Complex geometry and pseudo-analytic structures

Oberwolfach tutorial

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I.**Hrushovski construction of new structures** ([H1],[H2] and [Z0])

Given a class of structures $\mathbf{M}$ with a dimension notion d we want to consider a new function f on $\mathbf{M}$.
Then on $(\mathbf{M}, \mathrm{f})$ we can calculate a *predimension*

$$\delta(X) = \mathrm{d}(X \cup \mathrm{f}(X)) - \mathrm{size}(X).$$

Consider subclass of structures $(\mathbf{M}, \mathrm{f})$ which satisfy (Hrushovski inequality):

$$\delta(X) \geq 0 \text{ for any finite } X \subset \mathbf{M}.$$

Amalgamate all such structures to get a *universal and homogeneous* structure in the class.
The resulting structure $(\tilde{\mathbf{M}}, \mathrm{f})$ will have a good dimension notion and a nice geometry.
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Observation: If $\mathbf{M}$ is a field and we want $\mathrm{f} = \mathrm{ex}$ to be a group homomorphism

$$\mathrm{ex}(x_1 + x_2) = \mathrm{ex}(x_1) \cdot \mathrm{ex}(x_2)$$

then the corresponding predimension must be

$$\delta(X) = \mathrm{tr.deg}(X \cup \mathrm{ex}(X)) - \mathrm{lin.dim}(X) \geq 0.$$

The Hrushovski inequality, in the case of complex numbers and ex = exp, is equivalent to

$$\mathrm{tr.deg}(x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n}) \geq n$$

assuming that $x_1, \ldots, x_n$ are linearly independent.
This is **the Schanuel conjecture.**
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Pseudo-exponentiation

Consider the class of fields of characteristic 0 with a function ex:

$$\mathbf{K}_{\mathrm{ex}} = (K, +, \cdot, \mathrm{ex})$$

satisfying

EXP1: $\mathrm{ex}(x_1 + x_2) = \mathrm{ex}(x_1) \cdot \mathrm{ex}(x_2)$
EXP2: $\ker \mathrm{ex}$ is a cyclic additive subgroup.

Consider the subclass satisfying the Schanuel-Hrushovski condition

$$\delta(X) = \mathrm{tr.deg}(X \cup \mathrm{ex}(X)) - \mathrm{lin.dim}(X) \geq 0.$$

Amalgamation process produces a universal and homegeneous (generic) structure of any given infinite cardinality $\tilde{\mathbf{K}}_{\mathrm{ex}}(\lambda)$: an algebraically closed **field with pseudo-exponentiation**.
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More generally
Pseudo-analytic functions F :

The scheme can be repeated with other (classical) analytic functions (or systems of functions) F provided we know

I. The functional equation for F.

II. The Generalised Schanuel condition for F (GSCH).

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Questions.

1. Is there a *canonical choice* of $\tilde{\mathbf{K}}_{\rm ex}(\lambda)$ for each $\lambda > \aleph_0$?

2. What is the stability status of $\tilde{\mathbf{K}}_{\rm ex}$?

3. Does

$$\mathbb{C}_{\rm exp} \cong \tilde{\mathbf{K}}_{\rm ex}(2^{\aleph_0}), \text{ or } \mathbb{C}_{\rm exp} \equiv \tilde{\mathbf{K}}_{\rm ex}$$

hold? Compare the two theories.

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Two consequences of the construction of a generic $\tilde{\mathbf{K}}_F$.

GSCH: The (generalised) Schanuel condition.

EC: For any *free* and *normal* (non-overdetermined) irreducible system of polynomial equations over A

$$P(x_1, \ldots, x_n, y_1, \ldots, y_n) = 0$$

there exists a generic solution satisfying

$$y_i = \mathrm{ex}(x_i) \quad i = 1, \ldots, n.$$

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Admissible transformations T:

$$x_i^T := m_{i,1}x_1 + \ldots + m_{i,n}x_n$$

$$y_i^T = y_1^{m_{i,1}} \cdot \ldots \cdot y_n^{m_{i,n}}.$$

Free: $P=0$ does not imply

$$x_1^T = const,\ \text{nor}\ y_1^T = const$$

under any admissible transforamtion
Normal

$$\text{tr.deg}(x_1^T,\ldots,x_k^T,y_1^T,\ldots,y_k^T/\{a,\text{ex}\,a\}) \geq k$$

for any admissible transformation T of the generic solution (x,y) of $P=0$.

Proof of EC. Given P over K construct a 'free' extension $K' \geq K$ with solutions for
$P(x,y)=0\ \&\ y=\text{ex}(x).\Box$

. . . .

Other consequences of the construction:

Pregeometry and closure operator:
$\tilde{K}$ comes equipped with a **dimension notion**

$$\text{d}(X) = \min\{\delta(XY):\ Y \subseteq_{fin} \tilde{K}\}$$

and the corresponding closure operator

$$y \in \text{cl}(X) \Leftrightarrow \text{d}(Xy) = \text{d}(X).$$

One can construct $\tilde{K}$ with the extra assumption of **infinite dimensionality**

ID For each n there is X, $\text{d}(X) \geq n$.
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Lemma Suppose, for some class $\mathcal{H}_F$, conditions GSCH, EC, and the class $\mathcal{H}_F$ are first-order axiomatisable.
Then Th($\mathcal{H}_F$)+ GSCH + EC axiomatises the generics of $\mathcal{H}_F$.
If also ID is axiomatisable, then the theory of every infinite dimensional $\tilde{K}$ is near model complete and κ-stable, provided the number of quantifier-free types over $\tilde{K}$ is at most card $K + \kappa$.

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Corollary. The following instances of $\tilde{K}$ are ω-stable, [Z7]:

Generic function $(K, +, \cdot, f)$

$$\delta(X) = \text{tr.deg}(X \cup f(X)) - \text{size}(X) \geq 0.$$

Generic function with derivations
$(K, +, \cdot, f^{(i)} \ _{i \in \mathbb{Z}})$

$$\delta_{i_1,\dots,i_k}(X) =$$
$$= \text{tr.deg}(X \cup f^{i_1}(X) \cup \dots \cup f^{i_k}(X)) - k \cdot \text{size}(X) \geq 0,$$
$$\delta(X) = \min_{i_1,\dots,i_k} \delta_{i_1,\dots,i_k}(X).$$

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Raising to powers ([Z1],[Z2] and [Z2'])
analogue of $(\mathbb{C}, +, \cdot, x^r \ _{r \in R})$, $x^r = \exp(r \ln x)$.

Formalisation:
two-sorted structure $K^R = (V, K)$, V is a R-vector space, $K = (K, +, \cdot)$ a field of characteristic 0,

$$\text{ex} : V \to K^\times \text{ a group homomorphism.}$$

GSCH: $\delta(X) =$

$$= \text{lin.dim}_R(X) + \text{tr.deg}(\text{ex}\, X) - \text{lin.dim}_Q(X) + d \geq 0,$$

(e.g. $d = \mathrm{tr.deg}\, R$).

EC: For any *free* and *normal* system

$$L(x_1, \ldots, x_n) = a \ \&\ P(y_1, \ldots, y_n) = 0$$

(L is R-linear, P a polynomial over $\mathbb{Q}(\mathrm{ex}\, a)$)
there exists a solution satisfying

$$y_i = \mathrm{ex}(x_i) \quad i = 1, \ldots, n.$$

. . . .
Free: $L = a$ does not imply $x_1^T = const$ and $P = 0$ does not imply $y_1^T = const$ under any admissible transforamtion
Normal:

$$\mathrm{lin.dim}_R(x_1^T, \ldots, x_k^T/a) + \mathrm{tr.deg}(y_1^T, \ldots, y_k^T/\mathrm{ex}\, a) \geq k$$

for any admissible transformation T of the generic solutions x, y of $L = a$, $P = 0$.
. . . .

Remark For any finitely generated subgroup Γ of $(K^\times)^n$ there is a K^R with Γ definable. Hence the Lang property for $K^\times$ is expressible.

The **Lang property** for an semi-abelian variety A states:
for any algebraic subvariety W of A there are finitely many semi-abelian subvarieties $A_i \leq A$ such that

$$\Gamma \cap W = \cup_i (\Gamma \cap A_i).$$

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Proposition EC is first-order definable in K^R.

Proof. Use Ax's theorem on Schanuel's conjecture for function fields to get:

Claim A. Given $W(e) \subseteq K^n$, with $W \subseteq K^{n+\ell}$ algebraic subvariety defined over $\mathbb{Q}$ and $e \in K^\ell$, there is a finite collection $\mu(W) = \{M_1, \ldots, M_m\}$ of proper $\mathbb{Q}$-linear subspaces of V^n such that
for any $\mathbb{Q}$-linear subspace $N \subseteq V^n$, $b \in K^\ell$, and any positive dimensional atypical component S of the intersection $W(e) \cap \mathrm{ex}(N) \cdot b$ there is $M \in \mu(W)$ and $s \in S$ such that $S \subseteq \mathrm{ex}(M) \cdot s$.

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Theorem 1[Z2'] EC, GSCH and ID are first-order definable in class K^R. The theory of a generic member $\tilde{K}^R$ of the class is near model complete and superstable.

Proof. Use the Lang property, Claim A (Ax's Thm) and an analysis similar to one in [Z2].

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Theorem 1.1 Assume Schanuel's conjecture. Let $R \subseteq \mathbb{R}$. Then the theory $\mathbb{C}^R$ ($V = \mathbb{C}$, $K = \mathbb{C}$, $\mathrm{ex} = \exp$) is superstable and near model complete.
In particular it follows from the theory that any normal free system of exponential sums equations with real powers has a solution in $\mathbb{C}$.

Proof. Uses the proof of Theorem 1 and Khovanski's Theorem as in [Z1].

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A.Wilkie has recently proved the following.
Wilkie's Theorem Let $\mathbb{R}_{\exp,\sin_|}$ be the expansion of the real field by the exponentiation and the restricted sin and $S \prec \mathbb{R}_{\exp,\sin_|}$ its minimal elementary submodel. Let $k_1, \ldots, k_n \in \mathbb{R} + i\mathbb{R}$ be complex numbers represented in $\mathbb{R}_{\exp,\sin_|}$ such that $k_1, \ldots, k_n$ are independent over S in the sense of the pregeometry of o-minimal structure $\mathbb{R}_{\exp,\sin_|}$. Let R be the subfield of $\mathbb{C}$ generated by $k_1, \ldots, k_n$. Then the structure $\mathbb{C}^R$ of raising to powers satisfies the inequality

$$\mathrm{lin.dim}_R(X) + \mathrm{tr.deg}(\exp X) - \mathrm{lin.dim}_Q \geq 0.$$

Moreover, it satisfies the uniform version of Schanuel's conjecture.

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Theorem 1.2 Let $R \subseteq \mathbb{R}$ and satisfy the assumptions of Wilkie's Theorem. Then the theory $\mathbb{C}^R$ is superstable and near model complete.
Any normal free system of exponential sums equations with powers in R has a solution in $\mathbb{C}$.
Proof. This is a special case of Theorem 1.1 combined with Wilkie's Theorem.

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Problems 1. Extend Ax's Theorem on Schanuel's conjecture for differential fields to GSCH for Weierstrass functions (see W.D.Brownawell and K.K.Kubota, *Algebraic independence of Weierstrass functions*, Acta Arithmetica, 33 (1977), 113-148).

2. Find the functional equation, GSCH and EC for the Weierstrass function $\mathfrak{p}(\omega, x)$ as a function of two variables.

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II.

The canonical generics.

Non-elementary context.

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We consider again the fields with pseudo-exponentiation $K_{\mathrm{ex}} = (K, +, \cdot, \mathrm{ex})$.

Remark EXP, GSCH, EC and ID are $L_{\omega_1,\omega}$-definable.

Suggestion For the choice of a 'canonical' $\tilde{K}$ of cardinality λ consider the one that is prime over a (pregeometry) basis $B \subseteq \tilde{K}$, $\mathrm{card}\, B = \lambda$.

Tasks (see [Z0])

1.Prove the existence and uniqueness of 'the canonical' structure of cardinality λ.

2. Compare the canonical $\tilde{K}_{\mathrm{ex}}$ with $\mathbb{C}_{\mathrm{exp}}$.

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Observation. Any analytic structure on $\mathbb{C}$ has the following property:

If S is an analytic subset of $\mathbb{C}^n$ of dimension 0 then $\operatorname{card} S \leq \aleph_0$.

Define, for $S \subseteq \tilde{K}$ definable over A

$$\dim S = \max\{\mathrm{d}(s/A) : \ s \in S\}.$$

Countable closure property

CCP: If $\dim S = 0$ then $\operatorname{card} S \leq \aleph_0$.

Equivalently, $\mathrm{cl}(A)$ is countable, for finite A.

Remark. CCP is $L_{\omega_1,\omega}(Q)$-definable.

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Theorem (Shelah). If the theory of $\tilde{K}$ in some class is ω-stable then the prime over a basis exists and is unique for each λ.

Moreover, the canonical model has CCP.

This is the case for the theory of a generic function.

Problem Is this the case in general?

Is the non-elementary theory
...+GSCH+EC+ID+CCP
categorical in uncountable cardinals?
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Theorem 2 (Uniqueness Theorem)
1. [Z4] EXP+GSCH+EC+ID+CCP is categorical in uncountable cardinals, so the canonical field with pseudo-exponentiation K_{ex} exists and is unique in each uncountable cardinal. Moreover, its $L_{\omega_1,\omega}$-theory is 'nearly model complete'.

2. The canonical field with raising to powers K^R exists and is unique in each uncountable cardinal. Moreover, its $L_{\omega_1,\omega}$-theory is 'nearly model complete'.

Problem Prove similar statements for pseudo-Weierstrass functions and the operation of raising to powers on elliptic curves.
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Proof of the Uniqueness Theorem.

Ingredients:

1. Non-elementary stability theory and the Categoricity Theorem for quasiminimal excellent classes.

2. The group cover of $K^\times$ is quasiminimal excellent. Arithmetic analysis.

3. K_{ex} is quasiminimal excellent.
. . . .

Definition A class $\mathcal{H}$ is quasiminimal excellent if for every $M \in \mathcal{H}$ there is a closure operator cl

(i) For any $X \subseteq M$ the substructure $\mathrm{cl}(X) \in \mathcal{H}$.

(ii) (M, cl) is a pregeometry.

(iii) M is ω-homogeneous over submodels.

Definition A subset $C \subseteq M$ will be called **special** if there is a cl-independent $A \subseteq M$ and $A_1, \dots A_k \subseteq A$ such that

$$C = \bigcup_i \mathrm{cl}(A_i).$$

(iv) Suppose $C \subseteq M$ is special and $x \in \mathrm{cl}(C)$. Then the type of x over C is defined over a finite subset $C_0 \subseteq C$.

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Categoricity Theorem for quasiminimal excellent classes [Z8]. If $\mathcal{H}$ is quasiminimal excellent and has an infinite-dimensional model, then for every uncountable λ there is unique model of $\mathcal{H}$ satisfying CCP.

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The group cover of $K^\times$ is a two-sorted structure (V, K) with $V = (V, +)$, $K = (K, +, \cdot)$ and

$$\mathrm{ex} : V \to K$$

a homomorphism onto $K^\times$.

Theorem [Z3]. Each sort of group covers of $K^\times$ with standard kernel is quasiminimal excellent. (Hence the class is uncountably categorical.)

Proof. Uses Kummer's Theory and the theory of heights for (iii) and the theory of linear disjointness for (iv).

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Problem Prove similar statement for group covers of elliptic curves and abelian varieties.

Gavrilovich's Theorem 1. For elliptic curves ω-stability (non-elementary) and ω-homogeneity over models holds.
2. For abelian varieties ω-stability holds.
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The Liouville function

Definition (Wilkie) A function

$$f : \mathbb{C} \to \mathbb{C}$$

given by

$$f(x) = \sum_{i \geq 1} c_i x^i$$

is called a Liouville function, if $c_i^{-1} = a_i$ are integers satisfying

$$\text{for every } l \geq 1, \text{ for all large } i, \quad |a_{i+1}| > |a_i|^{i^l}$$

Definition (Koiran and others) **The theory of generic polynomial** is a limit of theories $\mathbb{C}_p$, where

$$p(x) = \sum_{1 \leq i \leq n} b_i x^i, \quad n \to \infty$$

are polynomial with generic coefficients.
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Theorem. Let $\mathbb{C}_f = (\mathbb{C}, +, \cdot, f)$ then

1.(Wilkie [W]) $\mathbb{C}_f$ satisfies GSCH: For any distinct nonzero complex numbers $z_1, \ldots, z_n$

$$\mathrm{tr.deg}(z_1, \ldots, z_n, f(z_1), \ldots, f(z_n)) \geq n;$$

2.(Koiran [K]) $\mathbb{C}_f$ satisfies EC and is a model of the (complete) theory ACF_0+GSCH+EC = the theory of generic polynomials;

3. $\mathbb{C}_f$ can be characterized as the unique canonical (prime over a basis) model of ACF_0+GSCH+EC of cardinality $2^{\aleph_0}$

(and the theory is ω-stable and near model complete).
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III. Poizat's fields and other exotics

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Definition A bad field is a structure $(K, +, \cdot, G)$ with $\mathrm{MR}(K) = N > 1$ and $G < K^{\times}$, a multiplicative subgroup, $\mathrm{MR}(G) = 1$.

Problem (197?) Do bad field exist?

Theorem ([BL] Baldwin and Holland, 2002) Yes, for each $N > 1$, if we drop the requirement that G is a group.

Theorem ([P2] B.Poizat, 2000) There exist an almost bad fields $(K, +, \cdot, G)$ with $\mathrm{MR}(K) = \omega \times N$ and
$G < K^{\times}$, a multiplicative subgroup, $\mathrm{MR}(G) = \omega$.
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Proof. Use Hrushovski construction.

Easy case, $G = P$ is just a subset (of black points).

$$\delta(X) = N\text{tr.deg}(X) - \text{size}(X \cap P).$$

Axioms:
GSCH: For distinct $x_1, \ldots, x_k \in P$,

$$N\text{tr.deg}(x_1, \ldots, x_k) - k \geq 0.$$

EC: Let $V \subseteq \mathbb{C}^k$ be an irreducible algebraic variety defined over a finitely generated subfield $\mathbb{Q}(C)$ which has a point
$\langle a_1, \ldots, a_k \rangle \in V$ satisfying:

$$a_i \neq a_j \text{ and } a_i \notin \text{acl}(C),\ i \neq j, i \leq k \quad (P\textbf{-free})$$

and

$$\text{tr.deg}(a_{i_1}, \ldots, a_{i_m}) \geq \frac{m}{N}, \text{ for any } i_1 < \cdots < i_m \leq k$$

$(P$**-normal**$)$.

Then there is $\langle a_1, \ldots, a_k \rangle \in V \cap P^k$.

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Claim. GSCH and EC for $(K, +, \cdot, P)$ are first order.
Proof. Easy □

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Difficult case, G is a subgroup (green points):

$$\delta(X) = N\text{tr.deg}(X) - \text{mult.rk}(X \cap G).$$

Axioms:
GSCH: For multiplicatively independent $x_1, \ldots, x_k \in G$,

$$N\text{tr.deg}(x_1, \ldots, x_k) - k \geq 0.$$

EC: Let $V \subseteq \mathbb{C}^k$ be an irreducible algebraic variety defined over a finitely generated subfield $\mathbb{Q}(C)$ and which is G-free and G-normal.

Then there is $\langle a_1, \ldots, a_k \rangle \in V \cap G^k$.

Claim. GSCH and EC for $(K, +, \cdot, P)$ are first order.
Proof. Uses Ax's Theorem. $\square$

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Problem. Explain these examples analytically.

Solutions, for $N = 2$.

$K = \mathbb{C}$.

Case, P, subset of $\mathbb{C}$.
Let ϵ, h be algebraic numbers, $\epsilon \notin \mathbb{R} \cup i\mathbb{R}$, $h \notin \mathbb{Q}$, $h\mathbb{R} \neq \epsilon\mathbb{R}$. Let

$$P = \{f(\epsilon t + hq) : t \in \mathbb{R},\ q \in \mathbb{Q}\}$$

Theorem 3.1 [Z9] $(\mathbb{C}, +, \cdot, P)$ is a model of Poizat's "black points" theory.

. . . .

Let

$$G = \{\exp(\epsilon t + hq) : t \in \mathbb{R},\ q \in \mathbb{Q}\}$$

Theorem 3.2 [Z9]. Assume Schanuel's conjecture. Then $(\mathbb{C}, +, \cdot, G)$ is a model of Poizat's "green points" theory.

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Problem. Do the general case N.

Remark Necessarily, the real dimension of the bad subset (P or G) must be $\frac{2}{N}$. Thus, the theory of fractional dimensions needs to be involved in the construction.

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Let

$$G = \{\exp(\epsilon t + hm) : t \in \mathbb{R},\ m \in \mathbb{Z}\}$$

Theorem 3.3 [Z6] Assume Schanuel's conjecture. Then

1.$\mathrm{Th}(\mathbb{C}, +, \cdot, G)$ is superstable,

$$\mathrm{U}(\mathbb{C}) = \omega \cdot 2,\ \mathrm{U}(G) = \omega.$$

2. The Miller-Speissegger spiral $G^0 = \exp(\epsilon\mathbb{R})$ is type-definable in $(\mathbb{C}, +, \cdot, G)$ as the *connected component* of G.

3. The field of reals is $L_{\omega_1,\omega}$-definable in $(\mathbb{C}, +, \cdot, G)$.

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Problems Do *canonical* models exist in class $(K, +, \cdot, G)$?

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Let $\epsilon = 1 + i$, $\bar{\epsilon} = 1 - i$, $h \in \mathbb{R} \setminus \mathbb{Q}$

$$G = \{\exp(\epsilon t + 2\pi i h m) : t \in \mathbb{R},\ m \in \mathbb{Z}\},$$

$$G^{\perp} = \{\exp(\bar{\epsilon} t + 2\pi i h m) : t \in \mathbb{R},\ m \in \mathbb{Z}\}$$

$$\Gamma = \exp(2\pi i h \mathbb{Z}).$$

Theorem 3.4 [Z6]. Assume Schanuel's conjecture. Then $\mathrm{Th}(\mathbb{C}, +, \cdot, G,\ G^{\perp},\ \Gamma)$ is superstable.
$\mathrm{U}(\mathbb{C}) = \omega \cdot 2$, $\mathrm{U}(G) = \omega$, $\mathrm{U}(G^{\perp}) = \omega$, $\mathrm{U}(\Gamma) = 1$.

Remark The 'torus' $\mathbb{C}^{\times}/\Gamma$ with the induced structure has superstable theory and possesses a type-definable involution J (an 'almost complex' conjugation) corresponding to the involution

$$J:\ \epsilon t + \bar{\epsilon} s \mapsto \epsilon t - \bar{\epsilon} s, \quad t, s \in \mathbb{R}.$$

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