CATEGORICITY

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Gődel Lecture

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1.Uniqueness, completeness and categoricity

Description of an object in a language – informally

Naive assumption on a language: *we can describe an object of our interest* **fully and completely**, *uniquely determining the object.*

Newtonian physics: such a description, a theory, is possible.

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What about **modern physics?**

In a formal logic we have

Completeness and *Categoricity*.

Completeness = 'fully and completely' in terms of the formal language itself.

Categoricity = unique interpretation in *reality*.

Categoricity – easier *acceptable* to practical mathematicians and physicists.

Completeness – easier *achievable* on the theoretical level.

Do the objects described by a formal theory exists in reality?

Depends on the type of the description, *complete* or *categorical*.

2. Categoricity. What categoricity?

Must not be based on a list of all possible configuration in all possible locations of the structure.

We want a *concise* description of a *large* structure.

Thus the (absolute) categoricity for first order languages is uninteresting.

Definition A structure $\mathbf{M}$ in a language L is said to have **λ-categorical** theory if there is exactly one, up to isomorphism, structure of cardinality λ satisfying the L-theory $\mathrm{Th}(\mathbf{M})$ of $\mathbf{M}$.

Theorem (M.Morley, 1964 confirming a conjecture by J.Los)

If a countable first-order T is λ-categorical for some $\lambda > \aleph_0$ then it is categorical for all $\lambda > \aleph_0$.

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Theorem (S.Shelah, 1983)

If an $L_{\omega_1,\omega}$*-sentence is categorical in* $\aleph_n$ *for all* n *then it is categorical (and has models) in all infinite* λ.

3. Stability, homogeneity and smoothness.

Theorem (M.Morley, S.Shelah)
Categoricity implies **stability**

Stability = dimension theory plus highly homogeneous models in all cardinalities).

Thesis *This is a weak form of smoothness*

4. **Trichotmy Conjecture**

Classical first-order λ-categorical structures for **uncountable** λ :

(1) **Trivial** structures (= only)

(2) **Linear structures** (Abelian divisible torsion-free groups; Vector spaces over a countable division ring, ...)

(3) **Algebraically closed fields**.

One can construct more complicated structures over the basic ones preserving the property of categoricity, e.g.

Algebraic groups

$$\mathrm{GL}(n, \mathbb{C}), \mathrm{PGL}(n, \mathbb{C}), ...$$

More generally,
complex **algebraic varieties** $V \subseteq \mathbb{C}^n$ **equipped with polynomially defined relations**

$$p(\bar{x}_1, \ldots, \bar{x}_m) = 0$$

in $n \times m$ variables).

Observation – example: **Compact complex spaces** in **the natural** language are ω-stable of finite Morley rank.

If a compact complex space is Kǎ̋hler it is saturated.

Trichotomy Conjecture: An uncountably categorical structure must be *classical*.

Trichotomy Conjecture (Z. 1982) *Given a f-o uncountably categorical structure* **M** *one and only one of the following holds:*

(i) the geometry of **M** is **trivial;**

(ii) the geometry of **M** is isomorphic to an **affine or projective** geometry over a countable division ring;

(iii) **M** is a structure of **algebraic geometry** over an algebraically closed field K.
(**M** is definably equivalent to the field K.)

The Trichotomy conjecture

refuted in general (E.Hrushovski, 1989)

proved under extra **Zariski** assumptions (E.Hrushovski and B.Zilber, 1994)

Also, proved in the **o-minimality – real algebraic geometry** context (Y.Peterzil, S.Starchenko, 1996).

Both the Zariski and o-minimality proofs exploit heavily *smoothness* assumptions.

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In the Zariski context

Pre-smoothness assumption on M (**dimension theorem**):

For any **closed irreducible**

$$S_1, S_2 \subseteq M^n$$

$\dim S_1 \cap S_2 \geq \dim S_1 + \dim S_2 - \dim M^n$

component-wise.

Hrushovski's counter-examples

Given a class of structures $\mathbf{M}$ with a dimension notions ∂_1, and ∂_2 we want to consider a *new function* f on $\mathbf{M}$.
On $(\mathbf{M}, \mathrm{f})$ introduce a **predimension**

$$\delta(X) = \partial_1(X \cup \mathrm{f}(X)) - \partial_2(X).$$

Consider structures $(\mathbf{M}, \mathrm{f})$ which satisfy the **Hrushovski inequality**:

$$\delta(X) \geq 0 \text{ for any finite } X \subset \mathbf{M}.$$

Amalgamate all such structures to get a *universal and homogeneous* structure in the class.
The resulting structure $(\tilde{\mathbf{M}}, \mathrm{f})$ *will have a good dimension notion and a nice geometry.*

Observation:

If **M** is a field and we want f = ex to be a group homomorphism

$$\mathrm{ex}(x_1 + x_2) = \mathrm{ex}(x_1) \cdot \mathrm{ex}(x_2)$$

then the corresponding predimension must be

$$\delta(X) = \mathrm{tr.deg}(X \cup \mathrm{ex}(X)) - \mathrm{lin.dim}_{\mathbb{Q}}(X) \geq 0.$$

The Hrushovski inequality, in the case of the complex numbers and ex = exp, is equivalent to

$$\mathrm{tr.deg}(x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n}) \geq n$$

assuming that $x_1, \ldots, x_n$ are linearly independent.

This is **the Schanuel conjecture.**

Pseudo-exponentiation

Consider the class of fields of characteristic 0 with a function ex:
$\mathrm{K}_{\mathrm{ex}} = (K, +, \cdot, \mathrm{ex})$ satisfying

EXP1: $\mathrm{ex}(x_1 + x_2) = \mathrm{ex}(x_1) \cdot \mathrm{ex}(x_2)$
EXP2: $\ker \mathrm{ex} = \pi\mathbb{Z}$

Consider the subclass satisfying the Schanuel condition

SCH : $\mathrm{tr.deg}(X \cup \mathrm{ex}(X)) - \mathrm{lin.dim}_{\mathbb{Q}}(X) \geq 0.$

Amalgamation process produces K_{ex},
an *Existentially Closed*
field with pseudo-exponentiation,

Existential Closedness property

EC: Any well-overdetermined system of equations in $+, \cdot, \mathrm{ex}$ has a solution in K_{ex}.

And

Countable Closure property

CC: 'Analytic' subsets of K^n of dimension 0 are countable.

Theorem *Given $\lambda > \aleph_0$, there is a unique model of axioms*
$\mathrm{ACF}_0 + \mathrm{EXP} + \mathrm{SCH} + \mathrm{EC} + \mathrm{CC}$
of cardinality λ.

This is a consequence of

Theorem A *The $L_{\omega_1,\omega}(Q)$-sentence*
$\mathrm{ACF}_0 + \mathrm{EXP} + \mathrm{SCH} + \mathrm{EC} + \mathrm{CC}$
is axiomatising a **quasi-minimal excellent class**.

and

Theorem B (Essentially S.Shelah 1983) *A quasi-minimal excellent class has models and is categorical in any uncountable cardinality.*

There are a series of further pseudo-analytic and analytic examples.

Theorem (A.Wilkie, P.Koiran, B.Z.)
There is an entire analytic function f which satisfies:
(i) Schanuel-type property (Hrushovski inequality)
$\text{tr.deg}(x_1, \ldots, x_n, f(x_1), \ldots f(x_n)) \geq n$
for distinct $x_1, \ldots, x_n \in \mathbb{C}$
(ii) existential closedness property,
(iii) the first-order theory T_f *of* $(\mathbb{C}, +, \cdot, f)$ *is* ω*-stable,*
(iv) the $L_{\omega_1,\omega}(Q)$*-sentence* $T_f + \text{CC}$ *is categorical in all uncountable cardinals.*

Conclusions.

1.Taking into account λ-categoricity and stability for stronger languages we can extend the list of basic geometries:

Basic geometries of stability theory:

(1) **Trivial geometry**

(2) **Linear geometry**

(3) **Algebraic geometry**

(*) **"Classical analytic" geometries** – *fusions* of (1)–(3).

2. We can predict for classical analytic geometries **Schanuel-type property** – a Hrushovski inequality.

3. We can predict for classical analytic geometries the **existential closeness** property –

– Any non-overdetermined and free (from obvious contradictions) system of equations has a solution.

Universal covers of semi-abelian varieties

$$0 \longrightarrow \Lambda \xrightarrow{i} \mathbb{C}^g \xrightarrow{\exp} \mathbb{A}(\mathbb{C}) \longrightarrow 1,$$

where exp is an analytic homomorphism from the additive group $(\mathbb{C}^g, +)$ and $\Lambda = \mathbb{Z}^N$ is a discrete subgroup of $\mathbb{C}^g$, exp a group homomorphism.

Question Is the universal cover uniquely determined by algebraic data **only**?

Universal covers of semi-abelian varieties

$$0 \longrightarrow \Lambda \xrightarrow{i} V \xrightarrow{\text{ex}} \mathbb{A}(F) \longrightarrow 1$$

$(V, +)$ a group, F an algebraically closed field, $\mathbb{A}(F)$ the F-points of semiabelian variety with the structure induced by F, ex a group homomorphism.

Reformulation Is the obvious $L_{\omega_1,\omega}$-sentence Σ_A describing the sequence uncountably **categorical?**

Theorem C Σ_A is uncountably categorical iff the following arithmetic conditions hold for **good** fields k:

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(i) (**Galois action on torsion points**) for all but finitely many prime p the group $\mathrm{Gal}(\tilde{k} : k)$ acts on the Tate module $T_p(\mathbb{A})$ as $\mathrm{GL}_N(\mathbb{Z}_p)$, and for remaining finite number of p the group acts as a subgroup of $\mathrm{GL}_N(\mathbb{Z}_p)$ of finite index;

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(ii) (**Kummer theory and heights**) given a mult-independent $a_1, \ldots, a_n$ there is an $l \in \mathbb{N}$ such that for any $m \in \mathbb{N}$ s.t. $\mathbb{A}(k)$ contains ml-torsion

$$\mathrm{Gal}(k(a_1^{\frac{1}{ml}}, \ldots, a_n^{\frac{1}{ml}}) : k(a_1^{\frac{1}{l}}, \ldots, a_n^{\frac{1}{l}})) \cong (\mathbb{Z}/m\mathbb{Z})^{Nn}.$$

Theorem C is a consequence of Keisler – Shelah theory of $L_{\omega_1,\omega}$-categoricity (*excellency*).

Galois action on roots of unity, Kummer and height theories are known for some $\mathbb{A}$.

This implies

Theorem A *The* $L_{\omega_1,\omega}(Q)$*-sentence* $\mathrm{ACF}_0 + \mathrm{EXP} + \mathrm{SCH} + \mathrm{EC} + \mathrm{CC}$ *is axiomatising a* **quasi-minimal excellent class**.

Conclusion 4 Categoricity can predict strong **arithmetic** properties.

Analytic context

Can the universal cover of the multiplicative group (torus) F^* be *compactified* as an **analytic Zariski** structure

$$0 \longrightarrow \Lambda \xrightarrow{i} V \xrightarrow{\text{ex}} F^* \longrightarrow 1?$$

This leads to the theory of **toric geometry**.

Toric geometry is the main model for **string theory and mirror symmetry**.

Hope

Uniqueness - categoricity criterion can help to find a true mathematical **model of physics**.