A categoricity theorem for quasi-minimal excellent classes

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Introduction This work stemmed from attempts to prove a *canonicity* of a certain type of structures, which we called *pseudo-analytic* (see [Z1]). These attempts, even for a particular class of structures, fields with pseudo-exponentiation, involved a substantial work in field arithmetic, and even after most of the algebraic problems were sorted out in [Z2], a problem remained, which we realized was of a pure model-theoretic nature. This is a categoricity statement for certain classes of structures which we now call quasi-minimal excellent. Initially we formulated the assumptions for the theorem as sufficient conditions applicable for some, and hopefully for all, pseudo-analytic structures, without being aware of Shelah's theory of excellent classes. Later, after discussing this with B.Hart, R.Grossberg and O.Lessmann, we realized that the assumptions are essentially a special case of Shelah's definition of excellent classes but formulated in the spirit of the Baldwin-Lachlan categoricity proof. There is no doubt that the proofs of two our main theorems follow Shelah's [S], yet (as was pointed out by the referee) their statements can not be derived directly from Shelah's particularly because they treat classes more general than $L_{\omega_1,\omega}$. This special form of a categoricity theorem has a much easier formulation customized for applications, and there is a rich and mathematically meaningful class of examples satisfying this special definition.

*With last correction 06.04.2005

The proof does not use any previous theory, though the technique is essentially Shelah's. The reader is encouraged to compare the present setting with Shelah's original work and carry out a further analysis, especially with regards to pseudo-analytic and analytic examples. We discuss shortly the applications in the second section of the paper.
Since the first version of the paper (July 2001) we have realized that a richer class of examples requires a slightly stronger version of the main theorem, this is a categoricity theorem for *almost quasi-minimal excellent* classes presented in the last section.

1 Quasi-minimal excellent structures

Let $\mathcal{C}$ be a class of structures closed under isomorphisms and satisfying the following assumptions 1-3:

Assumption 1 (pregeometry) *For any $H \in \mathcal{C}$ for any $X \subseteq H$ there is a superset $X \subseteq \mathrm{cl}(X) \subseteq H$ satisfying the following:*

(i) $\mathrm{cl}(X) \in \mathcal{C}$ *as a substructure of* H;

(ii) $\mathrm{cl}(Y) = \bigcup\{\mathrm{cl}(X) : \ X \subseteq Y, \ \ X \textit{ finite}\}$;

(iii) $X \to \mathrm{cl}(X)$ *is a monotone idempotent operator.*

We are also often interested in two extra conditions:

the countable closure property: $\mathrm{cl}(X)$ is countable for X finite;

the exchange principle: $z \in \mathrm{cl}(Xy) \setminus \mathrm{cl}(X) \Rightarrow y \in \mathrm{cl}(Xz)$.

Definition Let $H, H' \in \mathcal{C}$ and $G \subseteq H$, $G \subseteq H'$. Then a (partial) mapping, identical on G, $\varphi : H \to H'$ is called a G**-monomorphism** if it preserves quantifier-free formulas over G.

Definition A non-empty subset $X \subseteq H$ is called cl**-independent over** G if

$$\mathrm{cl}(G) \cap X = \emptyset$$

and for any $Y_1, Y_2 \subseteq X$

$$\mathrm{cl}(Y_1 \cup G) \cap \mathrm{cl}(Y_2 \cup G) = \mathrm{cl}(Y_1 \cap Y_2 \cup G).$$

X is said to be just cl-independent, if it is cl-independent over $\emptyset$.

Assumption 2 (ω-homogeneity over a submodel) [1] *Let $G \subseteq H$, $G \subseteq H'$, $G \in \mathcal{C}$ or $G = \emptyset$. Then*

(i) if X and X' are cl*-independent subsets over G in some $H, H' \in \mathcal{C}$, correspondingly, then any bijection $\varphi : X \to X'$ is a G-monomorphism;*

(ii) if a partial $\varphi : H \to H'$ is a G-monomorphism, $\mathrm{Dom}\,\varphi = X$*, with X finite, then for any $y \in \mathrm{cl}(X \cup G)$ there is an extension φ' of φ with* $\mathrm{Dom}\,\varphi' = X \cup \{y\}$.

(iii) if $\varphi : X \cup \{y\} \to X' \cup \{y'\}$ is a monomorphism, then

$$y \in \mathrm{cl}(X) \text{ iff } y' \in \mathrm{cl}(X').$$

Definition Given $X, C \subseteq H$ we say that **the type of X over C is defined over C_0** if any $\varphi : X \to H'$ which is a C_0-monomorphism is also a C-monomorphism.

Definition A subset $C \subseteq H$ will be called **special** if there is a cl-independent $A \subseteq H$ and $A_1, \ldots A_k \subseteq A$ such that

$$C = \bigcup_i \mathrm{cl}(A_i).$$

Assumption 3 *Suppose $C \subseteq H$ is special and X is a finite subset of* $\mathrm{cl}(C)$*. Then the type of X over C is defined over a finite subset $C_0 \subseteq C$.*

[1] with a correction in (ii) suggested by O.Roche, 3.11.04

Remark It follows from assumptions 1 and 2, by the Ehrenfeucht-Fraisse criterion, that, given finite $X \subseteq H$, any two points in $H \setminus \mathrm{cl}(X)$ are $L_{\omega_1,\omega}$-equivalent. If also an uncountable structure $H \in \mathcal{C}$ satisfies the countable closure property then any $L_{\omega_1,\omega}$-definable subset of H is either countable or has a countable complement.

A class $\mathcal{C}$ of structures satisfying assumptions 1-3 will be called in this paper **quasi-minimal excellent.** Also, each member of the class is said to be quasi-minimal excellent.

Below we usually use letters $\mathcal{A}$, $\mathcal{A}_0 \ldots$ to denote infinite independent sets, while $A, A_0, \ldots$ will stand for their finite subsets.

Lemma 1.1 *Let $\mathcal{A}_0 \subseteq H$, $\mathcal{A}'_0 \subseteq H'$ be independent subsets and suppose $G = \mathrm{cl}(\mathcal{A}_0)$ and $G' = \mathrm{cl}(\mathcal{A}'_0)$ are countable. Then, given a bijection $\varphi_0 : \mathcal{A}_0 \to \mathcal{A}'_0$, there is an isomorphism $\varphi : G \to G'$, extending φ_0.*

Proof Assuming $\mathcal{A}_0 = \{a_i : i \in \mathbb{N}\}$, $\mathcal{A}'_0 = \{a'_i : i \in \mathbb{N}\}$ and $\varphi_0(a_i) = a'_i$, by induction on i assume that there is a isomorphism

$$\psi_i : \mathrm{cl}(a_1, \ldots, a_i) \to \mathrm{cl}(a'_1, \ldots, a'_i),$$

sending a_j to a'_j for all $j \leq i$. Then by definition $\{a_{i+1}\}$ is cl-independent over $\mathrm{cl}(a_1, \ldots, a_i)$ and so is $\{a'_{i+1}\}$ over $\mathrm{cl}(a'_1, \ldots, a'_i)$. It follows from the ω-homogeneity over a submodel and the countability assumption that ψ_i can be extended to

$$\psi_{i+1} : \mathrm{cl}(a_1, \ldots, a_{i+1}) \to \mathrm{cl}(a'_1, \ldots, a'_{i+1}).$$

Then $\varphi = \bigcup_i \psi_i$ satisfies the requrements.$\Box$

Lemma 1.2 *Let $\mathcal{A}$ be a* cl*-independent subset in H, $\mathcal{A}_0 \subseteq \mathcal{A}$ infinite and $A_0, \ldots, A_m \subseteq \mathcal{A}$ finite subsets. For $X \subseteq H$ denote $\mathrm{cl}^*(X) = \mathrm{cl}(\mathcal{A}_0 \cup X)$, and let $B_0 \subseteq \mathrm{cl}^*(A_0)$ finite, $b_i \in \mathrm{cl}^*(A_i)$, $i = 1, \ldots, m$, and $p(x_1, \ldots, x_m)$ a quantifier-free type over B_0, such that $\models p(b_1, \ldots, b_m)$. Then there is a mapping*

$$\pi : \bigcup A_i \cup B_0 \cup \{b_1, \ldots, b_m\} \to \mathrm{cl}^*(A_0),$$

which is identity on B_0, $b_i^\pi \in \mathrm{cl}^*(A_i^\pi) \subseteq \mathrm{cl}^*(A_i \cap A_0)$ and $\models p(b_1^\pi, \ldots, b_m^\pi)$.

Proof Let A^* be a finite subset of $\mathcal{A}_0$, such that $B_0 \subseteq \mathrm{cl}(A_0 \cup A^*)$ and $b_i \in \mathrm{cl}(A_i \cup A^*)$ for all $i = 1, \ldots, m$.
Let $G_0 = \mathrm{cl}(A^* \cup A_0)$ and π_0 be the identity map on G_0. Then the extension of π_0 to an injection

$$\pi_1 : G_0 \cup \bigcup_{1 \le i \le m} A_i \setminus (A_0 \cup A^*) \to G_0 \cup \mathcal{A}_0 \setminus (A_0 \cup A^*)$$

is a monomorphism by the assumption 2(i). By ω-homogeneity extend π_1 to a monomorphism π with a domain covering $\{b_1, \ldots, b_m\}$. Then $\models p(b_1^\pi, \ldots, b_m^\pi)$, since π is a monomorphism over B_0. Also, $b_i^\pi \in \mathrm{cl}(A_i^\pi \cup A^*) \subseteq \mathrm{cl}^*(A_i^\pi)$ by the construction, and $\mathrm{cl}^*(A_i^\pi) \subseteq \mathrm{cl}^*(A_i \cap A_0)$ by the definition of π_1.$\square$

Theorem 1 *Let $\mathcal{C}$ be a quasi-minimal excellent class, $H, H' \in \mathcal{C}$, both with the countable closure property, $\mathcal{A} \subseteq H$, $\mathcal{A}' \subseteq H'$, independent and* $\mathrm{cl}(\mathcal{A}) = H$, $\mathrm{cl}(\mathcal{A}') = H'$. *Suppose that there is a bijection $\psi_0 : \mathcal{A} \to \mathcal{A}'$.*
Then ψ_0 extends to an isomorphism $\psi : H \to H'$.

Proof If $\mathcal{A}$ is finite or countable, the statement is proved in Lemma 1.1. So we assume $\mathcal{A}$ is infinite and uncountable, and so contains a proper countable subset $\mathcal{A}_0$. Let $\psi_0(\mathcal{A}_0) = \mathcal{A}'_0 \subseteq \mathcal{A}'$. W.l.o.g. we may assume that ψ_0 is identity on $\mathcal{A}_0$. By Lemma 1.1 we may also identify $\mathrm{cl}_H(\mathcal{A}_0) = \mathrm{cl}_{H'}(\mathcal{A}'_0) = G$.
We continue the proof of the theorem through a series of intermediate statements.

We now change the language by fixing G pointwise and so "monomorphism" from now on means "monomorphism over G" and $\mathrm{cl}(X)$ means $\mathrm{cl}(X \cup \mathcal{A}_0)$, the same as cl^* in Lemma 1.2 above. Notice that in the new language assumptions 1-3 still hold for structures containing $\mathcal{A}_0$.

Lemma 1.3 ψ_0 *is a monomorphism.*

Proof First notice that $\mathcal{A} \setminus A_0$ along with $\mathcal{A}' \setminus \mathcal{A}_0$ are independent. Then apply assumption 2(i).$\square$

Lemma 1.4 *If $X \subseteq \mathcal{A}$ is finite, then there is a isomorphism* $\mathrm{cl}(X) \to \mathrm{cl}(\psi_0(X))$ *extending* $\psi_0|_X$.

Proof Immediate from the countable closure property and the assumptions 2 by the back-and-forth method.$\Box$

Notation For $X \subseteq H$ denote

$$\mathrm{cl}^-(X) = \cup_{Y \subsetneq X} \mathrm{cl}(Y).$$

Lemma 1.5 *Let $A_0, \ldots, A_m \subseteq A \subseteq \mathcal{A}$, A finite and $A_0 \not\subseteq A_i$ for $i > 0$. Then any two tuples D, D' in* $\mathrm{cl}(A_0)$ *which are isomorphic over* $\mathrm{cl}^-(A_0)$, *are also isomorphic over* $\mathrm{cl}^-(A_0) \cup \bigcup_{1 \leq i \leq m} \mathrm{cl}(A_i)$.

Proof Assume not, and D, D' provide a counterexample. Then there is a quantifier-free formula $q(x, y)$ over $\mathrm{cl}^-(A_0)$, such that for some $B \subseteq \bigcup_{1 \leq i \leq m} \mathrm{cl}(A_i)$ both $q(D, B)$ and $\neg q(D', B)$ hold in H. By assumption 1.2 there is a map

$$\pi : \bigcup_{1 \leq i \leq m} A_i \cup D \cup D' \cup B \to \mathrm{cl}(A_0),$$

identical on $D \cup D'$, such that

$$\models \neg q(D', B^\pi) \ \&\ q(D, B^\pi)$$

and

$$\pi(B \cap \mathrm{cl}(A_i)) \subseteq \mathrm{cl}(A_0) \cap \mathrm{cl}(A_i^\pi) \subseteq \mathrm{cl}(A_0 \cap A_i) \subseteq \mathrm{cl}^-(A_0).$$

Thus $B^\pi \subseteq \mathrm{cl}^-(A_0)$. This contradicts the assumptions on D, D'. $\Box$

Definition Given a finite $A \subseteq \mathcal{A} \setminus \mathcal{A}_0$ we call a subset $D \subseteq \mathrm{cl}(A)$ **perfect over A in H** if (an inductive definition)
(i) $A = \emptyset$, or
(ii) for any $X \subsetneq A$ the sets $D_X = D \cap \mathrm{cl}(X)$ are perfect over X, and the type of D over $\mathrm{cl}^-(A)$ is defined over $\bigcup_{X \subsetneq A} D_X$.

Lemma 1.6 *Let $A \subseteq \mathcal{A} \setminus \mathcal{A}_0$ and $D \subseteq \mathrm{cl}(A)$ be finite, $n \geq 0$ and assume that for each $X \subseteq A$ with $|X| \leq n+1$ $D_X = D \cap \mathrm{cl}(X)$ is perfect over X and $\varphi_X : D_X \to H'$ is a monomorphism. Denote $\varphi = \bigcup_{X \subseteq A, |X|=n+1} \varphi_X$, and suppose also that the restriction of φ on $\bigcup_{Y \subseteq A, |Y|=n} D_Y$ is a monomorphism. Then φ is a monomorphism.*

Proof Let

$$\{X \subseteq A, |X| = n+1\} = \{A_i : i = 0, \dots, m\}, \text{ distinct.}$$

Notice that since, by cl-independence, for any $i \neq j$

$$D \cap \mathrm{cl}(A_i) \cap \mathrm{cl}(A_j) \subseteq \bigcup_{Y \subseteq A, |Y|=n} D_Y,$$

φ is a well-defined mapping.
Remember that

$$\bigcup_{Y \subseteq A, |Y|=n} D_Y = \bigcup_{i \leq m} D_{A_i} \cap \mathrm{cl}^-(A_i),$$

and apply Lemma 1.5 to see that φ preserves the quantifier-free formulas on

$$D_{A_0} \cup \bigcup_{0<i\leq m} D_{A_i} \cap \mathrm{cl}^-(A_i).$$

The same lemma by induction on $l \leq m$ yields that φ preserves the quantifier-free formulas on

$$\bigcup_{i \leq l} D_{A_i} \cup \bigcup_{l<i\leq m} D_{A_i} \cap \mathrm{cl}^-(A_i).$$

For $l = m$ we have the statement of the present lemma. $\square$

Lemma 1.7 *Given a finite $A \subseteq \mathcal{A} \setminus \mathcal{A}_0$ for any finite $D \subseteq \mathrm{cl}(A)$ there is a finite $\bar{D}$, $D \subseteq \bar{D} \subseteq \mathrm{cl}(A)$, perfect over A.*
More precisely,

$$\bar{D} = D \cup \bigcup_{X \subsetneq A} \bar{D}_X, \text{ where } D \cap \mathrm{cl}(X) \subseteq \bar{D}_X \subseteq \mathrm{cl}(X).$$

Moreover, if we relpace $\bar{D}_X$ *by* $\bar{D}'_X \supseteq \bar{D}_X$*, perfect over* X*, and* D *by* $D' \subseteq D$*, then*

$$\bar{D}' = D' \cup \bigcup_{X \subsetneq A} \bar{D}'_X,$$

is still perfect over A*.*

Proof (i) Induction on $|A|$. The statement is obvious for $A = \emptyset$. In the general case by assumption 3 the type of D over $G \cup \mathrm{cl}^{-}(A)$ is defined over a finite subset, say $\bigcup_{X \subsetneq A} D_X$, and G with each $D_X \subseteq \mathrm{cl}(X)$. By induction there are finite $\bar{D}_X$ perfect over X. Put

$$\bar{D} = D \cup \bigcup_{X \subsetneq A} \bar{D}_X,$$

which is perfect over A by the definition.
The 'moreover' statement is immediate by the definition. □

Proof of the theorem. We proceed by induction on n assuming that there is a family

$$\mathcal{D}_n = \{D_{A,i} \subseteq \mathrm{cl}(A) : \ A \subseteq \mathcal{A} \setminus \mathcal{A}_0, |A| \leq n, i \in \mathbb{N}\}$$

of finite sets perfect over A, such that

$$\mathrm{cl}(A) = \bigcup_{i \in \mathbb{N}} D_{A,i},$$

for each $B \subseteq A$

$$D_{A,i} \cap \mathrm{cl}(B) = D_{B,i},$$

and $\mathcal{D}_m \subseteq \mathcal{D}_n$ for $m \leq n$.

We also assume that there are monomorphisms for all $k \in \mathbb{N}$:

$$\psi_{n,k} : \bigcup_{A \subsetneq \mathcal{A}, |A| \leq n} D_{A,k} \to H', \quad \psi_0 \subseteq \psi_{n,k} \subseteq \psi_{n,k+1}$$

and for each $A \subseteq \mathcal{A}$ of size $\leq n$

$$\psi_A = \bigcup_{k \in \mathbb{N}} \psi_{n,k}|_{\mathrm{cl}(A)} : \mathrm{cl}(A) \to \mathrm{cl}(\psi_0(A)) \text{ is a bijection.}$$

Notice that for $D_{\emptyset,k}$ we can take any ascending chain of finite subsets covering $\mathrm{cl}(\emptyset)$, which is a substructure of common G, and $\psi_{0,k}$ be the identity mappings on the appropriate subsets.
Suppose that a chain $D_{X,i}$ has been constructed for each $X \subseteq \mathcal{A} \setminus \mathcal{A}_0$ of size $\leq n$ as well as monomorphisms $\psi_{n,k}$ for all $k \in \mathbb{N}$. For each $|A| = n+1$ put $D_{A,0} = \bigcup_{X \subsetneq A} D_{X,0}$, $\mathrm{cl}^0(A) = \mathrm{cl}(A) \setminus \mathrm{cl}^-(A)$, and for each $i \in \mathbb{N}$ by induction on i :

$$D_{A,i+1} = \begin{cases} D_{A,i} \cup \bigcup_{X \subsetneq A} D_{X,i+1} \cup \min\{\mathrm{cl}^0(A) \setminus D_{A,i}\} \text{ if this is perfect over } A \\ \\ D_{A,i} \cup \bigcup_{X \subsetneq A} D_{X,i+1} \text{ otherwise} \end{cases}$$

Since, given $a \in \mathrm{cl}^0(A)$, for some $j > i$ the type of $\{a\} \cup D_{A,i}$ over $G \cup \mathrm{cl}^-(A)$ is defined over $G \cup \bigcup_{X \subsetneq A} D_{X,j}$, we get by Lemma 1.7 $a \in D_{A,j}$, and thus $\bigcup_i D_{A,i} = \mathrm{cl}(A)$. Also for all j $D_{A,j} \setminus \bigcup_{X \subsetneq A} D_{X,j} \subseteq \mathrm{cl}^0(A)$, so for any $B \subseteq A$, by the induction hypothesis, $D_{A,j} \cap \mathrm{cl}(B) = D_{B,j}$.
By the ω-homogeneity for each $k \in \mathbb{N}$ we can extend $\psi_{n,k}|_{\mathrm{cl}(A)}$, whose domain is obviously a subset of $\bigcup_{X \subseteq A, |X|=n} D_{X,k}$, to

$$\psi_{A,k} : D_{A,k} \to H'.$$

More precisely, when $\psi_{A,k}$ is constructed, $A' = \psi_0(A)$, and a' is the first element in $\mathrm{cl}^0(A')$, of quantifier free type, say p, over $\psi_{A,k}(D_{A,k})$, let $j \geq k$ be such that the corresponding type $\psi^{-1}_{A,k}(p)$ over $D_{A,k}$ is realised in $D_{A,j}$, say by a. Extend by the ω-homogeneity $\psi_{A,k}$ to $\psi_{A,j} : D_{A,j} \to H'$ so that $\psi_{A,j}(a) = a'$. By doing this we have also defined $\psi_{A,m}$ for $k \leq m \leq j$ as the corresponding restrictions of $\psi_{A,j}$. Also, this process guarantees that the image of $\bigcup_{i \in \mathbb{N}} \psi_{A,i}$ is the whole of $\mathrm{cl}(A')$.
Define

$$\psi_{n+1,k} = \bigcup_{A \subseteq \mathcal{A}, |A|=n+1} \psi_{A,k}.$$

Since for any distinct $A_1, A_2 \subseteq \mathcal{A}$ of size $n+1$

$$D_{A_1,i} \cap D_{A_2,i} = D_{A_1 \cap A_2,i},$$

and $|A_1 \cap A_2| \leq n$, the definition is consistent.

Given a finite $B \subseteq \mathcal{A} \setminus \mathcal{A}_0$, the mapping $\varphi = \psi_{n+1,k}|_{\mathrm{cl}(B)}$, by Lemma 1.6, is a monomorphism. So $\psi_{n+1,k}$ is a a monomorphism.

Finally, put $\psi = \bigcup_{n \in \mathbb{N}} \psi_{n,n}$.
Obviously, the domain of ψ is $\mathrm{cl}(\mathcal{A}) = H$ and the range is

$$\bigcup_{A' \subseteq \mathcal{A}' \text{ finite}} \mathrm{cl}(A') = H'.$$

It finishes the construction of an isomorphism.□

Corollary 1 *If under the assumptions of the theorem* cl *satisfies also the exchange principle, then $\mathcal{C}$ is categorical in every uncountable cardinality.*

Proof Under the exchange principle cl-independence is equivalent to the independence in the usual pregeometry sence. In other words $\mathcal{A}$ is just a basis of H, which exists for any structure H, and by assumptions 1(ii) and 1(iv) has the same cardinality as H modulo $\aleph_0$. It follows from the theorem that any H and H' in $\mathcal{C}$ of the same uncountable cardinality are isomorphic.□

Theorem 2 *Suppose a quasi-minimal excellent class $\mathcal{C}^{\#}$ is axiomatizable by an $L_{\omega_1,\omega}$-sentence Σ and the relations $y \in \mathrm{cl}(x_1, \ldots, x_n)$ are $L_{\omega_1,\omega}$-definable for all n. Suppose also that there is an $H \in \mathcal{C}^{\#}$ containing an infinite* cl-*independent subset $\mathcal{A}$.*
Then for any infinite cardinal κ there is a structure $H_\kappa \in \mathcal{C}^{\#}$ of cardinality κ, satisfying the countable closure property. A structure with these properties is unique in $\mathcal{C}^{\#}$, provided the class satisfies the exchange principle.[2]

Proof Let $\mathcal{C}$ be the subclass of $\mathcal{C}^{\#}$ satisfying the countable closure property. Choose $A_0 \subseteq \mathcal{A}$ countable. By assumption 1 $\mathrm{cl}(A_0) \in \mathcal{C}^{\#}$. By the downward Lowenheim-Skolem theorem choose a countable submodel $H_0 \subseteq \mathrm{cl}(A_0)$ of Σ.

[2]The uniqueness statement was formulated incorrectly in the previous version of the article. I am grateful to O.Roche for pointing out the error to me.

H_0 obviously satisfies the countable closure property. So, we may assume $\mathrm{cl}(A_0) = H_0 \in \mathcal{C}$.
Suppose now $\mathrm{cl}(\mathcal{A}) = H \in \mathcal{C}$ for some $\mathcal{A}$ of cardinality $\lambda \geq \aleph_0$. Notice that by assumption 2 any cl-independent set is indiscernible. Let a be a point in $\mathcal{A}$, $\mathcal{A}' = \mathcal{A} \setminus \{a\}$ and $H' = \mathrm{cl}(\mathcal{A}')$.
Obviously, $H' \subsetneq H$ and by Theorem 1 a bijection between $\mathcal{A}'$ and $\mathcal{A}$ extends to an isomorphism $H' \cong H$. This gives rise to a sequence

$$H' = H_0 \subseteq H_1 \subseteq \ldots H_\alpha \subseteq \ldots$$

of isomorphic structures and embeddings of the form $H_\alpha = \mathrm{cl}(\mathcal{A}_\alpha)$, for α ordinals of cardinality λ, and

$$\mathcal{A}' = \mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \ldots \mathcal{A}_\alpha \subseteq \ldots$$

with unions on limit steps.
Taking

$$H_{\lambda^+} = \bigcup_{\alpha < \lambda^+} H_\alpha,$$

we get a structure of cardinality λ^+, which is still in $\mathcal{C}^\#$, since by the ω-homogeneity and the Ehrenfeucht-Fraisse criterion

$$H \equiv_{L_{\omega_1,\omega}} H_{\lambda^+}.$$

Also the countable closure property is satisfied for H_{λ^+} since

$$H_{\lambda^+} = \bigcup_{\alpha < \lambda^+} \mathrm{cl}(\mathcal{A}_\alpha).$$

Thus $H_{\lambda^+} \in \mathcal{C}$.
We can then carry on the same process up to κ. $\square$

Remark The countable closure property is not $L_{\omega_1,\omega}$-axiomatizable (yet, it is $L_{\omega_1,\omega}(Q)$ in many cases) so, the theorem proves categoricity for not necessarily $L_{\omega_1,\omega}$-axiomatizable subclass of $\mathcal{C}^\sharp$.

The following question has been suggested by the referee.

Question Does Theorem 2 hold when Σ is an $L_{\omega_1,\omega}(Q)$-sentence?

2 Examples and applications

Let F be an algebraically closed field of characteristic 0 and $F^\times$ its multiplicative group. Then, by obvious group theoretic reasons there exists an abelian divisible torsion free group H and a homomorphism ex such that the sequence

$$0 \to \mathbb{Z} \to^i H \to^{\mathrm{ex}} F^\times \to 0 \tag{1}$$

is exact. We call this sequence **a cover of the multiplicative group of the field** F. In logical terms the sequence is a two-sorted structure (H, F) with a group structure on sort H and a field structure on F and the mapping ex between the sorts.

It is more convenient in the context of the present paper to represent the sequence of the form (1) as a one sorted structure

$$H = (H, +,\ E(x,y),\ S(x,y,z),\ P(x,y,z))$$

with domain H, a group operation $+$, a binary equivalence relation E interpreted as the pullback of the equality on F by ex, and ternary relations S and P which are pullbacks of the basic ternary additive and multiplicative equations on F. In fact, P is redundant, since $\mathrm{ex}(x) \cdot \mathrm{ex}(y) = \mathrm{ex}(z)$ is a consequence of $x + y = z$ in the group H, so we omit it.

In particular, for the classical structure, the complex numbers with exponentiation: $H = F = \mathbb{C}$ and $\mathrm{ex} = \exp$, we have

$$E(x,y) \equiv x - y \in 2\pi i\mathbb{Z} \quad \text{and} \quad S(x,y,z) \equiv e^x + e^y = e^z.$$

It can easily be seen, as in [Z2], that

There is an $L_{\omega_1,\omega}$-sentence Σ (which is a finite diagram in Shelah's sense) such that any model H of Σ represents a sequence (1) with some algebraically closed field F, and conversely, any H corresponding to a sequence of the form (1) is a model of Σ.

The main result of [Z2] is the following

Theorem 3 *The class of models of Σ is quasi-minimal excellent. Moreover, the class satisfies the exchange principle and the countable closure property. Uncountable models of Σ are not homogeneous.*

Corollary 2 *$L_{\omega_1,\omega}$ theory of the structure*

$$(\mathbb{C}, +, e^x + e^y = e^z)$$

is categorical in all uncountable cardinalities.

Fields with pseudo-exponentiation. These are structures of the form $(F, +, \cdot, \mathrm{ex})$, where $(F, +, \cdot)$ is an algebraically closed field of characteristic zero. Further assumptions are:

1. ex is a homomorphism from the additive group onto the multiplicative group of the field, with cyclic kernel.

2. "The Schanuel conjecture" holds: for any additively independent $x_1, \dots, x_n \in F$ the transcendence degree of $x_1, \dots, x_n, \mathrm{ex}(x_1), \dots, \mathrm{ex}(x_n)$ is at least n.

3. Any *normal, free* system of equations over F has a generic solution in F. (We skip here the definitions of 'normal' and 'free'.)

The class of fields with pseudo-exponentiation was studied in [Z3]. A closure operator cl was introduced, and the main result of the paper states that
The class satisfying the $L_{\omega_1,\omega}$-conditions (1)-(3) above is quasi-minimal excellent.
The countable closure property (which can be formulated by a sentence of $L_{\omega_1,\omega}(Q)$) does not follow from (1)-(3). But by Theorem 2 we get that in any uncountable cardinality there is a unique field with pseudo-exponentiation satisfying the countable closure property. The one of cardinality continuum is a very close analogue of the complex numbers with exponentiation $(\mathbb{C}, +, \cdot, \exp)$ and, conjecturally, is isomorphic to it.

3 Almost quasi-minimal classes

We slightly generalize here the main result of the paper by extending the main definition. A good reason for this is the generalizations of the example of the previous section studied in [Z4], group covers of semi-Abelian varieties. Under certain arithmetic assumptions group covers of semi-Abelian varieties are almost quasi-minimal in the sense below.

Let $\mathcal{B}$ be a class of structures with a closure operator cl satisfying Assumption 1 and with an extra unary predicate U such that
for all $H \in \mathcal{B}$ *the closure operator* cl *on* H *has the property*

$$\mathrm{cl}(U(H)) = H.$$

We call such a class **almost quasi-minimal excellent** if it satisfies Assumptions 1-3 with the following correction to Assumption 2(i) (with the same meaning of H, H' and G):

(i) *if* $X \subseteq U(H)$ *and* $X' \subseteq U(H')$ *are* cl*-independent over* G *in some* $H, H' \in \mathcal{B}$, *correspondingly, then any bijection* $\varphi : X \to X'$ *is a* G*-monomorphism;*

Part (ii) and (iii) of the assumption remain unchanged.

We also redefine

Definition A subset $C \subseteq H$ is called **special** if there is a cl-independent $A \subseteq U(H)$ and $A_1, \dots A_k \subset A$ such that

$$C = \bigcup_i \mathrm{cl}(A_i).$$

With this definition in mind Assumption 3 stays unchanged.

Theorem 4 *Let* $\mathcal{B}$ *be an almost quasi-minimal excellent class,* $H, H' \in \mathcal{B}$, *both with the countable closure property,* $\mathcal{A} \subseteq U(H)$, $\mathcal{A}' \subseteq U(H')$, *independent and* $\mathrm{cl}(\mathcal{A}) = H$, $\mathrm{cl}(\mathcal{A}') = H'$. *Suppose that there is a bijection* $\psi_0 : \mathcal{A} \to \mathcal{A}'$.
Then ψ_0 *extends to an isomorphism* $\psi : H \to H'$.

Proof Same as for Theorem 1.□

Theorem 5 *Suppose a quasi-minimal excellent class* $\mathcal{B}^{\#}$ *is axiomatizable by an* $L_{\omega_1,\omega}$*-sentence* Σ *and the relations* $y \in \mathrm{cl}(x_1, \dots, x_n)$ *are* $L_{\omega_1,\omega}$*-definable*

for all n. Suppose also that there is an $H \in \mathcal{B}^{\#}$ containing an infinite cl-*independent subset $\mathcal{A} \subseteq U(H)$.*
Then for any infinite cardinal κ there is a unique structure $H_\kappa \in \mathcal{B}^{\#}$ of cardinality κ, satisfying the countable closure property.

Proof Same as for Theorem 2.$\Box$

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