1991 example by Hrushovski in the light of non-commutative geometry. A case study.

B.Zilber

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In 1991, after the main result of [HZ] identifying any ample Zariski geometry as a finite cover of an algebraic curve was obtained, Hrushovski produced an example of a non-algebraic Zariski geometry ([HZ], section 10). A more specific version of the example was discussed in an earlier version of [Z1] (with Y.Peterzil). There an interpretation of the example as a 'compactification of a non-compact Riemann surface' was given.
Here I come back to this example to give a more detailed construction of the Riemann surface on the one hand, and argue that the object in question is a curve in the context of non-commutative algebraic geometry, on the other. Moreover, we show that in a richer natural language, the structure is *analytic Zariski.*
The really new material is in section 2 of the paper treating the Hrushovski structure D as a non-commutative curve. We first construct an infinite dimensional F-vector space $\mathcal{H}$ of *locally Zariski functions* $D \to \mathrm{F}$, where F is the algebraically closed field in the base of the construction. These functions are not definable in D in general but the construction is almost canonical. We note that making our choices in the course of the construction we endow the field F with a function sgn with values $+1, -1$ (in general can be values in the group of roots of unity of order n).
Next we define a natural algebra $\mathrm{A}(D)$ of linear operators on $\mathcal{H}$ and consider its irreducible representations with the spectrum of a basic operator in the sector of F marked by sgn $= +1$. The set of *polarised* irreducible representations endowed with the natural Zariski topology given by ideals of $\mathrm{A}(D)$ gives us the Hrushovski D back.
In the next section we extend this construction to a series of coordinate rings

with sings and show that they give rise to non-algebraic Zariski structures, in fact included in examples of section 10 of [HZ].
This version of the article does not answer many natural questions, and should be treated as a draft of a prospective paper. In particular, we would like to show that the relations in the Zariski structures arising from the non-commutative rings are in correspondence with certain ideals of the rings.

1 The example

1.0.1 Let $\mathbb{Z}$ act nontrivially on $\mathbb{Z}/4\mathbb{Z}$, and let G be the semidirect product: f, h are the generators of $\mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}$, respectively,

$$f^4 = 1; \quad fh = hf^{-1} \tag{1}$$

are the defining relations.
Notice, that h^2 commutes with f, and hence, the centre $Z(G)$ of G is generated by h^2 and f^2, so has index 4 in G.
Fix $a \in \mathbb{R}$, $a > 1$. We claim that there is a strongly minimal double cover $p : D \to \{\mathbb{C} - (0)\}$ in the language containing the relations induced by the field structure on $\mathbb{C}$ and unary operations f, h, defining a faithful action of G on D such that:
f, h respect the kernel of p (that is $p(x) = p(x')$ implies $p(f(x)) = p(f(x'))$ and $p(h(x)) = p(h(x'))$ and induce the (unfaithful) action on $\{\mathbb{C} - (0)\}$:

$$p(h(x)) = a^2 p(x), \quad p(f(x)) = -p(x). \tag{2}$$

Notation For $g \in G$ reduced to the form $g = f^k h^n$, $n, k \in \mathbb{Z}$, let $|g| = (-1)^k a^{2n}$.

Notice that $|g| \in \mathbb{R} \setminus (0)$ and the unfaithful action of G on $\mathbb{C} \setminus (0)$ described above can be represented as

$$yg = |g|y.$$

The final example is a structure with the universe

$$\bar{D} = D \cup \{0, \infty\}$$

and an equivalence relation E given by the kernel of p on D and 0 and ∞ are one-element equivalence classes, in particular, we can consider some $\hat{p} : \bar{D} \to \bar{D}/E = \mathbf{P}^1$;
f, h are operations, acting on D as described above and identical on 0 and ∞;
for any Zariski-closed relation S on $\mathbf{P}^1$ we have $p^{-1}(S)$ as a relation on $\bar{D}$.
Call the relations introduced above closed and introduce the dimension notion in D as induced from $\mathbb{C}$ by p. Then using elimination of quantifiers following from the Lemma above one gets

1.0.2 Theorem C of [HZ] *$\bar{D}$ is an irreducible complete 1-dimensional pre-smooth Zariski structure*

Proof Suppose $\bar{D}$ is interpretable in a pure field. We first prove that $\bar{D}$ would then be birational with an irreducible curve C. Indeed, C is a double cover of the projective line $\mathbf{P}^1$, so $\dim C = 1$ and C has at most two irreducible components, say C_1 and C_2. Supposing that they are distinct we get that $\hat{p} : C_i \to \mathbf{P}^1$ is injective for both $i = 1$ and $i = 2$. Thus, the equivalence relation E links a generic element x of C_1 to a generic element x' of C_2. By construction $x' = f^2(x)$.
Necessarily, the setwise stabiliser G^0 of C_1 (and C_2) in G is of index 2 in G, so f^2 has to be in G^0, contradicting the construction. We have proven the irreducibility.
Since every algebraic curve is birationally equivalent to a smooth one, G embeds into the subgroup of the group of birational transformations of a smooth curve C'. Now remember that any birational transformation of a smooth algebraic curve is biregular. Notice also that both f and h fix two different points of C'. But G is not such a subgroup of the automorphism group of any smooth (irreducible) algebraic curve:
The automorphism group is finite in genus 2 or higher; in genus 0, it is $\mathrm{PGL}(2, \mathbb{C})$, and any subgroup of the group fixing two different points can be embedded into the multiplicative group of the complex numbers, while G is non-commutative. In genus 1 (the elliptic curve) the group of automorphisms of the curve fixing a point is given by complex multiplication (when the elliptic curve is viewed as $\mathbb{C}/L$, L a lattice), and one sees again the contradiction in the non-commutativity of G.$\square$

1.0.3 Interpretation in the reals.

Denote $K_1 = \{x \in \mathbb{C} : \mathrm{Re}(x) \geq 0 \ \&\ \mathrm{Im}(x) > 0\}$,
$K_2 = iK_1$, $K_3 = -K_1$, $K_4 = -iK_1$, $D = K_1 \cup K_2 \cup K_3 \cup K_4$, $\bar{D} = D \cup \{0, \infty\}$.
$p(x) = x^2$, $f(x) = ix$,

$$h(x) = \begin{cases} ax, & \text{if } x \in K_1 \cup K_3 \\ -ax, & \text{if } x \in K_2 \cup K_4 \end{cases} \tag{3}$$

Notice that $f^4(x) = x$, $f(h(x)) = h(f^{-1}(x))$ for any $x \in \mathbb{C}$ and $p(h(x)) = a^2 p(x)$, $p(f(x)) = -p(x)$ and, more generally, $p(g(x)) = |g|p(x)$. It follows, by (1) and (2) that the construction is isomorphic to the Example above.

1.0.4 A Riemann surface and its "compactification"

Now we construct an open, in the real topology, subset D^0 of D.
Let

$$A = \{r \in \mathbb{R} : a^{2n} < r < a^{2n+1}, n \in \mathbb{Z}\},\ B = \{r \in \mathbb{R} : a^{2n-1} < r < a^{2n}, n \in \mathbb{Z}\},$$

$$\langle a \rangle = \{a^m : m \in \mathbb{Z}\},$$

$$\Gamma = \langle a \rangle \cup i\langle a \rangle \cup -\langle a \rangle \cup -i\langle a \rangle,$$

$$D^0 = D \setminus \Gamma.$$

We claim that D^0 can be interpreted as a non-compact Riemann manifold.
Fix a positive real ϵ and let

$$U_1 = \{u + iv :\ -\epsilon < u < 0,\ v \in A\},$$

$$V_1 = \{u + iv :\ -\epsilon < u < 0,\ v \in B\}.$$

These are open subsets of the complex plane and so are

$$K_1^\epsilon = K_1 \cup U_1 \cup V_1 \setminus \Gamma,$$

$$K_2^\epsilon = iK_1^\epsilon,\ K_3^\epsilon = -K_1^\epsilon,\ K_4^\epsilon = -iK_1^\epsilon.$$

Now we construct a Riemann manifold patching the four open charts K_i^ϵ by biholomorphic maps. First let

$$\phi_{12} :\ x \in V_1 \mapsto x \in K_2^\epsilon,$$

$$\phi_{14} : x \in U_1 \mapsto -x \in K_4^\epsilon,$$

$$\phi_{23} :\ x \in iV_1 \mapsto x \in K_3^\epsilon,$$

$$\phi_{21} :\ x \in iU_1 \mapsto -x \in K_1^\epsilon,$$

$$\phi_{34} :\ x \in -V_1 \mapsto x \in K_4^\epsilon,$$

$$\phi_{32} :\ x \in -U_1 \mapsto -x \in K_2^\epsilon,$$

$$\phi_{41} :\ x \in -iV_1 \mapsto x \in K_1^\epsilon,$$

$$\phi_{43} :\ x \in -iU_1 \mapsto -x \in K_3^\epsilon.$$

Finally, the patching maps are

$$\psi_{ij} = \phi_{ij} \cup \phi_{ji}^{-1},\ 1 \le i < j \le 4$$

with the natural domains. This is well-defined since $V_1 \cap jU_1 = \emptyset$, for $j \in \{1, -1, i, -i\}$, as well as $V_1 \cap jV_1 = \emptyset$, for $j \in \{-1, i, -i\}$.
Notice that h is holomorphic on D^0, which can be seen by identifying h on each chart with a holomorphic function, namely with ax on $K_1^\epsilon \cup K_3^\epsilon$, and $-ax$ on $K_2^\epsilon \cup K_4^\epsilon$:
near boundary of a chart, say when $x \in V_1 \subseteq K_2$, we have, by the definition (3), $h(x) = -ax \in K_4$. On the other hand

$$x \in V_1 \subseteq K_1^\epsilon \xrightarrow{a} ax \in U_1 \xrightarrow{\phi_{14}} -ax \in K_4,$$

and analogously with the other charts.
So, the Zariski structure $\bar{D}$ can be considered a compactification of the Riemann manifold D^0.

1.0.5 The relationship between the structure on the Riemann manifold D^0 and the Zariski curve $\bar{D}$ is even more interesting.
Let $D^\sharp$ be the Zariski structure $\bar{D}$ enriched by unary predicate for Γ. Then $D^\sharp$ is an analytic Zariski structure with the quasi-compact topology generated by the Zariski topology of $\bar{D}$ plus $\bar{\Gamma} = \Gamma \cup \{\infty, 0\}$, as a closed set.
The theory of $D^\sharp$ is superstable with U-rank of Γ equal to 1 and U-rank of D equals to ω.
D^0 is definable in $D^\sharp$.
Proof The structure on $D^\sharp/E$, the compactification $\mathbf{P}^1$ of the complex torus $\mathbb{C}^\times$ with the distinguished multiplicative subgroup $|G| = \langle a^2 \rangle \cup -\langle a^2 \rangle$ is well-known, see e.g. section 5 of [Z11] for the case when a is a transcendental number. It is superstable, allows elimination of quantifiers to the level of existential formulas of a simple form and as a result, one can use the specialisation criterion to prove that the topology is quasi-compact. Now we can use the fact that the finite cover does not change essential properties (Proposition 10.1 of [HZ]) to prove the same properties for $D^\sharp$ itself.
The reader is asked to check the analytic Zariski axioms using elimination of quantifiers himself. Obviously, Γ is a 0-dimensional analytic subset of the open set D^0 and h is holomorphic on D^0 (the graph of the mapping is analytic) in the sense of [PZ].

2 A coordinate algebra $\mathrm{A}(D)$ of D.

2.1 Coordinate functions on D.

We consider an abstract algebraically closed field F of characteristic zero rather than $\mathbb{C}$.

2.1.1 Given a $c \in \mathrm{F}^\times$ let ce be one of the two points in D such that $p(ce) = c^2$.
Let

$$\sqrt{\ } : \mathcal{V}_{c^2} \to \mathcal{V}_c$$

be the unique inverse of the map $v \mapsto v^2$ between the two local neighborhoods in ${}^*\mathrm{F}$, and similarly

$$\sqrt{\ } : \mathcal{V}_{-c^2} \to \mathcal{V}_{ic}.$$

Since any point $c \in \mathrm{F}^\times$ is regular for p, function p maps $\mathcal{V}_{ce}$ bijectively on $\mathcal{V}_{c^2}$ and $\mathcal{V}_{f(ce)}$ bijectively on $\mathcal{V}_{-c^2}$. Hence we get the unique bijection

$$\sqrt{p} : \begin{array}{l} \mathcal{V}_{ce} \mapsto \mathcal{V}_c \\ \mathcal{V}_{f(ce)} \mapsto \mathcal{V}_{ic} \end{array}.$$

Since $f : D \to D$ induces via p function $y \mapsto -y$ on F, local Zariski-continuous bijection $\sqrt{pf}$ satisfies

$$\sqrt{p(f(t))} = i\sqrt{p(f(t))},$$

for $t \in \mathcal{V}_c$.
We let

$$\mathcal{D}_{|c|} = \mathcal{V}_{ce} \cup \mathcal{V}_{f(ce)} \cup \mathcal{V}_{f^2(ce)} \cup \mathcal{V}_{f^3(ce)}$$

and denote $x_{|c|}$ the unique function

$$x_{|c|} : \mathcal{D}_{|c|} \to \mathcal{V}_c \cup \mathcal{V}_{ic} \cup \mathcal{V}_{-c} \cup \mathcal{V}_{-ic} \subset {}^*\mathrm{F}$$

satisfying

$$x_{|c|}(f^k(t)) = i^k \sqrt{p}(t) \ \text{ for } \ t \in \mathcal{V}_{ce}, \ k = 0, 1, 2, 3. \tag{4}$$

This identifies $\mathcal{D}_{|c|}$ with the subset $\mathcal{V}_c \cup \mathcal{V}_{ic} \cup \mathcal{V}_{-c} \cup \mathcal{V}_{-ic} = \sqrt{\mathcal{V}_{c^2} \cup \mathcal{V}_{-c^2}}$ of ${}^*\mathrm{F}$, so $x_{|c|}$ is a **local coordinate function**.
The construction depends on the choice of c and ce and so we need to satisfy

$$\mathcal{D}_{|c|} = \mathcal{D}_{|ic|} = \mathcal{D}_{|-c|} = \mathcal{D}_{|-ic|}. \tag{5}$$

For each $c \in \mathrm{F}^\times$ we arrange our correspondence

$$\{c, ic, -c, -ic\} \mapsto \{ce, (ic)e, (-c)e, (-i)c\} \subset D$$

so that $(ic)e = f(ce), (-c)e = f^2(ce), (-ic)e = f^3(ce)$. This is coherent with the definition of ce. Moreover, it implies (5) and also guarantees that the definition of $x_{|c|}$ does not depend on the choice of the representation of $\mathcal{D}_{|c|}$.

Now we assume that $D_{|c|}$ and $x_{|c|}$ are defined by the above construction (non canonically) for every $c \in \mathrm{F}^\times$.
Let

$$\mathcal{D} = \bigcup\{\mathcal{D}_{|c|} : \ c \in \mathrm{F}^\times\}$$

and

$$x(t) = x_{|c|}(t) \text{ for each } c \in \mathrm{F}^\times, t \in \mathcal{D}_{|c|},$$

a bijective function $\mathcal{D} \to {}^*\mathrm{F}$.
Denote $\check{x}$ the function $\mathcal{D}_{|c|} \to \mathrm{F}^*$ defined as

$$\check{x}(t) = a^{-1}x(h(t)).$$

This is the **dual coordinate function**. It is bijective as well since h is bijective.
In particular, it follows that $\bar{h}(t) = x^{-1}(\check{x}(t))$ is a bijection on $\mathcal{D}$ and on each $\mathcal{D}_{|c|}$.
By definitions the identity

$$x(t) \cdot x(t) = p(s) = \check{x}(t) \cdot \check{x}(t) \tag{6}$$

holds on $\mathcal{D}$.
It follows that in each neighborhood $\mathcal{V}_c$ the restriction of the functions satisfy

$$x_c = \check{x}_c \text{ or } x_c = -\check{x}_c.$$

But (1) forces that exactly for two out of four neighborhoods in $\mathcal{D}_{|c|}$ the second happens, that is we have $\check{x}_c = -x_c = x_{-c}$. It follows that applying $a^{-1}x(h(t))$ twice we get

$$\check{\check{x}}(t) = a^{-2}x(h^2(t)) = x(t), \text{ any } t \in \mathcal{D} \tag{7}$$

and

$$\bar{h}(\bar{h}(t)) = t. \tag{8}$$

2.1.2 We now note that our construction of x and $\check{x}$ have some effect on F. Define the sign function

$$\mathrm{sgn}(t) := \frac{\check{x}(t)}{x(t)}.$$

By (6) the value of $\mathrm{sgn}(t)$ is either 1 or -1.
We then have

$$\mathrm{sgn}(f(t)) = \frac{\check{x}(ft)}{x(ft)} = \frac{-i\check{x}(t)}{ix(t)} = -\mathrm{sgn}(t).$$

In other words, the sign function of $p(f^k t)$ changes the sign when k is odd and has the same sign as $p(t)$ when k is even. So, it indeed depends on the value of $p(t)$ rather than on t. We let $\mathrm{sgn}\, p(t) = \mathrm{sgn}(t)$. Hence, we obtained a sign function on $\mathrm{F}^\times$

$$\mathrm{sgn} : \mathrm{F}^\times \to \{1, -1\}.$$

Using (2) we get the properties, for all $\lambda \in \mathrm{F}^\times$:

$$\mathrm{sgn}(-\lambda) = -\mathrm{sgn}(\lambda),$$

$$\mathrm{sgn}(a^2\lambda) = \mathrm{sgn}(\lambda).$$

By (6)

$$\mathrm{sgn}\, x^2(t) = \mathrm{sgn}\, p(t) = \mathrm{sgn}(t).$$

Hence

$$\mathrm{sgn} x(t)\check{x}(t) = \mathrm{sgn}(x^2(t)\mathrm{sgn}(t)) = \mathrm{sgn}(x^2(t))\mathrm{sgn}(t) = \mathrm{sgn}(t)\mathrm{sgn}(t)) = 1,$$

that is

$$\mathrm{sgn}\, x(t)\check{x}(t) = 1, \text{ all } t \in D. \tag{9}$$

2.1.3 The space of locally Zariski functions.
A function $\psi : \mathcal{D} \to {}^*\mathrm{F}$ will be called **locally Zariski** if for any $s \in D$ the function

$$\psi_{|\mathcal{V}_s} : \ \mathcal{V}_s \to {}^*\mathrm{F}$$

is a local Zariski continuous function (in particular, definable).
By construction coordinate functions x and $\check{x}$ are locally Zariski.
The sum and and the product of two locally Zariski functions are locally Zariski.

Let $\mathcal{H}$ be the F-algebra of locally continuous functions on $\mathcal{D}$ and $\mathcal{H}_0$ its subalgebra generated by x and $\check{x}$. We may also regard both as an F-vector space and its subspace with some linear operators on them.
We define linear operators X, $\check{X}$ and F on $\mathcal{H}$ as follows:

$$\begin{aligned} X:&\ \psi(t) \mapsto x(t)\psi(t);\\ \check{X}:&\ \psi(t) \mapsto \check{x}(t)\psi(t);\\ F:&\ \psi(t) \mapsto \psi(f(t)).\end{aligned}$$

We thus have the following relations between the operators:

$$XX^{-1} = X^{-1}X = 1, \tag{10}$$

$$X^2 = \check{X}^2, \tag{11}$$

$$FX = iXF,\ \ F\check{X} = -i\check{X}F \tag{12}$$

$$F^4 = 1 \tag{13}$$

So, we define the algebra $\mathrm{A}(D) = \mathrm{A}$

$$A = \langle X, X^{-1}, F, H\rangle$$

the F-algebra with given generators subject to relations (10)-(13).
Identity (11) follows from (6). Identity (12) follows from (5).
and (13) follow from (1).

Remark. The operator identities (10)-(13) give enough information on f and h we want to recover later. Indeed, we are going to show that even "smaller" algebra is sufficient for this purpose, since it has equivalent category of irreducible modules.

2.1.4 Remark. Consider the structure $(\mathcal{D}, x)$, that is $\mathcal{D}$ in the language expanded by the locally Zariski functions of 2.1.3 ($\check{x}$ is definable using x). Obviously, its substructure (D, x) gives rise, by the procedure of 2.1.3 to the same algebra A. We claim (without giving a proof here) that $(D, x) \preccurlyeq (\mathcal{D}, x)$. Moreover, the result of the construction in 2.1.3 is independent of the choices we had to make in the course of defining $x(t)$. This brings the following

Question Consider a structure $(\mathcal{D}, x)$ which is existentially closed in the class of structures satisfying (4) and (5). What is the model-theoretic status of the theory of this structure? Is it simple?

Since $(\mathrm{F}, \mathrm{sgn})$ is definable in (D, x) we ask the same question about existentially closed theories of the field endowed by a sign function.

2.1.5 Natural irreducible representations

Given $t \in D$, consider the subspace of $\mathcal{H}$

$$\mathcal{H}_t^0 = \{\psi \in \mathcal{H} : \ \psi(gt) = 0 \ \text{ for all } g \in \langle f, \bar{h} \rangle\},$$

where $\langle f, \bar{h} \rangle$ is the 8-element group generated by f and $\bar{h}$ (see subsection 2.1.1 and (7)).

Notice that $\mathcal{H}_t^0$ is invariant under the action of $X, \check{X}$ and F and so is an A-submodule of $\mathcal{H}$. Checking definitions we see that the function $x\check{x} - x(t)\check{x}(t)$ is in $\mathcal{H}_t^0$. Hence the quotient-space is the 4-dimensional A module

$$M_t = \mathcal{H}/\mathcal{H}_t^0 = \langle \mathbf{1}, \mathbf{x}, \check{\mathbf{x}}, \mathbf{x^2} \rangle$$

where $\mathbf{1}, \mathbf{x}, \check{\mathbf{x}}, \mathbf{x^2}$ are the elements of the quotient corresponding to $1, x, \check{x}.x^2$. Denote $\lambda(t) := x(t)\check{x}(t)$. The natural representation determined by t is given by the following action on the basis of M_t :

$$X : \begin{array}{l} \mathbf{1} \mapsto \mathbf{x} \\ \mathbf{x} \mapsto \mathbf{x}^2 \\ \check{\mathbf{x}} \mapsto \lambda(t)\mathbf{1} \\ \mathbf{x}^2 \mapsto \lambda(t)\check{\mathbf{x}}. \end{array}$$

Similarly,

$$\check{X} : \begin{array}{l} \mathbf{1} \mapsto \check{\mathbf{x}} \\ \mathbf{x} \mapsto \lambda(t)\mathbf{1} \\ \check{\mathbf{x}} \mapsto \mathbf{x}^2 \\ \mathbf{x}^2 \mapsto \lambda(t)\mathbf{x}, \end{array}$$

$$F : \begin{array}{l} \mathbf{1} \mapsto \mathbf{1} \\ \mathbf{x} \mapsto i\mathbf{x} \\ \check{\mathbf{x}} \mapsto -i\check{\mathbf{x}} \\ \mathbf{x}^2 \mapsto -\mathbf{x}^2. \end{array}$$

One can easily find that the following are independent eigenvectors for X

$$\psi_0(t) = x(t)^2\mathbf{1} + x(t)\mathbf{x} + \check{x}(t)\check{\mathbf{x}} + \mathbf{x}^2,$$

with eigenvalues $x(t)$,

$$\psi_1(t) = x(t)^2\mathbf{1} + ix(t)\mathbf{x} - i\check{x}(t)\check{\mathbf{x}} - \mathbf{x}^2,$$

with eigenvalues $ix(t)$,

$$\psi_2(t) = x(t)^2\mathbf{1} - x(t)\mathbf{x} - \check{x}(t)\check{\mathbf{x}} + \mathbf{x}^2,$$

with eigenvalues $-x(t)$, and

$$\psi_3(t) = x(t)^2\mathbf{1} - ix(t)\mathbf{x} + i\check{x}(t)\check{\mathbf{x}} - \mathbf{x}^2,$$

with eigenvalues $-ix(t)$.

The same vectors are eigenvectors of $\check{X}$ with eigenvalues $\check{x}(t),\ -i\check{x}(t),\ -\check{x}(t),\ i\check{x}(t)$, correspondingly.

The module M_t is annihilated by $X\check{X} - x(t)\check{x}(t)$.

In other words, all the eigenvalues of the operator $X\check{X}$ obtained by natural representations in correspondence with (9) are of the sign $+1$. We formulate it as the extra assumption on algebra $\mathrm{A}(D)$.
We assume that F is endowed with a sign function

$$\mathrm{sgn}:\ \mathrm{F}^\times \to \{1, -1\}$$

satisfying the properties, for all $\lambda \in \mathrm{F}^\times$:

$$\mathrm{sgn}(-\lambda) = -\mathrm{sgn}(\lambda),$$

$$\mathrm{sgn}(a^2\lambda) = \mathrm{sgn}(\lambda).$$

We now declare that we restrict ourselves to those representations of A only for which

$$\textit{the spectrum of operator } X\check{X} \textit{ is of sign } +1. \tag{14}$$

2.1.6 Comments The spectrum of an operator Y in a Banach algebra B is defined in terms more general than eigenvalues. More precisely, $\mathrm{spec}\, Y$ is the set of $\lambda \in \mathbb{C}$ such that $Y - \lambda$ is not invertible. This notion strongly depends on the norm of B. Thus, our assumtion (14) is a very weak version of a norm on $\mathrm{A}(D)$.

In bigger generality we would like to use an **n-sign function**

$$\mathrm{sgn}_n:\ \mathrm{F}^\times \to \{1, \epsilon, \ldots, \epsilon^{n-1}\},$$

where ϵ is a primitive root of unity of power n, with the properties

$$\mathrm{sgn}_n(\epsilon\lambda) = \epsilon \cdot \mathrm{sgn}_n(\lambda),$$
$$\mathrm{sgn}_n(a^n\lambda) = \mathrm{sgn}_n(\lambda).$$

Example 1. The partition of $\mathbb{C}^\times$ in 4 sectors in subsection 1.0.3 gives rise to a sign function

$$\mathrm{sgn}_4(t) = i^m, \text{ if } t \in K_m, \ m = 1, 2, 3, 4.$$

Similarly we can use the partition in (3) to define sgn_2.

There are natural operators acting on spaces of functions satisfying the assumption on the spectrum above or similar to it.

Example 2. Consider the space $\mathcal{G}_\prime$ of bounded C^∞ functions $\psi(t) : \mathbb{R}_+ \to \mathbb{C}$ on the positive reals with values in $\mathbb{C}$. Consider the differentiation operator $\frac{\mathrm{d}}{\mathrm{d}t}$. Then the eigenvectors of the operators can be described as multiples of $\exp(a + bi)t$ for $a, b \in \mathbb{R}$, $a \leq 0$. And the spectrum consists of all such $a + bi$, that is equals to the closed left half plane

$$\{z \in \mathbb{C} : \ \frac{\pi}{2} \leq \arg z \leq \frac{3\pi}{2}\}.$$

Example 3. Given a real $0 \leq \alpha < 2\pi$, let

$$\mathbb{C}_\alpha = \{z \in \mathbb{C} : \ 0 \leq \arg z \leq \alpha\}.$$

Let $\mathcal{G}_{>\alpha}$ be the space of C^∞ functions $\psi(t) : \mathbb{C}_\alpha \to \mathbb{C}$ which have a bounded extension to $\psi(t) : \mathbb{C}_\beta \to \mathbb{C}$ for some real $\beta > \alpha$.
The differentiation operator $\frac{\mathrm{d}}{\mathrm{d}t}$ again has the eigenvectors of the form $\exp(ct)$ for $c \in \mathbb{C}$, $\pi/2 \leq \arg c + \alpha < 3\pi/2$. So the spectrum equals to this sector.
Choosing $\alpha = (n-2)\pi/n$ we get the spectrum

$$S_n = \{c \in \mathbb{C}^\times : \ \frac{\pi}{2} \leq \arg c < \frac{\pi}{2} + \frac{2\pi}{n}\}.$$

Let ϵ be a primitive root of unity of power n. Then

$$\mathbb{C}^\times = S_n \dot{\cup} \epsilon S_n \dot{\cup} \ldots \dot{\cup} \epsilon^{n-1} S_n$$

is a partition of $\mathbb{C}^\times$ and so determines the function

$$\mathrm{sgn}_n : \ \mathbb{C}^\times \to \{1, \epsilon, \ldots, \epsilon^{n-1}\}.$$

2.1.7 Irreducible representations of A.
By definitions (see e.g. [BG, III]) A is a PI-algebra with center $Z = \mathrm{F}[Y, Y^{-1}]$, where $Y = X\check{X}$, and is a finite extension over the center. The action of A on $\mathcal{H}$ is an infinite-dimensional representation.
All the irreducible representations of A are 4-dimensional over F and can be described as follows.
By [BG], section III.1 (a consequence of Kaplansky's theorem and Artin-Tate's lemma) every irreducible A-module M is of finite-dimension over F and is annulated by a maximal ideal of the center Z. But $Z \cong \mathrm{F}[Y, Y^{-1}]$ and so a maximal ideal I of Z is of the form $\langle Y - \lambda \rangle$, for some $\lambda \in \mathrm{F}^{\times}$. In other words $Z/I \cong \mathrm{F}$ and $Y = X\check{X}$ acts on M as the multiplication by λ.
It follows that the action of A on M reduces to the a action of the F-algebra

$$\bar{\mathrm{A}}_{\lambda} = \langle \mathbf{X}, \check{\mathbf{X}}, \mathbf{F} : \ \mathbf{X}\check{\mathbf{X}} = \lambda, \mathbf{F^4} = \mathbf{1}, \mathbf{XF} = \mathbf{iFX}, \check{\mathbf{X}}\mathbf{F} = -\mathbf{iF}\check{\mathbf{X}} \rangle,$$

where $\mathbf{X}, \check{\mathbf{X}}, \mathbf{F}$ correspond to X, $\check{X}$ and F.
Let $e \in M$ be a common eigenvector of commuting operators X and $\check{X}$, and let $X \cdot e = \mu \cdot e$. Then by (12) Fe, $F^2 e$, and $F^3 e$ are eigenvectors of X with eigenvalues $i\mu$, $-\mu$ and $-i\mu$ correspondingly.
We now want to determine the corresponding $\check{X}$-eigenvalues, in fact it suffices to determine the eigenvalue $\check{\mu}$ for e_0 only, since then the $\check{X}$-eigenvalues for e_1, e_2 and e_3 will be determined by (12) as $-i\check{\mu}, -\check{\mu}, i\check{\mu}$ correspondingly.
So let

$$\check{X} e_0 = \check{\mu} e_0.$$

Since $X\check{X} e_0 = \mu\check{\mu} e_0 = \lambda e_0$ and $\mu^4 = \lambda^2$, we have two cases here:

$$\mu^2 = \lambda, \ \check{\mu} = \mu \tag{15}$$

and

$$\mu^2 = -\lambda, \ \check{\mu} = -\mu. \tag{16}$$

Both cases are consistent with the action of A, hence we have two nonisomorphic **polarised** A**-modules** annihilated by $X\check{X} - \lambda$. Here polarised means that the module is given with its basis e_0, e_1, e_2, e_3 and two polarised modules are considered isomorphic if the correspondence between the bases is an isomorphism of modules.
We denote the modules above $M^{\sigma}(\mu)$ where $\sigma = +1$, in case (15), and $\sigma = -1$, in case (16). We denote their canonical bases as

$$\{e_0^{\sigma}(\mu), \ldots, e_3^{\sigma}(\mu)\}.$$

Remark $M^+(\mu)$ and $M^-(\mu)$ are isomorphic as modules without polarisations.

2.1.8 Note that for each module $M^\sigma(\mu)$ constructed in 2.1.7

$$I = I^\sigma(\mu) = \{L \in \mathrm{A} : \ Le^\sigma(\mu)_0 = 0\}, \tag{17}$$

is a maximal left ideal containing $X\check{X} - \mu\check{\mu}$.
Moreover, with each such ideal we associate uniquely the polarised A-module

$$M(I) = \mathrm{A}/I$$

with the choice of eigenvectors for X,

$$e_0 = 1 + I, \ e_1 = F + I, \ e_2 = F^2 + I, \ e_3 = F^3 + I$$

with eigenvalues $\mu, i\mu, -\mu, -i\mu$, correspondingly.
There is one-to-one correspondence between polarised modules $M(I)$ and maximal ideals of the form (17).

The ideals I corresponding to $M^+(\mu)$ and $M^-(\mu)$ are uniquely defined by their generators

$$I^+(\mu) = \langle (X - \mu), \ (\check{X} - \mu) \rangle$$

and

$$I^-(\mu) = \langle (X - \mu), \ (\check{X} + \mu) \rangle.$$

2.1.9 Let $2D = 2D(\mathrm{A})$ be the set of all the left modules above, that is

$$2D(\mathrm{A}) = \{I^+(\mu), I^-(\mu) : \ \mu \in \mathrm{F}^\times\} = \{I^\sigma(\mu) : \ \mu \in \mathrm{F}^\times, \ \sigma = 1, \ -1\}.$$

We consider the following functions on $2D$:

$$f : \ I^\sigma(\mu) \mapsto I^{-\sigma}(i\mu)$$

and

$$h : \ I^\sigma(\mu) \mapsto I^\sigma(\sigma a\mu).$$

We then have

$$f^2 : I^\sigma(\mu) \mapsto I^\sigma(-\mu),$$

$$fh : I^{\sigma}(\mu) \mapsto^{h} I^{\sigma}(\sigma a\mu) \mapsto^{f} I^{-\sigma}(ia\mu),$$
$$hf^3 : I^{\sigma}(\mu) \mapsto^{f} I^{-\sigma}(-i\mu) \mapsto^{h} I^{-\sigma}(ia\mu).$$

We may conclude that f^4 is identity and $fh = hf^3$ in accordance with (1).

We also consider the left ideals of A of the form

$$J(\nu) = \langle X^2 - \nu \rangle, \quad \nu \in \mathrm{F}^{\times},$$

that is generated by elements of the form $X^2 - \nu$.
We may identify $\tilde{\mathrm{F}}^{\times}$ with $\mathrm{F}^{\times}$ and introduce the usual (algebraic) Zariski relations and functions on it. In particular, we write $cJ(\nu)$ for $J(c\nu)$ and so on.
Let

$$\tilde{\mathrm{F}}^{\times} = \{J(\nu) : \ \nu \in \mathrm{F}^{\times}\}$$

and consider the map $p : \ D \to \tilde{\mathrm{F}}^{\times}$ given as

$$p : \ I^{\sigma}(\mu) \mapsto J(\mu^2).$$

This is a 4-1 map satisfying

$$pf : I^{\sigma}(\mu) \mapsto^{f} I^{-\sigma}(i\mu) \mapsto^{p} J(-\mu^2)$$

and

$$ph : I^{\sigma}(\mu) \mapsto^{h} I^{\sigma}(\sigma a\mu) \mapsto^{p} J(a^2\mu^2).$$

That is we have

$$pf = -p, \ ph = a^2 p$$

in correspondence with (2).

2.1.10 Representations with a given sign.
We finally come to use the last part of the definition of $\mathrm{A}(D)$ given by assumption (14).
This corresponds to the requirement that in the definition of A_λ in 2.1.7 we assume $\mathrm{sgn}(\lambda) = +1$. By (15) and (16),

$$\lambda = \sigma\mu^2 \text{ iff } \sigma = \mathrm{sgn}(\mu^2).$$

Consequently, we let

$$D(\mathrm{A}) = \{I^\sigma(\mu) \in 2D(\mathrm{A}) : \ \mathrm{sgn}(\mu^2) = \sigma\}.$$

Note that $D(\mathrm{A})$ is closed under f and h, by the definitions in 2.1.9.
Also, for $\lambda \in \mathrm{F}^\times$ with $\mathrm{sgn}(\lambda) = \sigma$,

$$p^{-1}(\lambda) \cap D(\mathrm{A}) = \{I^\sigma(\sqrt{\lambda}), I^\sigma(-\sqrt{\lambda})\}$$

that is the restriction of p to $D(\mathrm{A})$ is a 2-1 map satisfying (2).
This proves

Proposition $D(\mathrm{A})$ *is a model of Hrushovski's example.*

3 A generalisation

3.0.11 A coordinate algebra B_ϵ.
Let ϵ be a primitive root of unity of order n. We define an F-algebra $B_\epsilon = B$

$$B_\epsilon = \langle U, U^{-1}, V, V^{-1} \rangle$$

as the algebra given by the generators and relations (18)-(20).

$$UU^{-1} = 1, \quad VV^{-1} = 1, \tag{18}$$

$$U^n = V^n, \tag{19}$$

$$UV = \epsilon VU. \tag{20}$$

Comment. Consider $U = XF$ and $V = \check{X}F$ in algebra $\mathrm{A}(D)$ of section 2. Then (18)-(20) hold for $\epsilon = -1$, $n = 2$. That is B_{-1} is a subalgebra of A.

3.0.12 Irreducible representations of B_ϵ
The centre of $B = B_\epsilon$ obviously contains $U^n = V^n$, so the algebra is a finite over its centre Z. Hence any irreducible B-module M is finite dimensional and is annihilated by a maximal ideal of the centre. It is easy to check that $Z = \mathrm{F}[Y, Y^{-1}]$, $Y = U^n = V^n$, so a maximal ideal of Z is of the form $\langle U^n - \lambda \rangle$, $\lambda \in \mathrm{F}^\times$.
Hence M is a $\bar{B}_\lambda$-module, where

$$\bar{B}_\lambda = \langle \bar{U}, \bar{V} : \ \bar{U}^n = \bar{V}^n = \lambda \rangle.$$

Let $e \in M$ be an eigenvector of U with eigenvalue μ. Then $\mu^n = \lambda$. Let $\check{\mu}$ satisfy $\check{\mu}^n = \lambda$ and define

$$e_k = \check{\mu}^{-k} V^k e.$$

Since

$$UV^k e = \epsilon^k V^k U e = \mu \epsilon^k V^k e,$$

the e_k are U-eigenvectors for $k = 0, \ldots, n-1$ with distinct eigenvalues $\epsilon^k \mu$. It follows M is spanned by $\{e_0, \ldots, e_{n-1}\}$ and its isomorphism type as a B-module with a polarisation is determined by μ and $\check{\mu}$. Let $\sigma = \check{\mu}/\mu$. This can be any root of unity of power n. We denote this polarised module $M^\sigma(\mu)$.

Remark $M^\sigma(\mu)$ and $M^\tau(\nu)$ are isomorphic as B_ϵ-modules without polarisations, for any σ, τ and μ, ν satisfying $\sigma^n = 1 = \tau^n$ and $\mu^n = \nu^n$.

3.0.13 Let $nD = nD(B_\epsilon)$ be the set of all the polarised modules above, that is

$$nD = \{M^\sigma(\mu) : \ \sigma^n = 1, \ \mu \in \mathrm{F}^\times\}.$$

Let $\delta^n = \epsilon$, that is δ is a root of ϵ of order n. We consider the following functions on nD :

$$f_n : \ M^\sigma(\mu) \mapsto M^{\epsilon\sigma}(\delta\mu)$$

and

$$h_n : \ M^\sigma(\mu) \mapsto M^\sigma(\sigma a\mu).$$

We then have,

$$f_n^n : M^\sigma(\mu) \mapsto M^\sigma(\epsilon\mu),$$

$$f_n^{n^2} : M^\sigma(\mu) \mapsto M^\sigma(\mu),$$

that is the order of f_n is n^2.

$$f_n h_n : M^\sigma(\mu) \mapsto^h M^\sigma(\sigma a\mu) \mapsto^f M^{\epsilon\sigma}(\delta\sigma a\mu),$$

$$h_n f_n^{n^2-n+1} : M^\sigma(\mu) \mapsto^{f^{n^2-n+1}} M^{\epsilon\sigma}(\epsilon^{-1}\delta\mu) \mapsto^h M^{\epsilon\sigma}(\delta\sigma a\mu).$$

Hence

$$f_n h_n = h_n f^{n^2-n+1},$$

that is the action by conjugation by h_n on the finite cyclic group of order n^2 is determined as $f_n \mapsto f_n^{n^2-n+1}$.

We define

$$D(B_\epsilon) = \{M^\sigma(\mu) \in nD : \ \sigma = \mathrm{sgn}_n \mu^n\}.$$

It is immediate from definitions that $D(B_\epsilon)$ *is closed under* f_n *and* h_n *and that the map*

$$D(B_\epsilon) \to \mathrm{F}^\times : \quad M^\sigma(\mu) \mapsto \mu^n$$

is n-*to*-1.
The same arguments as in 1.0.2 (more details in [HZ] section 10, especially Proposition 10.1) show that $D(B_\epsilon)$ *is an irreducible Zariski curve, not definable in an algebraically closed field.*

References

[HZ] E.Hrushovski and B.Zilber, *Zariski Geometries.* Journal of AMS, 9(1996), 1-56

[Z1] B.Zilber, *Lecture notes on Zariski structures*, 1994-2004, web-page