Bi-coloured fields on the complex numbers

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August, 2003 *

Abstract

We consider two theories of "bad fields" constructed by B.Poizat using Hrushovski's amalgamation and show that these theories have natural models representable as the field of complex numbers with a distinguished subset given as a union of countably many real analytic curves. One of the two examples is based on the complex exponentiation and the proof assumes Schanuel's conjecture.

1 Introduction

Following the programme in [Z1] we attempt to find an 'analytic' interpretation of two-coloured fields of Morley rank $\omega \times 2$ with definable multiplicative subgroup [P2] and the similar example with a generic subset from [P1], which have been constructed by B.Poizat using Hrushovski's method (see also Baldwin and Holland [BH1] and [BH2]). As is usual for Hrushovski's construction, Poizat produces two sets of axioms: the first axiom scheme states the *generalized Schanuel property* (in the terminology of [Z1]), the other one is the statement that the structure is existentially closed (in the category of strong embeddings). These axioms along with the other, obvious axioms of the structure determine the complete theory of the bicoloured field. We introduce here two real analytic structures $(\mathbb{C}, +, \times, P)$ and $(\mathbb{C}, +, \times, G)$, with P and G countable unions of Jordan curves, dense in $\mathbb{C}$, where P is a 'generic' subset of $\mathbb{C}$ and G a multiplicative subgroup. We construct P, the set of 'black' points, using the Liouville

*with later improvements

function introduced by Wilkie [W] and also studied by Koiran [K]. We prove that the structure satisfies Poizat's axioms. Interestingly, we use Wilkie's theorem on the generalized Schanuel condition for the Liouville function but don't need Koiran's theorem which states that the corresponding structure is existentially closed.

For G, the subgroup of 'green' points, we prove the generalized Schanuel condition assuming that the standard Schanuel conjecture is true. By a method very similar to the one for 'black' points we also prove the existential closedness of the second structure, thus showing that the structures are indeed the same as Poizat's.

In the notes preceding the present paper I had 'smaller' versions of P and G both being *connected* Jordan curves and was able to show the Schanuel conditions for these versions. But Dave Marker pointed out that the second half of the axioms cannot be true with this construction, since a necessary condition for a predicate $R \subseteq \mathbb{C}$ containing a Jordan curve to give rise to an ω-stable structure is that R is dense in $\mathbb{C}$. Our present predicates satisfy this condition.

A few comments about the analytic component of the proof are in order. The reader will see that the existential closedness theorem states the solvability of certain systems of analytic equations definable in the structure. For the generic choice of coefficients such a system is solvable for trivial reasons. So, it is enough to find a method to analytically continue the solution from a generic point to a large enough neighborhood. One can do it if the corresponding Jacobian matrix is nondegenerate. Now, crucially, as a consequence of the generalized Schanuel condition, we prove that the set of points where the Jacobian degenerates is very sparse and hence the continuation is possible. Notice similarly the Schanuel conjecture is used in the proof of the solvability of certain systems of exponential equations in [Z4] and indeed for the exponentiation in the reals in [MW]. This presents a good case to treat the general existential closedness condition for analytic and pseudo-analytic structures (see [Z1]) as a dual to the generalized Schanuel condition.

I would also like to mention that the two structures discussed in the paper are very interesting in the real model theory context as well. As a matter of fact, our G contains a connected subgroup G^0 and is a countable union of cosets of G^0, which are Jordan curves. Indeed G^0 can be identified as the 'spiral'

$$\{e^t(\cos t + i \sin t) : t \in \mathbb{R}\}$$

studied by C.Miller and P.Speissegger [M] as a subset of $\mathbb{R}^2$. The theorem of C. Miller and P. Speissegger states that the real field with the spiral is d-minimal; that is any definable subset of $\mathbb{R}$ is a union of an open set and finitely many discrete sets.

This link between real and complex versions of stability seems not to be incidental at all. Another, simpler example is provided by the predicate for the complex roots of unity in the complex field, with or without the reals. In the second case the structure is ω-stable (see [Z3]) while in the first case by a similar proof it has a nice elimination of quantifiers and properties similar to that of the Miller-Speissegger spiral (see [Z5]).

Finally, I would like to mention an important problem regarding other instances of "bad" fields.

Problem. Let T_P^n and T_G^n denote the theories of bicoloured fields, with black and green points, of Morley rank $\omega \times n$, $n > 2$, constructed in [P1] and [P2]. Construct explicit predicates P and G, of an 'analytic' nature, on the complex field $\mathbb{C}$, so that

$$(\mathbb{C}, +, \times, P) \vDash T_P^n \text{ and } (\mathbb{C}, +, \times, G) \vDash T_G^n.$$

Notice that if such examples are found, the real dimensions of P and G must be n times less than the real dimension of $\mathbb{C}$, that is equal to $\frac{2}{n}$. Thus the theory of subsets of $\mathbb{R}^m$ of fractional dimension needs to be involved in the construction.

2 Bi-coloured field with "black points". The generalized Schanuel property

A function

$$f : \mathbb{C} \to \mathbb{C}$$

given by

$$f(x) = \Sigma_{i\geq 0} c_i x_i$$

is called a **Liouville function** if $c_i^{-1} = a_i$ are integers satisfying

$$\text{for every } \ell \geq 1, \text{ for all sufficiently large } i, \quad |a_{i+1}| > |a_i|^{i^\ell}$$

Liouville functions were introduced by A.Wilkie [W] in regard to Problem 2 of [Z1] and the following **Schanuel condition** has been proved.

Fact 1 [W]. For any distinct nonzero complex numbers $z_1, \dots, z_n$

$$\text{tr.d.}(z_1, \dots, z_n, f(z_1), \dots, f(z_n)) \geq n,$$

where tr.d.(X) stands for the transcendence degree of X.

We fix a Liouville function f. Notice that since the coefficients in the definition of f are real, $f(x^c) = f(x)^c$, where y^c denotes here and below the complex conjugate.

Let $\epsilon = 1 + i$ and

$$P = \{f(\epsilon t + q) : \ t \in \mathbb{R}, q \in \mathbb{Q}, \ (t, q) \neq (0, 0)\}$$

considered as a unary predicate on $\mathbb{C}$.

Proposition 1 *For any distinct $p_1, \dots, p_n \in P$,*

$$2\text{tr.d.}(p_1, \dots, p_n) \geq n.$$

That is two times the transcendence degree is not less than the number of points.

Proof Let

$$T = \{\epsilon t_1 + q_1, \dots, \epsilon t_n + q_n\},$$

with $0 \notin T$, $\epsilon t_i + q_i \neq \epsilon t_j + q_j$, if $i \neq j$, $t_i \in \mathbb{R}$, $q_i \in \mathbb{Q}$, and let $f(\epsilon t_i + q_i) = p_i$. We want to prove that

$$2\text{tr.d.}(f(\epsilon t_1 + q_1), \dots, f(\epsilon t_n + q_n)) \geq n.$$

Now consider the system

$$T^* = T \cup T^c = \{\epsilon t_1 + q_1, \dots, \epsilon t_n + q_n, \epsilon^c t_1 + q_1, \dots, \epsilon^c t_n + q_n\}, \quad (1)$$

where $\epsilon^c = 1 - i$. We want to calculate the number of points in T^*.

The only form of equality that may happen between elements of T^* is

$$\epsilon t_i + q_i = \epsilon^c t_j + q_j, \text{ for some } i, j \leq n. \quad (2)$$

Since the sum $\epsilon\mathbb{R}+\mathbb{R}$ is direct the equality is equivalent to

$$t_i = -t_j \ \ \& \ \ t_i = \frac{1}{2}(q_j - q_i)$$

which implies that for each i there is at most one j such that (2) holds and that in this case all the elements involved are algebraic. Thus, if $k \leq n$ is the number of equalities of the form (2), then

$$|T^*| = 2n - k, \text{ and there is a subset } S \subseteq T \cap \mathrm{acl}\mathbb{Q} \text{ of size at least } k.$$

Thus,

$$\mathrm{tr.d.}T^* = \mathrm{tr.d.}T \leq n - k.$$

It follows, that

$$|T^*| - \mathrm{tr.d.}T^* \geq n.$$

By properties of Liouville function f

$$\mathrm{tr.d.}(T^* \cup f(T^*)) \geq |T^*|.$$

Hence

$$\mathrm{tr.d.}f(T^*) \geq |T^*| - \mathrm{tr.d.}T^* \geq n.$$

Recalling that $f(\epsilon t_i + q_i) = p_i$ and $f(\epsilon^c t_i + q_i) = p_i^c$, the latter can be rewritten as

$$\mathrm{tr.d.}(p_1, \dots, p_n, p_1^c, \dots, p_n^c) \geq n.$$

Since complex conjugation is a field isomorphism we have by symmetry

$$2\mathrm{tr.d.}(p_1, \dots, p_n) \geq \mathrm{tr.d.}(p_1, \dots, p_n, p_1^c, \dots, p_n^c) \geq n.$$

□

3 Existential Closedness Theorem for "black points"

In order to prove that $(\mathbb{C}, +, \times, P)$ is a model of Poizat's theory it remains to prove the following

Existential Closedness Theorem *Let $V \subseteq \mathbb{C}^k$ be an irreducible algebraic variety defined over a finitely generated subfield $\mathbb{Q}(C)$ which has a point $\langle a_1, \ldots, a_k \rangle \in V$ satisfying:*

$$a_i \neq a_j \quad \textit{and} \quad a_i \notin \mathrm{acl}(C), \ \textit{for all } i \neq j, i \leq k \qquad \textbf{(}P\textbf{-freeness)}$$

and

$$\mathrm{tr.d.}(a_{i_1}, \ldots, a_{i_m}) \geq \frac{m}{2}, \ \textit{for any } i_1 < \cdots < i_m \leq k \qquad \textbf{(}P\textbf{-normality)}.$$

Then there is $\langle a_1, \ldots, a_k \rangle \in V \cap P^k$ generic in V over C.

The rest of the section is devoted to the proof of the theorem.

We assume that V satisfies the assumptions of the theorem. We may assume that k is even, for if $k = 2n - 1$ we can replace V by $V \times \mathbb{C} \subseteq \mathbb{C}^{2n}$ which is both P-free and P-normal. We also assume $\dim V = n$ by the following

Lemma 3.1 *Let $V \subseteq \mathbb{C}^{2n}$ be a P-free and P-normal irreducible variety over some C_0 such that $\dim V = d > n$. Then, for any $c_1, \ldots, c_{2n} \in \mathbb{C}$ algebraically independent over C_0, an irreducible component V' of the variety $V \cap H_c$, for $H_c = \{x \in \mathbb{C}^{2n} : c_1 \cdot x_1 + \cdots + c_{2n} \cdot x_{2n} = 1\}$, is P-normal and P-free, and $\dim V' = d - 1$.*

Proof V' is defined over some $\mathrm{acl}(C_0 \cup \{c_1, \ldots, c_{2n}\})$. $\dim V' = d - 1$, as $V \nsubseteq H_c$. Let $c = \langle c_1, \ldots, c_{2n} \rangle$ and let $a = \langle a_1, \ldots, a_{2n} \rangle$ be a generic, over $C_0 c$, point in V'. Then a is generic in V over C_0 and $\mathrm{tr.d.}(c/C_0 a) = 2n - 1$, by elementary calculation.

Claim. Let $b = \langle b_1, \ldots, b_m \rangle \in \mathbb{C}^m$ with $b_i \in \mathrm{acl}(C_0 a)$, all $i \leq m$. Then

$\mathrm{tr.d.}(b/C_0) > \mathrm{tr.d.}(b/C_0 c)$ implies $\mathrm{tr.d.}(b/C_0) = \mathrm{tr.d.}(a/C_0) = d$.

Proof. Let $U \subseteq \mathbb{C}^{2n}$ be the minimal algebraic variety over $\mathrm{acl}(C_0 b)$ containing c coordinatewise. Suppose the inequality holds. Then $\mathrm{tr.d.}(c/C_0 b) < \mathrm{tr.d.}(c/C_0)$ and so $\dim U < 2n$. U must contain the hyperplane

$$H_a = \{x \in \mathbb{C}^{2n} : a_1 \cdot x_1 + \cdots + a_{2n} \cdot x_{2n} = 1\},$$

for otherwise $\dim U \cap H_a < \dim U$ contradicting $\mathrm{tr.d.}(c/C_0 a) = 2n - 1$. Hence H_a is a component of U and so $U = H_a$.

Notice also that H_a must contain a point b' with coordinates in $F_b = \mathrm{acl}(C_0 b)$, a field of definition of H_a. This follows from the elimination of quantifiers in the theory of algebraically closed fields.

But then a is the unique vector satisfying $(b', a) = 1$ and the equations $(z - b', a) = 0$ for all $z \in H_a$. Thus $a \in \mathrm{acl}(C_0 b)$ coordinatewise. This proves the claim.

Now we can prove that V' is P-free and P-normal.

Freeness. If $a_i \in \mathrm{acl}(C_0 c)$ then $\mathrm{tr.d.}(a_i/C_0 c) < \mathrm{tr.d.}(a_i/C_0) = 1$, since V is free. So, by the Claim, with $m = 1$, we will have $\mathrm{tr.d.}(a_i/C_0) = d$, which contradicts the condition $d > 1$.

Normality. Assuming, towards a contradiction, that $\mathrm{tr.d.}(a_{i_1}, \dots, a_{i_m}/C_0 c) < m/2$, we get by normality of V, $\mathrm{tr.d.}(a_{i_1}, \dots, a_{i_m}/C_0) > \mathrm{tr.d.}(a_{i_1}, \dots, a_{i_m}/C_0 c)$. Hence, by the claim, $\mathrm{tr.d.}(a_{i_1}, \dots, a_{i_m}/C_0 c) = \mathrm{tr.d.}(a_1, \dots, a_{2n}/C_0 c) = d - 1 \geq n \geq m/2$, the contradiction. □

Let $N = 2n$ and $\bar{V} \subseteq \mathbb{C}^N$ be an irreducible variety over some $\bar{C}$ such that, given generic $\langle y_1, \dots y_N \rangle \in V$ and a generic $\langle \bar{y}_1, \dots \bar{y}_N \rangle \in \bar{V}$, for any selection of indices $1 \leq i_1 < \dots i_m \leq N$,

$$\mathrm{tr.d.}_C(y_{i_1}, \dots y_{i_m}) = \mathrm{tr.d.}_{\bar{C}}(\bar{y}_{i_1}, \dots \bar{y}_{i_m}).$$

We call such a $\bar{V}$ **a formal conjugate** of V.

An example of a formal conjugate $\bar{V}$ would be V^c, the image of V under complex conjugation.

Let $W_s \subseteq \mathbb{C}^{5N}$ be the algebraic variety in variables $x_1, \dots x_N, \bar{x}_1, \dots \bar{x}_N$, $y_1, \dots y_N, \bar{y}_1, \dots \bar{y}_N, t_1, \dots t_N$ and parameters C and $s_1, \dots s_N \in \mathbb{C}$ given by the system of algebraic equations (3) and (4):

$$y = \langle y_1, \dots y_N \rangle \in V, \quad \bar{y} = \langle \bar{y}_1, \dots \bar{y}_N \rangle \in \bar{V}, \tag{3}$$

$$x_i = \epsilon t_i + s_i, \quad \bar{x}_i = \epsilon^c t_i + s_i, \ i = 1, \dots, N. \tag{4}$$

We are going to consider W_s in conjunction with

$$y_i = f(x_i), \quad \bar{y}_i = f(\bar{x}_i), \quad i = 1, \dots, N. \tag{5}$$

We say that an irreducible variety $Z \subseteq \mathbb{C}^{2m}$ over C in variables $x_1, \dots x_m, y_1, \dots y_m$ is f**-free** if there is a point $\langle a_1, \dots a_m, b_1, \dots b_m \rangle \in Z$ such that $a_i \neq a_j$ for $i \neq j$ and $a_i \notin \mathrm{acl}(C)$ for any $i \leq m$.

We say that Z is also f**-normal** if for the point above, for any $1 \leq i_1 < \cdots < i_k \leq m$,

$$\text{tr.d.}_C(a_{i_1}, \ldots, a_{i_k}, b_{i_1}, \ldots, b_{i_k}) - k \geq 0. \tag{6}$$

Notice that any solution

$$\langle t_1, \ldots, t_N, x_1, \ldots, x_N, \bar{x}_1, \ldots, \bar{x}_N, y_1, \ldots, y_N, \bar{y}_1, \ldots, \bar{y}_N \rangle$$

of (3)-(5) is uniquely and explicitly determined by the choice of $t_1, \ldots, t_N$. We thus will often refer to $t = \langle t_1, \ldots, t_N \rangle$ as a solution of (3)-(5). We call a solution t **semi-generic** if in the full solution corresponding to t, $\langle y_1, \ldots y_N \rangle$ is generic in V over C and $\langle \bar{y}_1, \ldots \bar{y}_N \rangle$ generic in $\bar{V}$ over $\bar{C}$.

Lemma 3.2 *Given $s_1, \ldots, s_N \in \mathbb{C}$, let for some U and any $u \in U$*
$t_i(u) \in \mathbb{C}$, $x_i(u) = \epsilon t_i(u) + s_i$, $\bar{x}_i(u) = \epsilon^c t_i(u) + s_i$, for $i = 1, \ldots, N$,
$x(u) = \langle x_1(u), \ldots x_N(u) \rangle$, $\bar{x}(u) = \langle \bar{x}_1(u), \ldots \bar{x}_N(u) \rangle$.
Assume that $f(x(u)) \in V$, $f(\bar{x}(u)) \in \bar{V}$,
$t_i(u) \neq t_j(u')$ if $i \neq j$ or $u \neq u'$, for any $u, u' \in U$ and $i, j \leq N$, and suppose also that $t_1(u) \in \text{acl}(\mathbb{Q})$, for all $u \in U$.
Then U is at most finite.

Proof Assume towards a contradiction that U is infinite, say $U = \mathbb{N}$.

We also may assume that $\epsilon t_i(u) + s_i \neq \epsilon^c t_j(u') + s_j$ for all $u \neq u'$ and $i, j \leq N$, since, given u, i and j, such an equality happens only for at most finitely many values of u'.

Recall that Hrushovski predimension for structure $(\mathbb{C}, +, \times, f)$ is

$$\delta(X) = \text{tr.d.}(X, f(X)) - |X|, \quad \delta(X/C) = \delta(X \cup C) - \delta(C).$$

We can extend C and assume $s_1, \ldots, s_N \in C$, $\bar{C} = C$. Then

$$\text{tr.d.}(f(x(u))/C) \leq n, \quad \text{tr.d.}(f(\bar{x}(u))/C) \leq n$$

and

$$\text{acl}(x(u), \bar{x}(u)) \subseteq \text{acl}(t_2(u), \ldots, t_N(u), C),$$

hence

$$\delta(x(u), \bar{x}(u)/C) \leq N - 1 + 2n - 2N < 0$$

for all u. Moreover, by the same argument,

$$\delta(x(u+1), \bar{x}(u+1)/C \cup \{x(1), \bar{x}(1), \ldots, x(u), \bar{x}(u)\}) < 0.$$

Hence inductively

$$\delta(C \cup \{x(1), \bar{x}(1), \dots, x(u), \bar{x}(u)\}) \leq \delta(C) - u.$$

For large enough u this contradicts the Schanuel property of the Liouville function f.$\square$

Corollary 1 *There is no continuos function $u \mapsto t(u)$ from an open subset of $\mathbb{C}^p$, $p > 0$, non-constant in all coordinates $t_i(u)$, such that*

$$\langle f(\epsilon t_1(u) + s_1), \dots, f(\epsilon t_N(u) + s_N) \rangle \in V \setminus \bigcup_{i \neq j} \{y_i = y_j\}$$

and

$$\langle f(\epsilon^c t_1(u) + s_1), \dots, f(\epsilon^c t_N(u) + s_N) \rangle \in \bar{V} \setminus \bigcup_{i \neq j} \{y_i = y_j\}.$$

Proof Otherwise, for every finite $A \subseteq \mathbb{C}$,

$$\bigcap_{a \in A,\ i \leq N} \{u \in U : t_i(u) \neq a\}$$

is open in U. Thus we can have infinitely many instances of u with $t_1(u) \in \mathrm{acl}(\mathbb{Q})$ and distinct coordinates, contradicting Lemma 3.2.$\square$

Lemma 3.3 *Given $s_1, \dots, s_N$, let $t^0 = \langle t_1^0, \dots, t_N^0 \rangle$ be a semi-generic solution of (3)-(5). Then there is a neighborhood of t^0 in $\mathbb{C}^N$ such that t^0 is the only solution in the neighborhood.*

Proof We use induction on n, $2n = N$.

Assume towards a contradiction that t^0 is not isolated. Notice that full solutions of (3)-(5) form an analytic set $\mathcal{A}$ in $\mathbb{C}^{5N}$. It follows from our assumption that the set is of a positive dimension p at the point $a^0 \in \mathcal{A}$ corresponding to t^0. Representing $\mathcal{A}$ as a local analytic covering of a unit disc in $\mathbb{C}^p$ (see e.g. Theorem 3.7 of [C]) we get a local biholomorphic isomorphism $\phi : U \to \mathcal{A}$ between an open subset U of $\mathbb{C}^p$ and an open subset $\phi(U)$ of $\mathcal{A}$ such that the closure of

$\phi(U)$ contains a^0. We may assume that 0 is in the closure of U and $a^0 \in \lim_{u\to 0}\varphi(u)$, that is a^0 belongs to the set of limit points. We write $t(u), x(u), \bar{x}(u), y(u), \bar{y}(u)$ for the corresponding coordinates of the solution $\phi(u)$, $u \in U$. In particular, with some abuse of notation we right $t^0 = t(0)$ and $a^0 = \phi(0)$ is a semi-generic solution.

By Corollary 1 at least one of $t_i(u)$ is constant, equivalently, $t_i' = 0$. It corresponds to the fact that y_i is constant in Corollary 1. Notice that this is not possible if $n = 1$, for then both y_1 and y_2 are constant because of algebraic interdependence. This would imply both $t_1(u)$ and $t_2(u)$ constant, contradicting the assumptions. Thus we assume below that $n > 1$ (and so $N > 2$).

Let r be the minimal even number such that

$$\text{tr.d.}_C(y_{i_1}(0), \ldots, y_{i_r}(0)) = \frac{r}{2}. \tag{7}$$

We may assume $\{i_1, \ldots, i_r\} = \{1, \ldots, r\}$. Let $V^1 \subseteq \mathbb{C}^r$ be the locus of $\langle y_1(0), \ldots, y_r(0)\rangle$ over C and $\bar{V}^1$ the locus of $\langle \bar{y}_1(0), \ldots, \bar{y}_r(0)\rangle$ over $\bar{C}$. It is obvious from assumptions and definitions that $\dim V^1 = r/2$, half the number of variables, and that V^1 is P-free and P-normal. Thus V^1 satisfies the condition of the Lemma. So, by induction, either $r = N$ or $t_1(u), \ldots, t_r(u)$ are constant and $r \le N - 2$.

Consider the first case, $r = N$.

By Corollary 1 we assume t_1 and y_1 are constants. We also may assume that t_{N-1} and y_{N-1} are not constants.

Notice that

$$\text{tr.d.}_C(y_2(0), \ldots, y_{N-1}(0)) = n = \text{tr.d.}_C(y_1(0), y_2(0), \ldots, y_{N-1}(0)) \tag{8}$$

by our assumption and P-normality.

Let $V' \subseteq \mathbb{C}^{N-2}$ be the locus of $\langle y_2(0), \ldots, y_{N-1}(0)\rangle$ over $\text{acl}(C \cup \{y_1(0)\})$ and let C' be a finite subset of parameters defining the variety.

Claim 1. $\dim V' = n - 1$ and V' is P-normal and P-free.

Proof. By the calculations in (8) $y_1(0) \in \text{acl}_C(y_2(0), \ldots, y_{N-1}(0))$. Hence

$$\text{tr.d.}_{C,y_1(0)}(y_2(0), \ldots, y_{N-1}(0)) = \text{tr.d.}_C(y_1(0), y_2(0), \ldots, y_{N-1}(0)) - 1 = n - 1.$$

Freeness. Suppose towards a contradiction that for some non-zero $1 < i \le N - 2$, $y_i(0) \in \text{acl}(C, y_1(0))$. Then $\text{tr.d.}_C(y_1(0), y_i(0)) = 1$, contradicting $r = N > 2$. This contradicts the assumption of the claim and proves that V' is P-free.

For normality we need to show that for any $1 < i_1 < \cdots < i_\ell \leq N-2$

$$\text{tr.d.}_{C'}(y_{i_1}(0), \ldots, y_{i_\ell}(0)) \geq \frac{\ell}{2}.$$

For any odd $\ell < N-2$ we have by assumption,

$$\text{tr.d.}_C(y_1(0), y_{i_1}(0), \ldots, y_{i_\ell}(0)) \geq \frac{\ell+1}{2} + 1 \geq \frac{\ell}{2} + 1.$$

For an even $\ell < N-2$, letting $\ell' = \ell + 1$ and considering above an extra term $y_{i_{\ell+1}}$ we then have

$$\text{tr.d.}_C(y_1(0), y_{i_1}(0), \ldots, y_{i_\ell}(0)) \geq \text{tr.d.}_C(y_1(0), y_{i_1}(0), \ldots, y_{i_\ell}(0), y_{i_{\ell+1}}(0)) - 1 \geq$$

$$\geq \frac{\ell'+1}{2} = \frac{\ell}{2} + 1.$$

Hence in either case

$$\text{tr.d.}_{C'}(y_{i_1}(0), \ldots, y_{i_\ell}(0)) \geq \frac{\ell}{2},$$

which proves the claim.

Since $t_2(u), \ldots, t_{N-1}(u), y_2(u), \ldots, y_{N-1}(u), \bar{y}_2(u), \ldots, \bar{y}_{N-1}(u)$ is a solution of the system corresponding to V', we get a contradiction with the induction hypothesis. Hence the case $r = N$ is eliminated and we are left with $r \leq N-2$ and $t_1(u), \ldots, t_r(u)$ all constant.

Let $w(u) = \{w_1(u), \ldots, w_{2m}(u)\rangle$ be $y_{r+1}(u), \ldots, y_N(u)$, $2m = N - r$, at least one of them non-constant by assumptions. Let $V'' \subseteq \mathbb{C}^{2m}$ be the locus of $\langle w_1(0), \ldots, w_{2m}(0)\rangle$ over $\text{acl}(C \cup \{y_1(0), \ldots, y_r(0)\})$ and C'' the finite set of parameters for V''.

Claim 2. $\dim V'' = m$, V'' is P-free and P-normal.

Proof. We have

$$\dim V'' = \text{tr.d.}(y_{r+1}(0), \ldots, y_N(0)/C \cup \{y_1(0), \ldots, y_r(0)) =$$

$$= \text{tr.d.}_C(y_1(0), \ldots, y_N(0)) - \text{tr.d.}_C(y_1(0), \ldots, y_r(0)) = N/2 - r/2 = m.$$

Freeness. We need to show that for any $i \le 2m$, $w_i(0) \notin \mathrm{acl}(C'')$. Assuming $w_i(0) = y_j(0) \in \mathrm{acl}(C'')$, some $j \ge r+1$, we will get

$$\mathrm{tr.d.}_C(y_1(0), \dots, y_r(0), y_j(0)) = \frac{r}{2},$$

contradicting P-normality of V.

Normality. For $i_1 < \dots < i_l \le 2m$ by P-normality of V

$$\mathrm{tr.d.}_C(y_1(0) \dots, y_r(0), w_{i_1}(0), \dots, w_{i_l}(0)) \ge \frac{l+r}{2},$$

hence

$$\mathrm{tr.d.}_{C''}(w_{i_1}(0), \dots, w_{i_l}(0)) \ge \frac{l+r}{2} - \mathrm{tr.d.}_C C'' \ge \frac{l+r}{2} - \frac{r}{2} = \frac{l}{2}.$$

Claim proved.

The claim contradicts the induction hypothesis. □

Lemma 3.4 *Given $s^0 \in \mathbb{C}^N$ and a corresponding semi-generic solution $t^0 \in \mathbb{C}^N$ of the system (3)-(5) there are neighborhoods S of s^0 and T of t^0 such that for every $s \in S$ there is a unique $t(s) \in T$, satisfying (3)-(5). Moreover $s \mapsto t(s)$ is a holomorphic function on S.*

Proof Let $W \subseteq \mathbb{C}^{6N}$ be the algebraic variety in variables $s_i, t_i, x_i, \bar{x}_i, y_i, \bar{y}_i$, $i = 1, \dots, N$, defined by

$$(w, s) \in W \text{ iff } w \in W_s.$$

Let

$$\mathcal{W} = W \cap \{f(x) = y \ \&\ f(\bar{x}) = \bar{y}\}.$$

We assume $C = \bar{C}$. By assumption a connected component $\mathcal{W}^0$ of $\mathcal{W}$ containing (w^0, s^0) with w^0 given by t^0 is not in any algebraic subvariety of W of the form $W \cap \{y \in Y \ \vee \ \bar{y} \in \bar{Y}\}$ for $Y, \bar{Y}$ formally conjugated proper algebraic subvarieties of V and $\bar{V}$ over C.

Let $\mathcal{W}^0_s \subseteq \mathbb{C}^{5N} \times \{s\}$ be the fiber over s of the projection map $(w, s) \mapsto s$ on $\mathcal{W}^0$ and let $(w_0, s) \in \mathcal{W}^0_s$ be an isolated generic point of $\mathcal{W}^0_s$. By Lemma 3.3 the set

$$\mathcal{W}^0_s \setminus \bigcup \{y \in Y \ \vee \ \bar{y} \in \bar{Y} : Y, \bar{Y} \text{ proper, defined over } C\}$$

consists of isolated points, hence is open in $\mathcal{W}_s^0$. Thus, for an open polydisc $D \subseteq \mathbb{C}^{5N}$ containing w_0, for any proper algebraic C-definable subvarieties $W_Y = \{y \in Y \vee \bar{y} \in \bar{Y}\}$ of W

$$(D \times \{s\}) \cap \mathcal{W}^0 \cap W_Y = \emptyset, \text{ but } (D \times \{s\}) \cap \mathcal{W}^0 \neq \emptyset.$$

By continuity, for some open S, for any W_Y,

$$(D \times S) \cap \mathcal{W}^0 \cap W_Y = \emptyset.$$

Hence $(D \times \{s\}) \cap \mathcal{W}^0$, for each $s \in S$, consists of isolated points of $\mathcal{W}_s^0$ and so the projection $(D \times S) \cap \mathcal{W}^0 \to S$ has 0-dimensional fibers. For D and S small enough we get singleton fibers and the projection becomes bijective. By [C], 1.3.3, Proposition 3, the map $s \mapsto w(s)$, giving the unique point of the fiber, is holomorphic. Finally recall that any point $(w, s) \in \mathcal{W}_s^0$ is determined by its projection on t-coordinates. $\square$

Proof of the Existential Closedness Theorem

Now let $\bar{V}$ be V^c. Let y^0 be a generic point of the algebraic variety V and $x^0 \in \mathbb{C}^N$, $f(x^0) = y^0$.

Since $\mathbb{C} = \epsilon\mathbb{R} + \mathbb{R}$, we can represent $x_i^0 = \epsilon t_i^0 + s_i^0$ for some $t_i^0, s_i^0 \in \mathbb{R}$, $i = 1, \ldots, N$. It follows that $t^0, x^0, \bar{x}^0, y^0, \bar{y}^0$, where $\bar{x}^0$ and $\bar{y}^0$ are the complex conjugates of x^0 and y^0 correspondingly, satisfy (3)-(5) and form a semi-generic solution.

For S and T of Lemma 3.4 consider $s \in \mathbb{R}^N \cap S$ and $t = t(s)$. We claim that t has to be real as well. Indeed, consider the full solution $(t, x, \bar{x}, y, \bar{y}, z)$ of (3)-(5) with parameters s. Then, by the form of the equation, $(t^c, \bar{x}^c, x^c, \bar{y}^c, y^c, z^c)$ is a solution of the same system.

For small S the t is close to t^0, a real number, hence t^c is close to t^0 as well. Thus $t^c \in T$ and is a solution of (3)-(5) corresponding to s. We then have by uniqueness $t^c = t(s)$, which proves that t is real.

It remains now to choose a rational point $q \in S$ and consider the solution $t(q)$ of the system (3)-(5). Since $t_i(q)$ are real and $q_i \in \mathbb{Q}$ we have that $y_i(q) = f(\epsilon t_i + q_i) \in P$ and $\langle y_1(q), \ldots, y_N(q)\rangle \in V$. $\square$

Corollary 2 *The complete theory of* $(\mathbb{C}, +, \times, P)$ *is axiomatised by Poizat's axioms. The theory is* ω*-stable with* $\mathrm{MR}(\mathbb{C}) = \omega \times 2$ *and* $\mathrm{MR}(G) = \omega$.

Proof Poizat's axiomatisation (Proposition 2.1 of [P1]) consists of an obvious description of an algebraically closed field of characteristic zero with a subset P along with two lists of non-obvious axioms, $\forall$-axioms and $\forall\exists$-axioms. $\forall$-axioms are equivalent to the statement of our Proposition 1. $\forall\exists$-axioms are equivalent to the statement that for any V satisfying the assumptions of the Existential Closedness Theorem and any algebraic subvariety $V' \subset V$ of smaller dimension there is a point $a \in (V \setminus V') \cap G^k$. Our Existential Closedness Theorem is formally stronger as a it provides is even generic over any given C, in fact, it is equivalent to $\forall\exists$-axioms when applied to an ω-saturated model.

The statements on Morley rank is the property of the first order theory shown to be $\omega 2$ by Poizat. $\Box$

4 The field with "green points"

Let

$$G = \{\exp(\epsilon t + q) : t \in \mathbb{R},\ q \in \mathbb{Q}\}$$

considered as a unary predicate on $\mathbb{C}$. This is a multiplicative torsion-free subgroup of $\mathbb{C}^*$. It has a subgroup

$$G^0 = \{\exp(\epsilon t) : t \in \mathbb{R}\}$$

which can be naturally considered as the "connected component" of G.

For a finite subset X of a torsion-free abelian group, the subgroup generated by X is obviously embeddable into a vector space over $\mathbb{Q}$ of a large enough dimension. The **group rank** $grk(X)$ is defined as the size of a maximal independent subset of X in the vector space, equivalently, the dimension of the divisible hull of X as a vector space.

We recall that the standard **Schanuel conjecture** states that for any additively independent complex numbers $z_1, \dots, z_n$

$$\text{tr.d.}(z_1, \dots, z_n, \exp(z_1), \dots, \exp(z_n)) \geq n.$$

Proposition 2 *Assume Schanuel's conjecture. Then, for any* $h_1, \dots, h_n \in G$,

$$2\text{tr.d.}(h_1, \dots, h_n) \geq grk(h_1, \dots, h_n).$$

Proof We may assume that $h_1, \dots, h_n$ are multiplicatively independent, $h_i = \exp(\epsilon t_i + q_i)$, thus $\epsilon t_1 + q_1, \dots, \epsilon t_n + q_n$ are additively independent. We want to prove that

$$2\text{tr.d.}(h_1, \dots, h_n) \geq n.$$

Notice that any $\mathbb{Z}$-linear transformations of the system

$$T = \{\epsilon t_1 + q_1, \dots, \epsilon t_n + q_n\} \tag{9}$$

corresponds to a multiplicative transformation of $h_1, \dots, h_n$. Thus, a $\mathbb{Z}$-linear non-degenerate transformation does not change the inequality in the statement of the proposition. By applying such a transformation we can assume that $q_1 = \dots q_{n-1} = 0$ in (9). It follows, in particular, that $t_1, \dots, t_{n-1}$ are $\mathbb{Q}$-linearly independent.

Now consider the system

$$T^* = \{\epsilon t_1, \dots, \epsilon t_{n-1}, \epsilon t_n + q_n, \epsilon^c t_1, \dots, \epsilon^c t_{n-1}, \epsilon^c t_n + q_n\} \qquad (10)$$

where $\epsilon^c = 1 - i$, the result of complex conjugation. We want to calculate the $\mathbb{Q}$-linear dimension lin.d. T^* as well as the transcendence degree of the system T^*.

Since ϵ and ϵ^c are $\mathbb{R}$-linearly independent,

$$\mathrm{span}_Q\{\epsilon t_1, \dots, \epsilon t_{n-1}, \epsilon^c t_1, \dots, \epsilon^c t_{n-1}\} = \mathrm{span}_Q\{t_1, \dots, t_{n-1}, it_1, \dots, it_{n-1}\}$$

is of $\mathbb{Q}$-linear dimension $2n-2$ and

$$\text{lin.d.}T^* = 2n - 2 \text{ iff } q_n, t_n \in \mathrm{span}_Q(t_1, \dots, t_{n-1}). \qquad (11)$$

Either of conditions in (11) imply that tr.d.$T^* \leq n - 2$.

Notice further that

$$\text{tr.d.}T^* = \text{tr.d.}(t_1, \dots, t_n)$$

hence, if lin.d. $T^* < 2n$ we get tr.d.$T^* < n$. It follows, that under all circumstances

$$\text{lin.d.}T^* - \text{tr.d.}T^* \geq n.$$

By the Schanuel conjecture

$$\text{tr.d.}(T^* \cup \exp T^*) \geq \text{lin.d.}T^*.$$

Hence

$$\text{tr.d.} \exp T^* \geq \text{lin.d. } T^* - \text{tr.d.}T^* \geq n.$$

Remembering that $\exp(\epsilon t_i + q_i) = h_i$ and $\exp(\epsilon^c t_i + q_i) = h_i^c$, the latter can be rewritten as

$$\text{tr.d.}(h_1, \dots, h_n, h_1^c, \dots, h_n^c) \geq n.$$

Since complex conjugation is a field isomorphism we have by symmetry

$$2\text{tr.d.}(h_1, \dots, h_n) \geq n.$$

□

5 Existential Closedness Theorem for "green points"

Let $V \subseteq \mathbb{C}^k$ be an irreducible algebraic variety defined over a finitely generated subfield $\mathbb{Q}(C)$ and $a = \langle a_1, \ldots, a_k \rangle \in V$ is a generic point of V over C.

We will say that V is **G-free** if

$$a_1^{m_1} \cdot \ldots a_k^{m_k} \notin \mathrm{acl}(C), \text{ for any } \langle m_1, \ldots, m_k \rangle \in \mathbb{Z}^k \setminus \{0\}.$$

We will say that V is **weakly G-normal** if for any $1 \leq i_1 < \cdots < i_r \leq k$

$$\mathrm{tr.d.}_C(a_{i_1}, \ldots, a_{i_r}) \geq \frac{r}{2}.$$

Given a non-degenerate $(k \times k)$-matrix M over $\mathbb{Z}$ (that is with linearly independent rows $m_1, \ldots, m_k \in \mathbb{Z}^k$) we consider a transformation

$$\phi_M : (\mathbb{C}^*)^k \to (\mathbb{C}^*)^k : \quad y \mapsto y^M$$

where

$$y = \langle y_1, \ldots, y_k \rangle, \quad y^M = \langle y^{m_1}, \ldots, y^{m_k} \rangle \text{ and } y^{m_i} = y_1^{m_{i1}} \cdot \cdots \cdot y_k^{m_{ik}}.$$

ϕ_M is a local analytic automorphism of the torus. The image $\phi_M(V)$ is a dense subset of an irreducible algebraic variety defined over C, which we denote V^M. We shall call a transformation of the form ϕ_M an **admissible transformation.**

A generic point a of V corresponds to a^M, a generic point of V^M.

Remark It follows immediately from definitions that V is G-free if and only if V^M is.

We will say that V is **G-normal** if V^M is weakly G-normal for every non-degenerate integer matrix M.

In order to prove that $(\mathbb{C}, +, \times, G)$ is a model of Poizat's theory we prove the following

Existential Closedness Theorem *Let $V \subseteq \mathbb{C}^k$ be an irreducible algebraic variety defined over a finitely generated subfield $\mathbb{Q}(C)$. Assume that V is G-normal and G-free. Then there is $\langle a_1, \ldots, a_k \rangle \in V \cap G^k$*

generic in V over C.

Below in this section V is a a variety satisfying the assumptions of the theorem. We may also assume as in the previous section that $k = 2n$. We also assume $\dim V = n$ by the following analog of Lemma 3.1

Lemma 5.1 *Let $V \subseteq \mathbb{C}^{2n}$ be a G-free and G-normal irreducible variety over some C_0 such that $\dim V = d > n$. Then for any $c_1, \dots, c_{2n} \in \mathbb{C}$ algebraically independent over C_0 an irreducible component V' of the variety $V \cap H_c$, for $H_c = \{y \in \mathbb{C}^{2n} : c_1 \cdot y_1 + \cdots + c_{2n} \cdot y_{2n} = 1\}$, is G-normal and G-free and $\dim V' = d - 1$.*

Proof V' is defined over some $C_1 \subseteq \mathrm{acl}(C_0 \cup \{c_1, \dots, c_{2n}\})$. Then $\dim V' = k - 1$, as $V \nsubseteq H_c$. Let $c = \langle c_1, \dots, c_{2n} \rangle$ and a be a generic over C_0c point in V'. Then a is generic in V over C_0 and $\mathrm{tr.d.}(c/C_0a) = 2n - 1$.

Let $M_1, \dots, M_k \in \mathbb{Z}^{2n}$ be linearly independent.

Claim. $\mathrm{tr.d.}(a^{M_1}, \dots, a^{M_k}/C_0) > \mathrm{tr.d.}(a^{M_1}, \dots, a^{M_k}/C_0c)$ implies $\mathrm{tr.d.}(a^{M_1}, \dots, a^{M_k}/C_0) = \mathrm{tr.d.}(a/C_0) = d$.

This is a special case of the claim in the proof of Lemma 3.1.

Now, to see that V' is G-free consider a non-zero $M \in \mathbb{Z}^{2n}$. If $a^M \in \mathrm{acl}(C_0c)$ then $\mathrm{tr.d.}(a^M/C_0c) < \mathrm{tr.d.}(a^M/C)$, since V is free, and so by the claim above, $\mathrm{tr.d.}(a^M/C_0) = d$, which contradicts the condition $d > 1$.

Normality: Assuming, towards a contradiction, that for some linearly independent $M_1, \dots, M_k \in \mathbb{Z}^{2n}$, $\mathrm{tr.d.}(a^{M_1}, \dots, a^{M_k}/C_0c) < k/2$ we get by normality of V $\mathrm{tr.d.}(a^{M_1}, \dots, a^{M_k}/C_0) > \mathrm{tr.d.}(a^{M_1}, \dots, a^{M_k}/C_0c)$. Hence by the claim $\mathrm{tr.d.}(a^{M_1}, \dots, a^{M_k}/C_0c) = \mathrm{tr.d.}(a_1, \dots, a_{2n}/C_0c) = d - 1 \geq n$, the contradiction. □

In full analogy to section 2 we use notation $N = 2n$ and $\bar{V}$ for a variety over some $\bar{C}$ with the property that given generic $\langle y_1, \dots y_N \rangle \in V$, a generic $\langle \bar{y}_1, \dots \bar{y}_N \rangle \in \bar{V}$, for any selection of integer vectors $M_1, \dots M_l \in \mathbb{Z}^N$,

$$\mathrm{tr.d.}_C(y^{M_1}, \dots y^{M_l}) = \mathrm{tr.d.}_{\bar{C}}(\bar{y}^{M_1}, \dots \bar{y}^{M_l}).$$

Again, an example of such a $\bar{V}$ would be V^c, the image of V under complex conjugation.

We also consider the variety $W_s \subseteq \mathbb{C}^{5N}$ over parameters C and $s \in \mathbb{C}^N$ given by equations (3) and (4) (with the new meaning of V and $\bar{V}$) and a system of equations

$$y_i = \exp(x_i), \quad \bar{y}_i = \exp(\bar{x}_i), \quad i = 1, \ldots, N. \tag{12}$$

Lemma 5.2 *Given $s_1, \ldots, s_N \in \mathbb{C}$, let for an index set U, for each $u \in U$, $x_i(u) = \epsilon t_i(u) + s_i$, $\bar{x}_i(u) = \epsilon^c t_i(u) + s_i$, for $i = 1, \ldots, N$,*

$x(u) = \langle x_1(u), \ldots x_N(u)\rangle$, $\bar{x}(u) = \langle \bar{x}_1(u), \ldots \bar{x}_N(u)\rangle$.

Assume that for all u

$\exp(x(u)) \in V$, $\exp(\bar{x}(u)) \in \bar{V}$, $\{t_1(u), \ldots, t_N(u)\}$ is $\mathbb{Q}$-linearly independent over $C \cup \bar{C} \cup \{s_1, \ldots, s_N\}$ and $x(u) \neq x(u')$ if $u \neq u'$, for any $u, u' \in U$.

Then U is at most countable.

Proof Let $C' = C \cup \bar{C} \cup \{s_1, \ldots, s_N\}$. Obviously, for each u

$$\text{lin.d.}(x(u), \bar{x}(u)/C') = \text{lin.d.}(\{\epsilon t_1(u), \ldots, \epsilon t_N(u), \epsilon^c t_1(u), \ldots, \epsilon^c t_N(u)\}/C') =$$

$$= \text{lin.d.}(\{t_1, \ldots, t_N, it_1, \ldots, it_N\}/C') = 2\text{lin.d.}(\{t_1, \ldots, t_N\}/C') = 2N$$

by assumptions.

We also have

$$(\exp(x(u)), \exp(\bar{x}(u)) \in V \times \bar{V}, \quad \text{acl}(x(u), \bar{x}(u)) \subseteq \text{acl}(t_1(u), \ldots, t_N(u), C').$$

Thus,

$$\text{tr.d.}(\exp(x(u)), \exp(\bar{x}(u)/C') \leq 2n = N, \quad \text{tr.d.}(x(u), \bar{x}(u)/C') \leq N.$$

Define a predimension for the structure $\mathbb{C}_{\exp} = (\mathbb{C}, +, \cdot, \exp)$ as in [Z2]

$$\delta(X) = \text{tr.d.}(X \cup \exp(X)) - \text{lin.d.}X, \quad \delta(X/C) = \delta(X \cup C) - \delta(C).$$

By above we have

$$\delta(x(u), \bar{x}(u)/C') \leq 0$$

for all u. It follows that $x(u) \in \text{cl}_F(C)$ as defined in [Z2], section 4, for some exponentially-algebraically closed field F with pseudo-exponentiation, extending $\mathbb{C}$. By Lemma 5.7 of [Z2] (countable closure property) which only needs the Schanuel conjecture for $\mathbb{C}$, the set U has to be at most countable. $\Box$

Lemma 5.3 *For a continuous function $u \mapsto t(u)$ from an open subset $U \subseteq \mathbb{C}^p$, non-constant in all coordinates $t_i(u)$, such that*

$$x(u) = \langle \exp(\epsilon t_1(u) + s_1), \dots, \exp(\epsilon t_N(u) + s_N) \rangle \in V$$

and

$$\bar{x}(u) = \langle \exp(\epsilon^c t_1(u) + s_1), \dots, \exp(\epsilon^c t_N(u) + s_N) \rangle \in \bar{V},$$

there is a proper $\mathbb{Q}$-affine subspace

$$L_m(d) = \{t \in \mathbb{C}^N : m_1 t_1 + \cdots + m_N t_N = d\},$$

some $d \in \mathrm{span}_Q C'$ $(C' = C \cup \bar{C} \cup \{s_1, \dots, s_N\})$ such that

$$J_m(d) = \{u \in U : \langle t_1(u), \dots, t_N(u) \rangle \in L_m(d)\}$$

contains an open subset.

Proof Suppose that for any $m \in \mathbb{Z}^N \setminus 0$ and $d \in \mathrm{span}_Q C'$ the set $J_m(d)$ is nowhere dense in U. Then the set

$$U' = U \setminus \bigcup \{J_m(d) : m \in \mathbb{Z}^N \setminus 0,\ d \in \mathrm{span}_Q C'\}$$

is the complement of the union of countably many nowhere-dense subsets, so is of cardinality continuum. For any $u \in U'$ by construction the coordinates of $t(u)$ are linearly independent over C'. This contradicts Lemma 5.2.□

Lemma 5.4 *Given $s_1, \dots, s_N$, let $t^0 = \langle t_1^0, \dots, t_N^0 \rangle$ be a semi-generic solution of (3),(4) and (12). Then there is a neighborhood of t^0 in $\mathbb{C}^N$ such that t^0 is the only solution in the neighborhood.*

Proof Assume towards a contradiction that t^0 is not isolated. By repeating the argument in the proof of Lemma 3.3 we then are given an open disc U, the point 0 belonging to its closure, and an analytic map from U into the set of solutions, such that $t^0 = t(0)$ and $(t(0), x(0), \bar{x}(0), y(0), \bar{y}(0))$ is a semi-generic solution of (3),(4) and (12).

We are going to show by induction on n, $2n = N$, that this is not possible.

By Lemma 5.3 there is a non-zero integer tuple m, a complex number d and an open $U_m(d) \subseteq U$ such that at least one of

$$\sum_{1 \le i \le N} m_i t_i(u) = d \text{ for all } u \in U_m(d),$$

or equivalently

$$\sum_{1 \le i \le N} m_i t' = 0 \text{ on the open set.}$$

It corresponds to the fact that

$$\prod_{1 \le i \le N} y_i^{m_i}(u) = \exp \sum_{1 \le i \le N} m_i(\epsilon t_i(u) + s^0)$$

is constant, where the string $y(u)$ is in V.

We can now replace variables in a non-degenerate way by introducing, say

$$\hat{t}_1 = \sum_{1 \le i \le N} m_i t_i \text{ and } \hat{y}_1 = \prod_{1 \le i \le N} y_i^{m_i}$$

and leaving the remaining unchanged. This corresponds to an admissible transformation

$$\phi : \langle y_1, \ldots, y_{n-1}, y_n, y_{n+1}, \ldots, y_N \rangle \mapsto \langle \prod_{1 \le i \le N} y_i^{m_i}, y_2, \ldots, y_N \rangle.$$

Thus we assume w.l.o.g. that $\hat{y}_1 = y_1$ is constant for u in the open set, and so is t_1. Notice that this is not possible if $n = 1$ by the same argument as in the proof of Lemma 3.3. Thus we assume below that $N > 2$.

Let r be the minimal even number such that for some $M_1, \ldots, M_r \in \mathbb{Z}^N$ linearly independent

$$\text{tr.d.}_C(y^{M_1}(0), \ldots, y^{M_r}(0)) = \frac{r}{2}. \tag{13}$$

The equality is satisfied by $r = N$, hence the minimal r exists.

By an admissible transformation of variables we may assume that $\langle y^{M_1}, \ldots, y^{M_r} \rangle = \langle y_1, \ldots, y_r \rangle$.

Let $V^1 \subseteq \mathbb{C}^r$ be the locus of $\langle y_1(0), \ldots, y_r(0)\rangle$ over C and $\bar{V}^1$ the locus of $\langle \bar{y}_1(0), \ldots, \bar{y}_r(0)\rangle$ over $\bar{C}$. It is obvious from assumptions and definitions that $\dim V^1 = r/2$, half the number of variables, and that V^1 is G-free and G-normal. Thus V^1 satisfies the condition of the Lemma. So, by induction, either $r = N$ or $t_1(u), \ldots, t_r(u)$ are constant and $r \leq N - 2$.

Consider the first case, $r = N$.

We assume t_1 and y_1 are constant and t_{N-1} and y_{N-1} are not constant.

Similarly to (8) we notice that

$$\text{tr.d.}_C(y_2(0), \ldots, y_{N-1}(0)) = n = \text{tr.d.}_C(y_1(0), y_2(0), \ldots, y_{N-1}(0)). \tag{14}$$

Let $V' \subseteq \mathbb{C}^{N-2}$ be the locus of $\langle y_2(0), \ldots, y_{N-1}(0)\rangle$ over $\text{acl}(C \cup \{y_1(0)\})$ and let C' be a finite subset of parameters defining the variety.

Claim 1. $\dim V' = n - 1$ and V' is G-normal and G-free.

Proof. $\dim V' = n - 1$ by the same reasons as in the proof of Lemma 3.3.

Freeness. Suppose towards a contradiction that for some non-zero $M \in \mathbb{Z}^N$, $m_1 = 0$, $y^M(0) \in \text{acl}(C, y_1(0))$. Then $\text{tr.d.}_C(y_1(0), y_M(0)) = 1$ and $r = 2$. This contradicts the assumption of the claim.

For normality we need to show that for any $M_1, \ldots, M_\ell \in \mathbb{Z}^{N-2}$ linearly independent ($\ell \leq N - 2$) and $z(0) = \langle y_2(0), \ldots, y_k(0)\rangle$

$$\text{tr.d.}_{C'}(z^{N_1}(0), \ldots, z^{N_\ell}(0)) \geq \frac{\ell}{2},$$

where $C' \subseteq \text{acl}(C \cup \{y_1(0)\})$ is a finite set of parameters needed to define V'.

We use the same calculations as in the proof of Claim 1 of Lemma 3.3 to show this and so prove the claim.

The Claim eliminates the case $r = N$ and we assume again that $r \leq N - 2$ and $t_1(u), \ldots, t_r(u)$ are all constant.

Let $w(u) = \{w_1(u), \ldots, w_{2m}(u)\rangle$ be $y_{r+1}(u), \ldots, y_N(u)$, $2m = N - r$, at least one of them non-constant by assumptions. Let $V'' \subseteq \mathbb{C}^{2m}$ be the locus of $\langle w_1(0), \ldots, w_{2m}(0)\rangle$ over $\text{acl}(C \cup \{y_1(0), \ldots, y_r(0)\})$ and C'' the finite set of parameters for V''.

Claim 2. $\dim V'' = m$ and V'' is G-free and G-normal.

Proof. $\dim V'' = m$ by the same calculation as in the proof of Claim 2 of 3.3.

Freeness. Up to transformation of variables we need to show that $w_i(0) \notin \mathrm{acl}(C'')$. Assuming $w_i(0) = y_j(0) \in \mathrm{acl}(C'')$, some $j \geq r+1$, we will get

$$\mathrm{tr.d.}_C(y_1(0), \dots, y_r(0), y_j(0)) = \frac{r}{2},$$

contradicting G-normality of V.

Normality. For linearly independent $M_1, \dots, M_\ell \in \mathbb{Z}^{2m}$ by G-normality of V

$$\mathrm{tr.d.}_C(y_1(0) \dots, y_r(0), w^{M_1}(0), \dots, w^{M_\ell}(0)) \geq \frac{\ell + r}{2},$$

hence

$$\mathrm{tr.d.}_{C''}(w^{M_1}(0), \dots, w^{M_\ell}(0)) \geq \frac{\ell + r}{2} - \mathrm{tr.d.}_C C'' \geq \frac{\ell + r}{2} - \frac{r}{2} = \frac{\ell}{2}.$$

Claim proved.

The Claim contradicts the induction hypothesis and finishes the proof of the lemma. □

Proof of the Existential Closedness Theorem

We first recall the statement of Lemma 3.4 with f replaced by exp, with the same proof.

In the same way the final part of the proof of the Existential Closedness Theorem for P is valid for G. □

Corollary 3 *The complete theory of* $(\mathbb{C}, +, \times, G)$ *is axiomatised by Poizat's axioms. The theory is* ω*-stable with* $\mathrm{MR}(\mathbb{C}) = \omega \times 2$ *and* $\mathrm{MR}(G) = \omega$.

Proof Poizat's axiomatisation consists of the obvious description of an algebraically closed field of characteristic zero with a torsion-free subgroup G along with two lists of non-obvious axioms, $\forall$-axioms and $\forall\exists$-axioms (section 3 of [P2]). $\forall$-axioms, as proved in his paper, are equivalent to the statement of our Proposition 2. $\forall\exists$-axioms are equivalent to the statement that for any V satisfying the assumptions of the Existential Closedness Theorem and any algebraic subvariety $V' \subset V$ of smaller dimension there is a point $a \in (V \setminus V') \cap G$. This is provided by our Existential Closedness Theorem. $\Box$

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