms of three specified previous values.)

*Proof.* Similar to the proof of the usual recursion theorem on  $\omega$ .  $\square$

Thus the definition of *Sat* in 8.1 is an application of 8.2 with  $\pi_1(n) = i$  if for some  $j, k < n$ ,  $[i, j, k] = n$ ,  $= 0$  otherwise; and  $\pi_2$  and  $\pi_3$  are defined similarly, picking out  $j$  and  $k$  respectively from  $[i, j, k]$ , and with  $H : V^4 \times \omega \rightarrow V$  defined so that

$$H(x, y, z, a, n) = \begin{cases} \{f \in {}^{<\omega}a : \pi_2(n), \pi_3(n) \in \text{dom } f \wedge f(\pi_2(n)) = f(\pi_3(n))\} & \text{if } \pi_1(n) = 0, \\ \{f \in {}^{<\omega}a : \pi_2(n), \pi_3(n) \in \text{dom } f \wedge f(\pi_2(n)) \in f(\pi_3(n))\} & \text{if } \pi_1(n) = 1, \\ y \cup z & \text{if } \pi_1(n) = 2, \\ ({}^{<\omega}a \setminus y) \cap \{g \in {}^{<\omega}a : \exists f \in y \text{ dom } f \leq \text{dom } g\} & \text{if } \pi_2(n) = 3, \\ \{f \in {}^{<\omega}a : \pi_2(n) \in \text{dom } f \wedge \forall x \in a \text{Sub}(a, f, \pi_2(n), x) \in z\} & \text{if } \pi_1(n) = 4, \\ 0 & \text{otherwise.} \end{cases}$$

(The  $F$  got from this  $H, \pi_1, \pi_2, \pi_3$  (in 8.2) is *Sat*.)

It is completely routine to show that *Sat* so defined satisfies the required statement (\*) (just before 8.1)—by induction on  $\phi$ .

Before defining  $G$  we must introduce a term that picks out the largest  $n \in \omega$  such that “ $v_n$  occurs free” in the “formula coded by  $m$ ”. We denote this  $n$  by  $\theta(m)$ . We first define  $Fr(m)$  (“the set of  $i$  such that  $v_i$  occurs free in the formula coded by  $m$ ”) as follows (again using 8.2):

$$\begin{aligned} Fr([0, i, j]) &= \{i, j\}; \\ Fr([1, i, j]) &= \{i, j\}; \\ Fr([2, i, j]) &= Fr(i) \cup Fr(j); \\ Fr([3, i, j]) &= Fr(i); \\ Fr([4, i, j]) &= Fr(j) \setminus i; \\ Fr(x) &= \emptyset, \text{ if } x \text{ not of the above form.} \end{aligned}$$

(7)

Clearly one can prove in ZF that  $Fr(x)$  is a finite set of natural numbers for any set  $x$ , and we defined

$$\theta(x) = \max(Fr(x)).$$

$\theta$  is  $\Delta_1^{ZF}$ .

It is easy to show that if  $\phi$  is any formula of LST and  $m = \lceil \phi \rceil$ , then  $\theta(m)$  is the largest  $n$  such that  $v_n$  occurs as a free variable in  $\phi$ , and that if  $f \in Sat(m, a)$ , for any  $a \in V$ , then  $\text{dom } f \geq 1 + \theta(m)$  (ie.  $0, 1, \dots, \theta(m) \in \text{dom } f$ ). This is proved by induction on  $\phi$  and it is for this reason that we defined  $Sat([3, j, k], a)$  as we did (rather than just as  ${}^{<\omega}a \setminus Sat(j, a)$ ).

We can now define  $G$  by

$$G(m, a, s) = \begin{cases} \{b \in a : (s \cup \{\langle \theta(m), b \rangle\}) \in Sat(m, a)\} & \text{if } s \in {}^{<\omega}a \text{ and } \text{dom } s = \theta(m) (= \{0, \dots, \\ \emptyset & \text{otherwise.} \end{cases}$$

Then  $G$  is easily seen to be  $\Delta_1^{ZF}$  (since  $\theta, Sat$  are), and has the required properties mentioned at the beginning of subsection 6, because of (\*) (just before 8.1).

## 9 ZF + V=L ⊢ AC

### 9.1

We first construct a class term  $H : V \rightarrow V$  such that if  $\langle a, R \rangle \in V$  and  $R$  is a well-ordering of the set  $a$ , then  $H(\langle a, R \rangle) = \langle \omega \times {}^{<\omega}a, R' \rangle$ , where  $R'$  is a well-ordering of  $\omega \times {}^{<\omega}a$ .

[We don't need absoluteness, though it holds]

We define  $H(x) = y$  iff  $x$  is not of the form  $\langle a, R \rangle$ , where  $R$  well-orders  $a$ , and  $y = \emptyset$ , or  $x$  is of this form, and  $y$  is an ordered pair the first coordinate of which is  $\omega \times {}^{<\omega}a$  and the second coordinate is  $R'$ , where  $R' \subseteq (\omega \times {}^{<\omega}a)^2$ , and satisfies:  $\langle \langle n, s \rangle, \langle n', s' \rangle \rangle \in R'$  iff

1.  $n < n'$ , or
2.  $n = n'$ , and  $\text{dom } s < \text{dom } s'$ , or
3.  $n = n'$ , and  $\text{dom } s = \text{dom } s' = k$ , say, and  $\exists j < k$  such that  $\forall l < j$   $\langle s(l), s'(l) \rangle \in R$ .

(This is basically lexicographic order within chunks based on domain size.)

Then it is easy to show that ZF proves that  $H$  has the required property.

Now let  $G : \omega \times V \times V \rightarrow V$  be as at the beginning of subsection 6.

Define  $J : On \rightarrow V$  so that  $J(0) = 0$ , and  $J(\alpha + 1)$  is the unique binary relation  $S$  on  $L_{\alpha+1}$  such that for all  $x, y \in L_{\alpha+1}$ ,

1. If  $x \in L_\alpha$  and  $y \notin L_\alpha$ , then  $\langle x, y \rangle \in S$ ;
2. If  $x \in L_\alpha$  and  $y \in L_\alpha$ , then  $\langle x, y \rangle \in S$  iff  $\langle x, y \rangle \in J(\alpha)$ ;
3. If  $x, y \in L_{\alpha+1} \setminus L_\alpha$  and  $H(\langle L_\alpha, J(\alpha) \rangle) = \langle \omega \times {}^{<\omega}L_\alpha, R \rangle$ , and  $\langle m, s \rangle \in \omega \times {}^{<\omega}a$  is  $R$ -minimal such that  $G(m, s, L_\alpha) = x$ , and  $\langle m', s' \rangle \in \omega \times {}^{<\omega}a$  is  $R$ -minimal such that  $G(m', s', L_\alpha) = y$ , then  $\langle x, y \rangle \in S$  iff  $\langle \langle m, s \rangle, \langle m', s' \rangle \rangle \in R$ .

And  $J(\delta) = \bigcup_{\alpha < \delta} J(\alpha)$  if  $\delta$  is a limit.

Then, from this definition, we immediately have by induction on  $\alpha$ :

**Lemma 9.1** (ZF)  $\forall \alpha \in On$ ,  $J(\alpha)$  is a well-ordering of  $L_\alpha$ , and  $J(\alpha) \subseteq J(\alpha+1)$ , and  $L_{\alpha+1}$  is an initial segment of  $L_{\alpha+1}$  under the ordering  $J(\alpha+1)$ .

**Corollary 9.2** (ZF) The formula  $\Phi(x, y) := \exists \alpha (\alpha \in On \wedge \langle x, y \rangle \in J(\alpha))$  is a well-ordering of  $L$ . (ie.  $\Phi$  satisfies the axioms for a total ordering of  $L$ , and every  $a \in L$  has a  $\Phi$ -least element. In particular  $\forall a \in L, \{\langle x, y \rangle \in a^2 : \Phi(x, y)\}$  is a well-ordering of  $a$ .)

**Theorem 9.3**  $ZF + V=L \vdash$  every set can be well-ordered, so  $ZF + V=L \vdash AC$ .

*Proof.* Immediate from 9.2.  $\square$

## 10 Cardinal Arithmetic

### 10.1

Recall  $A \sim B$  means there is a bijection between  $A$  and  $B$ .

**Definition 10.1** An ordinal  $\alpha$  is called a cardinal if for no  $\beta < \alpha$  is  $\beta \sim \alpha$ .

Cardinals are usually denoted  $\kappa, \lambda, \mu$ . *Card* denotes the class of all cardinals. Now every well-ordered set is bijective with an ordinal (using an order-preserving bijection). (Provable in ZF.) Hence if we assume ZFC, as we do throughout this subsection, then every set is bijective with an ordinal.

**Definition 10.2** (ZFC) The class term  $\text{card} : V \rightarrow On$  is defined so that  $\text{card } x$  is the least ordinal  $\alpha$  such that  $\alpha \sim x$ .

**Lemma 10.3** (ZFC) (1) The range of  $\text{card}$  is precisely the class of cardinals.

(2) For all cardinals  $\kappa$  there is a cardinal  $\mu$  such that  $\mu > \kappa$ . ( $\kappa^+$  is the least such  $\mu$ .)

(3) If  $X$  is a set of cardinals with no greatest element then  $\sup X$  is a cardinal.

(4)  $\text{card } \kappa = \kappa$  for all cardinals  $\kappa$ .

*Proof.* (1) Exercise

(2) Consider  $\text{card } \mathbb{P}\kappa$  (though this result is provable in ZFC)

(3) Let  $\beta = \sup X$ . Suppose  $\exists \gamma < \beta (\gamma \sim \beta)$ . Choose  $\kappa \in X, \kappa > \gamma$ . Then  $\text{id}_\gamma$  is an injection from  $\gamma$  to  $\kappa$ . However  $\kappa \in X$ , so  $\kappa < \beta$ , so by

the Schröder-Bernstein Theorem  $\kappa \sim \gamma$ —contradicting the fact that  $\kappa$  is a cardinal.

(4) Exercise.  $\square$

(2) and (3) allow us to make the following

**Definition 10.4** (ZFC) *The class term  $\aleph : On \rightarrow Card$  is defined by (writing  $\aleph_\alpha$  for  $\aleph_\alpha$ )*

1.  $\aleph_0 = \omega$  (ie.  $\text{card } \mathbb{N}$ )
2.  $\aleph_{\alpha+1} = \aleph_\alpha^+$
3.  $\aleph_\delta = \bigcup_{\alpha < \delta} \aleph_\alpha$  for  $\delta$  a limit.

**Lemma 10.5**  $\{\aleph_\alpha : \alpha \in On\}$  is the class of all infinite cardinals (enumerated in increasing order). Thus  $\aleph_1$  is the smallest uncountable cardinal.

*Proof.* Exercise.  $\square$

**Definition 10.6** *Suppose  $\kappa, \lambda$  are cardinals.*

1.  $\kappa + \lambda = \text{card}(\kappa \times \{0\}) \cup (\lambda \times \{1\})$ .
2.  $\kappa \cdot \lambda = \text{card } \kappa \times \lambda$ .
3.  $\kappa^\lambda = \text{card } {}^\lambda \kappa$ .

**Theorem 10.7** *Suppose  $\kappa, \lambda, \mu$  are non-zero cardinals. Then*

1.  $\kappa^{\lambda+\mu} = \kappa^\lambda \cdot \kappa^\mu$ .
2.  $\kappa^{\lambda \cdot \mu} = (\kappa^\lambda)^\mu$ .
3.  $(\kappa \cdot \lambda)^\mu = \kappa^\mu \cdot \lambda^\mu$ .
4. (ZFC)  $2^\kappa > \kappa$ .
5. (ZFC) If  $\kappa$  or  $\lambda$  is infinite,  $\kappa + \lambda = \kappa \cdot \lambda = \max\{\kappa, \lambda\}$ .
6.  $+$ ,  $\cdot$  and  $\text{exp}$  are (weakly) order-preserving.

*Proof.* See the books.  $\square$

**Definition 10.8** *The Generalized Continuum Hypothesis (GCH) is the statement of LST: for all infinite cardinals  $\kappa$ ,  $2^\kappa = \kappa^+$  (ie.  $\forall \alpha \in \text{On}(2^{\aleph_\alpha} = \aleph_{\alpha+1})$ ).*

**Definition 10.9** *Suppose  $\beta > 0$  is an ordinal and  $\sigma = \langle \kappa_\alpha : \alpha < \beta \rangle$  is a  $\beta$ -sequence of cardinals (ie.  $\sigma$  is a function with domain  $\beta$  and  $\sigma(\alpha) = \kappa_\alpha$  for all  $\alpha < \beta$ ). Then we define*

1.  $\sum_{\alpha < \beta} \kappa_\alpha = \text{card} \bigcup_{\alpha < \beta} (\kappa_\alpha \times \{\alpha\})$
2.  $\prod_{\alpha < \beta} \kappa_\alpha = \text{card} \{f : f : \beta \rightarrow \bigcup_{\alpha < \beta} \kappa_\alpha, \forall \alpha < \beta (f(\alpha) \in \kappa_\alpha)\}$ .

**Lemma 10.10** *These definitions agree with the previous ones for  $\beta = 2$ . Further, if  $\kappa, \lambda$  are cardinals, then  $\kappa^\lambda = \prod_{\alpha < \lambda} \kappa$ .*

*Proof.* Easy exercise.  $\square$

**Lemma 10.11** (1) *Suppose  $\gamma, \delta$  are non-zero ordinals and  $\langle \kappa_{\alpha, \beta} : \alpha < \gamma, \beta < \delta \rangle$  is a sequence of cardinals (indexed by  $\gamma \times \delta$ ). Then*

$$\prod_{\alpha < \gamma} \sum_{\beta < \delta} \kappa_{\alpha, \beta} = \sum_{f \in {}^\gamma \delta} \prod_{\alpha < \gamma} \kappa_{\alpha, f(\alpha)}.$$

(ie.  $\prod$  distributes over  $\sum$ .)

(2) *Suppose  $\beta$  is a non-zero ordinal and  $\langle \kappa_\alpha : \alpha < \beta \rangle$  is a  $\beta$ -sequence of cardinals and  $\kappa$  is any cardinal. Then*

(a)  $\kappa \cdot \sum_{\alpha < \beta} \kappa_\alpha = \sum_{\alpha < \beta} (\kappa \cdot \kappa_\alpha)$ .

(b) *If  $\kappa_\alpha = \kappa$  for all  $\alpha < \beta$ , then  $\sum_{\alpha < \beta} \kappa_\alpha = \sum_{\alpha < \beta} \kappa = \text{card } \beta \cdot \kappa$ .*

(3)  $\sum, \prod$  are (weakly) order-preserving.

*Proof.* Exercises.  $\square$

**Theorem 10.12** (“The König Inequality”) *Suppose  $\kappa_\alpha < \lambda_\alpha$  for all  $\alpha < \beta$ . Then*

$$\sum_{\alpha < \beta} \kappa_\alpha < \prod_{\alpha < \beta} \lambda_\alpha.$$

*Proof.* Define  $f : \bigcup_{\alpha < \beta} (\kappa_\alpha \times \{\alpha\}) \rightarrow \prod_{\alpha < \beta} \lambda_\alpha$  by

$$f(\langle \eta, \alpha \rangle)(v) = \begin{cases} \eta & \text{if } v = \alpha \\ 0 & \text{if } v \neq \alpha \end{cases}$$

Clearly  $f$  is injective, so  $\sum_{\alpha < \beta} \kappa_\alpha \leq \prod_{\alpha < \beta} \lambda_\alpha$ .

Now suppose that  $h : \bigcup_{\alpha < \beta} (\kappa_\alpha \times \{\alpha\}) \rightarrow \prod_{\alpha < \beta} \lambda_\alpha$ . We show that  $h$  is not onto.

For  $\gamma < \beta$ , define  $h_\gamma : \bigcup_{\alpha < \beta} (\kappa_\alpha \times \{\alpha\}) \rightarrow \lambda_\gamma$  by

$$h_\gamma(\langle \eta, \alpha \rangle) = (h(\langle \eta, \alpha \rangle))(\gamma) \quad (*)$$

Since  $\kappa_\gamma < \lambda_\gamma$ ,  $h_\gamma \upharpoonright \kappa_\gamma \times \{\gamma\}$  cannot map onto  $\lambda_\gamma$  so there is an  $a_\gamma \in \lambda_\gamma \setminus h_\gamma[\kappa_\gamma \times \{\gamma\}]$  (\*\*).

Define  $g \in \prod_{\alpha < \beta} \lambda_\alpha$  by  $g(\gamma) = a_\gamma$  (for  $\gamma < \beta$ ).

Then  $g \notin \text{ran } h$ , since if  $h(\langle \gamma, \alpha \rangle) = g$ , then  $h(\langle \gamma, \alpha \rangle)(\gamma) = g(\gamma)$  for all  $\gamma < \beta$ , so  $h(\langle \gamma, \alpha \rangle)(\alpha) = g(\alpha) = a_\alpha$ , ie  $h_\alpha(\langle \gamma, \alpha \rangle) = a_\alpha$ , so  $a_\alpha \in h_\alpha[\kappa_\alpha \times \{\alpha\}]$ , contradicting (\*\*).  $\square$

**Definition 10.13** (1) Let  $\alpha$  be a limit ordinal and suppose  $S \subseteq \alpha$ . Then  $S$  is unbounded in  $\alpha$  if  $\forall \beta < \alpha \exists \gamma \in S (\gamma > \beta)$ .

(2) Let  $\kappa$  be a cardinal. Then  $cf(\kappa)$  is the least ordinal  $\alpha$  such that there exists a function  $f : \alpha \rightarrow \kappa$  such that  $\text{ran } f$  is unbounded in  $\kappa$ .

**Remark 10.14** Suppose  $cf(\kappa) = \alpha$  and  $\gamma < \alpha$ ,  $\gamma \sim \alpha$ . Say  $p : \gamma \rightarrow \alpha$  is a bijection. Let  $f : \alpha \rightarrow \kappa$  be such that  $\text{ran } f$  is unbounded in  $\kappa$ . Now clearly  $\text{ran } f = \text{ran}(fp)$ , so  $fp : \gamma \rightarrow \kappa$  is a function whose range is unbounded in  $\kappa$ . Since  $\gamma < \alpha$  this contradicts the definition of  $cf(\kappa)$ . Hence no such  $\gamma$  exists, ie.  $cf(\kappa)$  is always a cardinal. Clearly  $cf(\kappa) \leq \kappa$ .

**Definition 10.15** An infinite cardinal  $\kappa$  is called regular if  $cf(\kappa) = \kappa$ .

**Examples 10.16** (a)  $cf(\aleph_0) = \aleph_0$  (obvious).

(b)  $cf(\aleph_1) = \aleph_1$ , since if  $cf(\aleph_1) < \aleph_1$ , then  $cf(\aleph_1) = \aleph_0$ . Say  $f : \aleph_0 \rightarrow \aleph_1$  is unbounded. Then  $\aleph_1 = \bigcup_{n < \aleph_0} f(n)$ , and is a countable union of countable sets, and thus (in ZFC) countable, which is impossible.

(c)  $cf(\aleph_\omega) = \aleph_0$ .  $\geq$  is clear. Consider  $f : \aleph_0 \rightarrow \aleph_\omega$  defined so that  $f(n) = \aleph_n$ .

**Theorem 10.17** For any infinite cardinal  $\kappa$ ,  $cf(\kappa)$  is the least ordinal  $\beta$  such that there is a  $\beta$ -sequence  $\langle \kappa_\alpha : \alpha < \beta \rangle$  of cardinals such that

1.  $\kappa_\alpha < \kappa$  for all  $\alpha < \beta$ ,
2.  $\sum_{\alpha < \beta} \kappa_\alpha = \kappa$ .

*Proof.* Exercise.  $\square$

**Theorem 10.18** For any infinite cardinal  $\kappa$ ,

1.  $\kappa^+$  is regular,
2.  $cf(2^\kappa) > \kappa$ .

*Proof.* (1) Let  $\beta = cf(\kappa^+)$  and suppose  $\beta < \kappa^+$ . Then  $\beta \leq \kappa$ . By 10.17, there are  $\kappa_\alpha < \kappa^+$  (for  $\alpha < \beta$ ) such that  $\sum_{\alpha < \beta} \kappa_\alpha = \kappa^+$ . Then  $\kappa_\alpha \leq \kappa$  for all  $\alpha$ . But  $\sum_{\alpha < \beta} \kappa_\alpha \leq \sum_{\alpha < \beta} \kappa \leq \kappa \cdot \kappa = \kappa^2 = \kappa$ —a contradiction.

(2) Suppose  $\mu = cf(2^\kappa)$ , and  $\mu \leq \kappa$ . Choose  $\langle \kappa_\alpha : \alpha < \mu \rangle$  such that  $\kappa_\alpha < 2^\kappa$  for all  $\alpha < \mu$  and such that  $\sum_{\alpha < \mu} \kappa_\alpha = 2^\kappa$ .

By König,  $\sum_{\alpha < \mu} \kappa_\alpha < \prod_{\alpha < \mu} 2^\kappa$ , ie.  $2^\kappa < \prod_{\alpha < \mu} 2^\kappa$ .

But  $\prod_{\alpha < \mu} 2^\kappa = (2^\kappa)^\mu = 2^{\kappa \cdot \mu} = 2^\kappa$  (since  $\mu < \kappa$ ). This is a contradiction.

$\square$

**Examples 10.19**  $cf(2^{\aleph_0}) > \aleph_0$ ; and this is the only provable constraint on the value of  $2^{\aleph_0}$ . —So, for example,  $2^{\aleph_0} \neq \aleph_\omega$ .

**Theorem 10.20** Suppose  $\alpha$  is an infinite ordinal. Then  $\text{card } L_\alpha = \text{card } \alpha$ .

*Proof.* Induction on  $\alpha$ .

For  $\alpha = \omega$ ,  $L_\omega = \bigcup_{n \in \omega} L_n$ . Since each  $L_n$  is finite, and  $\omega \subseteq L_\omega$  (so  $L_\omega$  is not finite),  $\text{card } L_\omega = \aleph_0 = \text{card } \omega$ .

Suppose  $\text{card } L_\alpha = \text{card } \alpha$ .

Now  $L_{\alpha+1} = \{G(m, a, s) : m \in \omega, s \in {}^{<\omega}L_\alpha\}$ .

However, for  $x$  infinite,  $\text{card } {}^{<\omega}x = \text{card } x$ .

So  $\text{card } L_{\alpha+1} \leq \aleph_0 \cdot \text{card } {}^{<\omega}L_\alpha = \aleph_0 \cdot \text{card } L_\alpha = \aleph_0 \cdot \text{card } \alpha = \text{card } \alpha = \text{card } (\alpha + 1)$ .

Also  $L_\alpha \subseteq L_{\alpha+1}$ , so  $\text{card } L_{\alpha+1} \geq \text{card } L_\alpha = \text{card } \alpha = \text{card } (\alpha + 1)$ .

For  $\delta$  a limit,  $\text{card } L_\delta = \text{card } \bigcup_{\alpha < \delta} L_\alpha \leq \sum_{\alpha < \delta} \text{card } L_\alpha \leq \aleph_0 + \sum_{\omega \leq \alpha < \delta} \text{card } L_\alpha = \aleph_0 + \sum_{\omega \leq \alpha < \delta} \text{card } \alpha$  (IH)  $\leq \aleph_0 + \sum_{\omega \leq \alpha < \delta} \text{card } \delta = \aleph_0 + \text{card } \delta^2 = \text{card } \delta$  (since  $\delta$  is infinite).

—and other way round too:  $\delta \subseteq L_\delta$ , so that works.  $\square$

## 11 The Mostowski-Shepherdson Collapsing Lemma

**Lemma 11.1** *Suppose  $X$  is a set and  $M_1, M_2$  are transitive sets. Suppose  $\pi_i : X \rightarrow M_i$  are  $\in$ -isomorphisms (ie.  $\forall x, y \in X (x \in y \leftrightarrow \pi_i(x) \in \pi_i(y))$ ). Then  $\pi_1 = \pi_2$  (and hence  $M_1 = M_2$ ).*

*Proof.* Define  $\phi(x) \Leftrightarrow x \notin X \vee \pi_1(x) = \pi_2(x)$ .

We prove  $\forall x \phi(x)$  by  $\in$ -induction (see 3.6).

Suppose  $x$  is any set, and  $\phi(y)$  holds for all  $y \in x$ . If  $x \notin X$  we are done. Hence suppose  $x \in X$ , and  $\pi_1(x) \neq \pi_2(x)$ . Then there is  $z$  such that (say)  $z \in \pi_1(x)$  and  $z \notin \pi_2(x)$ . Since  $M_1$  is transitive and  $\pi_1(x) \in M_1$ , we have  $z \in M_1$ . Hence (since  $\pi_1$  is onto),  $\exists y \in X$  such that  $\pi_1(y) = z$ . Since  $\pi_1(y) \in \pi_1(x)$ , we have  $y \in x$ , and hence (by IH),  $z = \pi_1(y) = \pi_2(y)$  and  $\pi_2(y) \in \pi_2(x)$ . So  $z \in \pi_2(x)$ —a contradiction.

Thus  $\phi(x)$  holds, hence result by 3.6.  $\square$

**Theorem 11.2** *Suppose  $X$  is any set such that  $\langle X, \in \rangle \models$  Extensionality (ie. if  $a, b \in X$  and  $a \neq b$ , then  $\exists x \in X$  such that  $x \in a \wedge x \notin b$  or vice versa.) Then there is a unique transitive set  $M$  and a unique function  $\pi$  such that  $\pi$  is an  $\in$ -isomorphism from  $X$  to  $M$ .*

*Proof.* Uniqueness is by 11.1. For existence, we prove by induction on  $\alpha \in On$ , that  $\exists \pi_\alpha : X \cap V_\alpha \sim M_\alpha$  for some transitive set  $M_\alpha$ .

Note that  $\forall \alpha \in On$ ,  $\langle X \cap V_\alpha, \in \rangle \models$  Extensionality (since  $V_\alpha$  is transitive). Now suppose  $\pi_\alpha, M_\alpha$  exist for all  $\alpha < \beta$ . It's easy to show (by 11.1) that they are unique and  $\forall \alpha < \alpha' < \beta$   $M_\alpha \subseteq M_{\alpha'}$ , and  $\pi_\alpha = \pi_{\alpha'} \upharpoonright M_\alpha$ . Hence if  $\beta$  is a limit ordinal, then take  $M_\beta = \bigcup_{\alpha < \beta} M_\alpha$  and  $\pi_\beta = \bigcup_{\alpha < \beta} \pi_\alpha$ .

So suppose  $\beta = \gamma + 1$ . We have  $\pi_\gamma : X \cap V_\gamma \sim M_\gamma$ . For  $x \in X \cap V_{\gamma+1}$ , note that  $y \in x \cap X \rightarrow y \in X \cap V_\gamma$ , so we may define

$$\pi_{\gamma+1}(x) = \{\pi_\gamma(y) : y \in x \cap X\}.$$

Let  $M_{\gamma+1} = \pi_{\gamma+1}[X \cap V_{\gamma+1}]$ . So,  $\pi_{\gamma+1} : X \cap V_{\gamma+1} \rightarrow M_{\gamma+1}$  is surjective.

We now claim that  $\pi_{\gamma+1}$  is injective. Suppose  $a, b \in X \cap V_{\gamma+1}$ ,  $a \neq b$ . Since  $\langle X \cap V_{\gamma+1}, \in \rangle \models$  Extensionality,  $\exists c \in X \cap V_{\gamma+1}$  such that (say)  $c \in a \wedge c \notin b$ .

Then  $\pi_{\gamma+1}(a) = \{\pi_\gamma(y) : y \in a \cap X\} \ni \pi_\gamma(c)$ .

Suppose  $\pi_\gamma(c) \in \pi_{\gamma+1}(b)$ . Then  $\pi_\gamma(c) = \pi_\gamma(t)$  for some  $t \in b \cap X$ . Since  $c \notin b \cap X$ , we have  $c \neq t$ , so  $\pi_\gamma$  is not injective—contradiction.

Thus  $\pi_\gamma(c) \notin \pi_{\gamma+1}(b)$ , so  $\pi_{\gamma+1}(a) \neq \pi_{\gamma+1}(b)$  and so  $\pi_{\gamma+1}$  is injective.

We now show that if  $x \in X \cap V_\gamma (\subseteq X \cap V_{\gamma+1})$ , then  $\pi_\gamma(x) = \pi_{\gamma+1}(x)$  (\*)

For,  $y \in \pi_\gamma(x)$  implies  $y \in \pi_\gamma(x) \in M_\gamma$  implies  $y \in M_\gamma$  (since  $M_\gamma$  is transitive), say  $\pi_\gamma(t) = y$  ( $t \in X \cap V_\gamma$ ).

Then  $\pi_\gamma(t) \in \pi_\gamma(x)$ , so  $t \in x$ , hence  $t \in x \cap X$ .

Thus  $\pi_{\gamma+1}(x) = \{\pi_\gamma(z) : z \in x \cap X\} \ni \pi_\gamma(t) = y$ .

This shows  $\pi_\gamma(x) \subseteq \pi_{\gamma+1}(x)$ .

Conversely, suppose  $y \in \pi_{\gamma+1}(x)$ . Then  $y = \pi_\gamma(t)$  for some  $t \in x \cap X$ . Since  $t \in x \in X \cap V_\gamma$ , we have  $\pi_\gamma(t) \in \pi_\gamma(x)$  (since  $\pi_\gamma$  is an  $\in$ -isomorphism). I.e.  $y \in \pi_\gamma(x)$ . So  $\pi_{\gamma+1}(x) \subseteq \pi_\gamma(x)$ , and we have (\*).

Now suppose  $a, b \in X \cap V_{\gamma+1}$ , and  $a \in b$  (so  $a \in X \cap V_\gamma$ ).

Then  $\pi_{\gamma+1}(b) = \{\pi_\gamma(y) : y \in b \cap X\}$ . But  $a \in b \cap X$ , so  $\pi_\gamma(a) \in \pi_{\gamma+1}(b)$ . Hence by (\*)  $\pi_{\gamma+1}(a) \in \pi_{\gamma+1}(b)$ .

Finally,  $M_{\gamma+1}$  is transitive, since if  $a \in b \in M_{\gamma+1}$ , then  $b = \pi_{\gamma+1}(x)$  for some  $x \in X \cap V_{\gamma+1}$ , and hence  $a = \pi_\gamma(y)$  for some  $y \in x \cap X$ . Since  $y \in X \cap V_\gamma$ , we have, by (\*),  $\pi_\gamma(y) = \pi_{\gamma+1}(y)$ , so  $a \in \text{ran } \pi_{\gamma+1} = M_{\gamma+1}$ , as required.  $\square$

## 12 The Condensation Lemma and GCH

**Theorem 12.1** (*The Condensation Lemma*) *Let  $\alpha$  be a limit ordinal and suppose  $X \preceq L_\alpha$  (ie.  $\forall a_1, \dots, a_n \in X$ , and formulas  $\phi(v_1, \dots, v_n)$  of LST,  $\langle X, \in \rangle \models \phi(a_1, \dots, a_n)$  iff  $\langle L_\alpha, \in \rangle \models \phi(a_1, \dots, a_n)$ , although we only need this when  $\phi$  is a  $\Sigma_1$  formula). Then there is unique  $\pi$  and  $\beta$  such that  $\beta \leq \alpha$  and  $\pi : X \sim L_\beta$  is an  $\in$ -isomorphism. Further if  $Y \subseteq X$  and  $Y$  is transitive, then  $\pi(y) = y$  for all  $y \in Y$ .*

We prove this in stages.

**Lemma 12.2**  $\forall m \in \omega, L_m \subseteq X$ .

*Proof.* Clear for  $m = 0$ . Suppose  $L_m \subseteq X$  and let  $a \in L_{m+1}$ , so  $a = \{a_1, \dots, a_n\} \subseteq L_m$ . Then  $L_\alpha \models \exists x((a_1 \in x \wedge \dots \wedge a_n \in x) \wedge \forall y \in x(y = a_1 \vee \dots \vee y = a_n))$ . Hence  $X \models \exists x((a_1 \in x \wedge \dots \wedge a_n \in x) \wedge \forall y \in x(y = a_1 \vee \dots \vee y = a_n))$ . Clearly such an  $x$  must be  $a$ , so  $a \in X$ . Hence  $L_{m+1} \subseteq X$ . Hence the result follows by induction.  $\square$

**Lemma 12.3**  $X \models \text{Extensionality}$ .

*Proof.* For suppose  $a, b \in X$  and  $a \neq b$ . Then  $\exists c, c \in a \wedge c \notin b$  (say), and  $c \in L_\alpha$  since  $L_\alpha$  is transitive. Thus  $L_\alpha \models \exists x(x \in a \wedge x \notin b)$ , so  $X \models \exists x(x \in a \wedge x \notin b)$ , as required.  $\square$

By 11.2 there is transitive  $M$  and  $\pi : X \sim M$ . Now since  $M$  is transitive,  $M \cap \text{On}$  is a transitive set of ordinals so is an ordinal,  $\beta$ , say. Then  $\beta \leq \alpha$  (exercise—suppose  $\beta > \alpha$ , so  $\pi^{-1}(\alpha) \in X$ . Show  $\pi^{-1}(\alpha) = \alpha$  to get contradiction). We show  $M = L_\beta$ .

*An admission!* For this proof we need the fact that most of the formulas that we have proven  $\Delta_1^{ZF}$  are in fact absolute between transitive classes satisfying much weaker axioms than ZF—in fact BS—basic Set Theory (see Devlin). BS is such that  $L_\alpha \models \text{BS}$  for any limit ordinal  $\alpha > \omega$ . In particular, the formula  $\text{On}(x)$ , and  $\Phi(x, y) := \text{On}(x) \wedge y = L_x$ , is  $\Delta_1^{ZF}$  and hence absolute between  $V$  and  $L_\alpha$  and between  $V$  and  $M$ . (Since  $M$  is transitive.) As an application, suppose  $\beta = \gamma \cup \{\gamma\}$ . Since  $\beta \notin M$ , and  $\gamma \in M$ , and  $M \models \text{On}(\gamma)$  (since  $\text{On}(\gamma)$  really is  $\Sigma_0$  and  $M$  is transitive), we have  $M \models \exists x(\text{On}(x) \wedge \forall y y \neq x \cup \{x\})$ . Now  $X \sim M$ , so  $X \models \exists x(\text{On}(x) \wedge \forall y y \neq x \cup \{x\})$ , hence  $L_\alpha \models \exists x(\text{On}(x) \wedge \forall y y \neq x \cup \{x\})$ , which is a contradiction, since  $\alpha$  is a limit ordinal. Hence, we have shown:

**Lemma 12.4**  $\beta$  is a limit ordinal.

**Lemma 12.5**  $L_\beta \subseteq M$ .

*Proof.* Since  $\beta$  is a limit,  $L_\beta = \bigcup_{\gamma < \beta} L_\gamma$ , so fix  $\gamma < \beta$ . Sufficient to show  $L_\gamma \subseteq M$ .

Now for any  $\eta < \alpha$ ,  $L_\eta \in L_\alpha$ . Since  $L_\alpha \cap \text{On} = \alpha$ , we have  $L_\alpha \models \underbrace{\forall x(\text{On}(x) \rightarrow \exists y \Phi(x, y))}_\sigma$ .

Hence  $X \models \sigma$ , since  $X \preceq L_\alpha$ , so  $M \models \sigma$ , since  $X \sim M$ .

Since  $\forall x \in M, M \models \text{On}(u) \Leftrightarrow u \in \text{On} \wedge u < \beta$ , we have in particular  $M \models \exists y \Phi(\gamma, y)$ —say  $a \in M$  and  $M \models \Phi(\gamma, a)$ . By absoluteness  $a = L_\gamma$ , so  $L_\gamma \in M$ , so  $L_\gamma \subseteq M$  since  $M$  is transitive.  $\square$

**Lemma 12.6**  $M \subseteq L_\beta$ .

*Proof.* Since  $L_\alpha = \bigcup_{\gamma < \alpha} L_\gamma$ , we have  $L_\alpha \models \underbrace{\forall x \exists y \exists z (\text{On}(y) \wedge \Phi(y, z) \wedge x \in z)}_\tau$ .

Hence  $X \models \tau$  (since  $X \preceq L_\alpha$ ), hence  $M \models \tau$  (since  $X \sim M$ ).  
Let  $a \in M$ . Then for some  $c, d \in M$ ,

$$M \models \text{On}(c) \wedge \Phi(c, d) \wedge a \in d.$$

By absoluteness,  $c \in \text{On}$ , and hence  $c < \beta$ , and  $d = L_c$  and  $a \in L_c$ . Hence  $a \in \bigcup_{\gamma < \beta} L_\gamma = L_\beta$ , as required.  $\square$

**Lemma 12.7** *Suppose  $Y \subseteq X$ ,  $Y$  transitive. Then  $\forall y \in Y \pi(y) = y$ .*

*Proof.* It's easy to show  $\pi[Y]$  is transitive and  $\pi : Y \sim \pi[Y]$ . However,  $\text{id} \upharpoonright Y \sim Y$ . Hence by 11.1,  $\pi = \text{id} \upharpoonright Y$ .  $\square$

We have now completed the proof of 12.1.

**Lemma 12.8** (ZFC) *Let  $A$  be any set and  $Y \subseteq A$ . Then there is a set  $X$  such that  $Y \subseteq X \subseteq A$  and  $\langle X, \in \rangle \preceq \langle A, \in \rangle$ , and  $\text{card } X = \max(\aleph_0, \text{card } X)$ .*

*Proof.* This is the downward Löwenheim-Skolem Theorem.  $\square$

**Theorem 12.9** (ZF+V=L) *Let  $\kappa$  be a cardinal, and suppose  $x$  is a bounded subset of  $\kappa$ . Then  $x \in L_\kappa$ .*

*Proof.* Clear if  $\kappa \leq \omega$ , so assume  $\kappa > \omega$ . Now  $x \subseteq \alpha$  for some  $\omega \leq \alpha < \kappa$ , so  $x \subseteq L_\alpha$ . Then  $L_\alpha \cup \{x\}$  is transitive.

Using V=L, let  $\lambda$  be a limit ordinal such that  $\lambda \geq \kappa$ , and  $L_\alpha \cup \{x\} \subseteq L_\lambda$ . By 12.8, with  $A = L_\lambda$  and  $Y = L_\alpha \cup \{x\}$ , let  $X$  be such that  $L_\alpha \cup \{x\} \subseteq X$  and  $X \preceq L_\lambda$ , with  $\text{card } X \leq \text{card } L_\alpha \cup \{x\} = \text{card } \alpha$ . Let  $\pi : X \sim L_\beta$  be as in 12.1. Then  $\text{card } \beta = \text{card } L_\beta = \text{card } X \leq \text{card } \alpha < \kappa$ , so  $\beta < \kappa$ . But  $L_\alpha \cup \{x\}$  is transitive so, in particular,  $\pi(x) = x$ , so  $x \in L_\beta \subseteq L_\kappa$ , as required.  $\square$

**Corollary 12.10** *ZF+V=L  $\vdash$  GCH. Hence if ZF is consistent, so is ZFC+GCH.*

*Proof.* By 12.9. ZF+V=L  $\vdash$  'for all infinite  $\kappa$ ,  $\mathbb{P}\kappa \subseteq L_{\kappa^+}$ '. But ZF  $\vdash$  'for all infinite  $\kappa$ ,  $\text{card } L_{\kappa^+} = \kappa^+$ ', hence ZF+V=L  $\vdash$  'for all infinite  $\kappa$ ,  $\text{card } \mathbb{P}\kappa \leq \kappa^+$ '. So  $2^\kappa \leq \kappa^+$ , and  $\geq$  is obvious.  $\square$