A remarkable pseudo-analytic structure

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preliminary version

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1 Spiral structures

We define **the** $(\alpha, \mathbb{R})$**-canonical two-spiral structure** $(\alpha \in \mathbb{R} \setminus \pi\mathbb{Q})$ to be the structure $\mathbb{C}_{G,\Gamma,2}$ on the complex numbers $\mathbb{C}$ in the language

$$L_{G,\Gamma,2} = \{+, \cdot, G, G^\dagger, \Gamma, \Gamma^\dagger\}, \quad G, G^\dagger, \Gamma, \Gamma^\dagger \text{ unitary predicates,}$$

with the interpretation

$$\Gamma(\mathbb{C}) = \exp(\alpha\mathbb{Z}), \ \Gamma^\dagger(\mathbb{C}) = \exp(i\alpha\mathbb{Z}), \tag{1}$$

$$G(\mathbb{C}) = \exp(\epsilon\mathbb{R}) \cdot \Gamma^\dagger(\mathbb{C}), \quad G^\dagger(\mathbb{C}) = \exp(\bar{\epsilon}\mathbb{R}) \cdot \Gamma(\mathbb{C}). \tag{2}$$

Notice that by easy calculations also

$$\Gamma(\mathbb{C}) \subseteq G(\mathbb{C}) \text{ and } \Gamma^\dagger(\mathbb{C}) \subseteq G^\dagger(\mathbb{C})$$

thus we can add to the above

$$G(\mathbb{C}) = \exp(\epsilon\mathbb{R}) \cdot \Gamma(\mathbb{C}), \quad G^\dagger(\mathbb{C}) = \exp(\bar{\epsilon}\mathbb{R}) \cdot \Gamma^\dagger(\mathbb{C}). \tag{3}$$

Given a $L_{G,\Gamma,2}$-definable group H (written additively, say) and an integer m we also assume that the **enriched language** $L_{G,\Gamma,2}$ contains symbols for the ($L_{G,\Gamma,2}$-definable) subgoup mH.

1.1

The $(\alpha, \mathbb{R})$-canonical structure is a model of the first-order theory $T_{G,\Gamma,2}$.

Proof See sections 5-6.

1.2

The theory is complete, near model complete, with respect to the enriched $L_{G,\Gamma,2}$, and superstable.

Proof See sections 5-6.

1.3

The 'spiral' subgroups

$$G_0(\mathbb{C}) = \exp(\epsilon\mathbb{R}) \text{ and } G_0^{\dagger}(\mathbb{C}) = \exp(\bar{\epsilon}\mathbb{R})$$

are type-definable in $\mathbb{C}_{G,\Gamma,2}$. Also

$$G_0 \cdot G_0^{\dagger} = \mathbb{C}^*$$

Identifying $\epsilon\mathbb{R}$ with G_0 and $\bar{\epsilon}\mathbb{R}$ with $G_0^{\dagger}$ via exp we can consider the additive group $\mathbb{C} = \epsilon\mathbb{R} + \bar{\epsilon}\mathbb{R}$ as definable in the structure, along with the mapping

$$\exp : \epsilon x + \bar{\epsilon} y \mapsto \exp \epsilon x \cdot \exp \bar{\epsilon} y$$

$$\exp : \ \mathbb{C} \to \mathbb{C}^*.$$

Proof Just notice that

$$z \in G_0 \text{ iff } \bigwedge_{m \in \mathbb{N}} \exists y \in G \ y^m = z. \tag{4}$$

Similarly with $G_0^{\dagger}$.
The rest follows from the facts that

$$\epsilon\mathbb{R} + \bar{\epsilon}\mathbb{R} = \mathbb{C}.$$

1.4

Notice by straightforward calculations that the type-definable subset

$$G_0(\mathbb{C}) \cap G_0^\dagger(\mathbb{C}) = \exp(\pi\epsilon\mathbb{Z})$$

$$G(\mathbb{C}) \cap G^\dagger(\mathbb{C}) = \exp(\pi\epsilon\mathbb{Z} + \alpha\mathbb{Z} + \alpha i\mathbb{Z}) = \exp\pi\epsilon\mathbb{Z}) \cdot \Gamma(\mathbb{C}) \cdot \Gamma^\dagger(\mathbb{C}).$$

We define a new sort

$$V = \frac{G \times G^\dagger}{\Gamma \cdot \Gamma^\dagger},$$

a quotient-group with the group operation written additively. As in 1.3 we can idetify $V(\mathbb{C})$ with $\mathbb{C}$ and the map

$$\mathrm{ex}\ :\ \langle \exp(\epsilon x + i\alpha m), \exp(\bar{\epsilon} y + i\alpha n)\rangle \mathrm{mod}(\Gamma \cdot \Gamma^\dagger) \mapsto \exp(\epsilon x + \bar{\epsilon} y)$$

as $\exp$.
In this representation we obviously have

$$\ker \mathrm{ex}\ = \frac{\exp(\pi\epsilon\mathbb{Z}) \cdot \Gamma(\mathbb{C}) \cdot \Gamma^\dagger(\mathbb{C})}{\Gamma(\mathbb{C}) \cdot \Gamma^\dagger(\mathbb{C})} \cong \mathbb{Z}.$$

In the sort $V(\mathbb{C})$ we define in the first-order way

$$A = \frac{G \times \{1\}}{\Gamma \cdot \Gamma^\dagger}, \quad A^\dagger = \frac{\{1\} \times G^\dagger}{\Gamma \cdot \Gamma^\dagger}$$

$$\Delta = \frac{\Gamma \times \{1\}}{\Gamma \cdot \Gamma^\dagger}, \quad \Delta^\dagger = \frac{\{1\} \times \Gamma^\dagger}{\Gamma \cdot \Gamma^\dagger}$$

and define by types

$$A_0 = \frac{G_0 \times \{1\}}{\Gamma \cdot \Gamma^\dagger}, \quad A_0^\dagger = \frac{\{1\} \times G_0^\dagger}{\Gamma \cdot \Gamma^\dagger}.$$

Notice that we may identify

$$A_0(\mathbb{C}) = \epsilon\mathbb{R}, \quad A_0^\dagger(\mathbb{C}) = \bar{\epsilon}\mathbb{R}$$

and

$$\Delta(\mathbb{C}) = \alpha\mathbb{Z}, \quad \Delta^\dagger(\mathbb{C}) = i\alpha\mathbb{Z}.$$

1.5

The involutive transformation of $\mathbb{C}$ (as the sort $V(\mathbb{C})$)

$$\mathcal{I} : \epsilon x + \bar{\epsilon} y \mapsto \epsilon x - \bar{\epsilon} y \qquad (x, y \in \mathbb{R})$$

is type-definable in $\mathbb{C}_{G,\Gamma,2}$.

Proof Let $I(z, \check{z})$ be the type

$$\bigwedge_{m\in\mathbb{N}} (\exists u, v)\ u \in G \ \&\ v \in G^{\dagger} \ \&\ u^m \cdot v^m = z \ \&\ u^m \cdot v^{-m} = \check{z}. \tag{5}$$

It is immediate from the definition of the type that $I(z, \check{z})$ implies $z \cdot \check{z}$ satisfies the type G_0 and $z \cdot \check{z}^{-1}$ the type $G_0^{\dagger}$. This proves that I restricted to $G_0 \times G_0^{\dagger}$ defines $\mathcal{I}$ (with the translation of $\cdot$ into $+$.)

1.6

The field $\mathbb{R} \subseteq \mathbb{C}$ of reals along with all C^{∞} functions are $L_{\omega_1,\omega}$-definable in $L_{G,\Gamma,2}$.

Proof By section 4.

1.7

If $\mathbb{C}_{G,\Gamma,2} \precnapprox M$ in the language with the definable 'metric' relation $|z_1| < |z_2|$ on $\mathbb{C}$ and the exponentiation exp on the reals then there exists $1 \neq \gamma_0 \in \Gamma(M)$ such that for every positive integer m the equation

$$\gamma^m = \gamma_0$$

is solvable in $\Gamma(M)$, in other words there is no group embedding of $\Gamma(M)$ into the additive group $\hat{\mathbb{Z}}$.

Proof It is well known and easy to see that when $|z_1| < |z_2|$ is in the language the reals are definable in $\mathbb{C}$, $\mathbb{C} = \mathbb{R} + i\mathbb{R}$ and the same holds in M with corresponding formal reals.

We also have a definable embedding

$$f_\alpha : \Gamma(M) \to \mathbb{R}(M) \text{ the reals of } M$$

corresponding to

$$f_\alpha : \exp(\alpha m) \mapsto m$$

on $\mathbb{C}_{G,\Gamma,2}$. Hence $f_\alpha(G_0(M)) = \mathbb{Z}(M)$, the ring of integers of M, which must be a proper extension of standard integers, since the reals of M have been properly extended. If $\eta \in \mathbb{Z}(M)$ is a non-standard (infinite) integer then obviously $\eta! \in \mathbb{Z}(M)$ is divisible by any standard integer and hence the element $\gamma_0 \in \Gamma(M)$ corresponding to $\eta!$ has the required property.$\square$

Question. Would the same be true without assuming the exponentiation definable?

1.8

We say that a model M of $T_{G,\Gamma,2}$ is **proper** if both groups $\Gamma(M)$ and $\Gamma^\dagger(M)$ are embeddable into $\hat{\mathbb{Z}}$.

For proper M the statements above about type-definable sets and relations hold.

1.9

Assuming the model M is proper define a type-definable bijection
$\cdot^i : \Gamma(M) \mapsto \Gamma^\dagger(M)$ as a relation between z and z^i :

$$z \in \Gamma \ \& \ z^i \in \Gamma^\dagger \ \& \ z \cdot z^i \in G^0.$$

This is a group isomorphism.
Also, by construction $z \mapsto z \cdot z^i$ is a map from $\Gamma(M)$ to $G_0(M)$ and similarly we have a definable embedding into $G_0^\dagger(M)$.

The same relation written in terms of sort V gives a type-definable map

$$i\cdot : \ \Delta \to \Delta^\dagger.$$

2 The $(e^\alpha, \hat{\mathbb{Z}})$-canonical structures

A **weakly** $(e^\alpha, \hat{\mathbb{Z}})$**-canonical structure** is a model M of $T_{G,\Gamma,2}$ with both $\Gamma(M)$ and $\Gamma^\dagger(M)$ isomorphic to the additive group $\hat{Z}$.

2.1

There exists a weakly $(e^\alpha, \hat{\mathbb{Z}})$-canonical structure with any fixed cardinality $\kappa = \operatorname{card} G(M) \geq 2^{\aleph_0}$.

Proof Start with the class of two sorted structures (V, F), with F a field of characteristic zero, V an additive group, $\mathrm{ex} : V \to F^*$ a homomorphism, $A, A^\dagger \leq V$ subgroups,

$$A = A_0 \dot{+} \Delta^\dagger, \quad A^\dagger = A_0^\dagger \dot{+} \Delta,$$

where

$$\Delta \cong \Delta^\dagger \cong \hat{Z} \cong \ker \mathrm{ex}\,,$$

Δ is pseudo-generated by some $\mathrm{ex}\,(\alpha)$ and $\Delta^\dagger$ by $\mathrm{ex}\,(i\alpha)$,

$$A \cap A^\dagger = \Delta + \Delta^\dagger, \quad \Delta \cap \Delta^\dagger = 0$$

and

$$A \cap \ker \mathrm{ex} \; = A^\dagger \cap \ker \mathrm{ex} \; = 0.$$

We also assume the property

$$\delta(X) \geq 0,$$

for the $(A, A^\dagger)$-predimension δ, all finite subsets X in the class.

Show that the class has the amalgamation over strong substructures property. Thus there is an universal structure in the class satisfying usual Hrushovski-style axioms which are exactly $T_{G,\Gamma,2}$ as far as the sort F with subgroups

$$G = \mathrm{ex}\,(A), \quad G^\dagger = \mathrm{ex}\,(A^\dagger), \quad \Gamma = \mathrm{ex}\,(\Delta) \quad \text{and} \quad \Gamma^\dagger = \mathrm{ex}\,(\Delta^\dagger).$$

Since ex is bijective on G and $G^\dagger$ and G_0, $G_0^\dagger$ are type-definable, by the argument 1.3, the sort V and its subgroups A and $A^\dagger$ in the constructed model is determined uniquely by the description of G and $G^\dagger$. See sections 5-6 for details. $\square$

2.2

By elementary equivalence and the fact that for a canonical M

$$\bigcap_{n\in\mathbb{N}} \Delta(M) = 0 = \bigcap_{n\in\mathbb{N}} \Delta^\dagger(M)$$

all the facts of 1.4 hold in M with $\mathbb{Z}$ replaced by $\hat{Z}$.

2.3

The predimension δ of 2.1 generates a dimension notion under which

$$\dim G(M) = 1 = \dim A(M), \quad \dim \Gamma(M) = 0 = \dim \Delta(M)$$

and moreover $G_0(M)$ (and $A_0(M)$) is **quasi-minimal**, i.e. the set can not be divided into two dimension 1 parts of dimension 1.
We thus can consider a pregeometry on $G_0(M)$ (which is probably locally modular) and hence define a basis and thereafter the **pseudo-linear dimension of** $G_0(M)$ as the cardinality of the (pseudo-linear) basis.
As usual, any two pseudo-linear bases of $G_0(M)$ are isomorphic by an elementary bijection over $\Gamma \cdot \Gamma^\dagger$.
Using an argument from [ZPE] (which is actually Shelah's, but in a weaker form than *excellency*) we can show that

Given a pseudo-linear basis B of cardinality κ there is a unique, up to isomorphism over $\Gamma \cdot \Gamma^\dagger$, prime over $\Gamma \cdot \Gamma^\dagger$ and B weakly $(e^\alpha, \hat{\mathbb{Z}})$-canonical model of $T_{G,\Gamma,2}$.

Definition A weakly $(e^\alpha, \hat{\mathbb{Z}})$-canonical model which is prime over $\Gamma \cdot \Gamma^\dagger$ and a pseudo-linear basis B of cardinality $2^{\aleph}_0$ we will call a $(e^\alpha, \hat{\mathbb{Z}})$-canonical model.

We fix notation $\mathbb{F}$ for an $(e^\alpha, \hat{\mathbb{Z}})$-canonical model of $T_{G,\Gamma,2}$.We also abbreviate, unless we mean otherwise, *strongly* $(e^\alpha, \hat{\mathbb{Z}})$*-canonical model of* $T_{G,\Gamma,2}$ to just *canonical model.*

3 A canonical model with topology

3.1

We now introduce a topology on a definable extension of the canonical model $\mathbb{F}$. We denote $\bar{\mathbb{F}}$ the structure with

the universe $\bar{F} = F \cup \{\infty\}$, where ∞ is an extra point named by a new constant, in fact set-wise $\bar{F}$ can be naturally identufied as the projective line $\mathbf{P}(F)$;
the basic predicates on $\bar{F}$ are going to be all the Zariski-closed subsets of $\bar{F}^n$, all n along with predicates G, Γ $G^\dagger$ and $\Gamma^\dagger$ are being declared closed in $\bar{\mathbb{F}}$ Fand interpreted in $\bar{\mathbb{F}}$ as follows

$$G(\bar{F}) = G(F) \cup \{0, \infty\}, \quad G^\dagger(\bar{F}) = G^\dagger(F) \cup \{0, \infty\}$$

$$\Gamma(\bar{F}) = \Gamma(F) \cup \{0, \infty\} \text{ and } \Gamma^\dagger(\bar{F}) = \Gamma^\dagger(F).$$

Using identification of 2.1 we have also

$$A(\bar{F}) = A(F) \cup \{\Box, \Diamond\}, \quad A^\dagger(\bar{F}) = A(F) \cup \{\Box, \Diamond\}$$

where

$$\mathrm{ex}\,(\Box) = \infty \text{ and } \mathrm{ex}\,(\Diamond) = 0.$$

Also

$$\Delta(\bar{F}) = \Delta(F) \cup \{\Box, \Diamond\} \text{ and } \Delta^\dagger(\bar{F}) = \Delta^\dagger(F).$$

This leads us to

$$V(\bar{F}) = V(F) \cup \{\Box, \Diamond\}$$

with the addition becoming a ternary relation $\mathrm{SUM}(x, y, z)$ on $V(\bar{F})$ as in [Z?] which is the union of semigroups $V(F) \cup \{\Box\}$ and $V(F) \cup \{\Diamond\}$ with $\Box + \Diamond = V(\bar{F})$.

Finally, we postulate ex in the extended $\bar{\mathbb{F}}$ to be continuos. In particular

$$\ker \mathrm{ex}\,(\bar{F}) = \ker \mathrm{ex}\,(F)$$

is closed.

3.2

$A, A^{\dagger}, \Delta$ and $\Delta^{\dagger}$ are closed as images of closed sets under continuous map ex.

$$A_0(\bar{F}) = \bigcup_{m\in\mathbb{N}} mA(\bar{F}) = A_0(F) \cup \{\Box, \Diamond\}$$

and

$$A_0^{\dagger}(\bar{F}) = \bigcup_{m\in\mathbb{N}} mA^{\dagger}(\bar{F}) = A_0^{\dagger}(F) \cup \{\Box, \Diamond\}$$

are closed as intersection of closed sets.

The closure of the map $i\cdot$ of 1.9 becomes the binary relation $K \subseteq \Delta \times \Delta^{\dagger}$

$K(x, y)$ iff $(\exists z \in A_0)\,\mathrm{SUM}(x, y, z)\ \&\ x \in \Delta\ \&\ y \in \Delta^{\dagger}$ iff $y = i{\cdot}x \vee x = \Box \vee x = \Diamond$,

In particular $i\cdot$ is not a continuos mapping.

Similarly, the involutive transformation $\mathcal{I}$ (see 1.5) is not continuos, since by (5) the closure of its graph is

$$I(\bar{F}) = I(F) \cup\ \{\Box, \Diamond\} \times V(\bar{F})\ \cup\ V(\bar{F}) \times \{\Box, \Diamond\}.$$

Define a closed subset Z of A corresponding to $2\alpha\mathbb{Z}$

$$z \in Z \text{ iff } (\exists x, y)\ x \in \Delta\ \&\ y \in \Delta^{\dagger}\ \&\ \mathrm{SUM}(x, y, z)$$

it follows

$$Z(\bar{F}) = Z(F) \cup \{\Box, \Diamond\} \text{ where } Z(F) \cong \hat{Z}.$$

On the other hand $iZ(\bar{F})$, the closure of the subset corresponding to $2i\alpha\mathbb{Z}$ is defined by

$$z \in iZ \text{ iff } (\exists x, y)\ x \in \Delta\ \&\ y \in \Delta^{\dagger}\ \&\ \mathrm{SUM}(z, x, y)$$

and

$$iZ(\bar{F}) = iZ(F) \cong \hat{Z}.$$

Thus, one can say that the topology considers the 'real direction' to be compactified with extra end points $\Box$ and $\Diamond$, while the 'imaginary direction' is compact in $\mathbb{F}$ already. So, there is no continuous map between the two, though there is the type-definable involutive transformation $\mathcal{I}$ exchanging the directions.

3.3 Questions and conjectures

Prove that the canonical structure with the topology is analytic Zariski.

Prove that the topology is compatible with toric geometry.

Prove that the involutive transformation $\mathcal{I}$ of the canonical structure transforms the topology $X \mapsto X^\dagger$.

Appendix 1

4 Jordan curve on the complex plane

Let $C \subseteq \mathbb{C}$ be a Jordan curve (without endpoints) on the complex plane, that is the image of a continuous differentiable map $f : \mathbb{R} \to \mathbb{C} = \mathbb{R} + i\mathbb{R}$ which has a local differentiable inverse almost everywhere on $\mathbb{R}$. We also assume that C is not dense in $\mathbb{C}$. Then

4.1

By applying a conformal transformation

$$z \mapsto \frac{az - b}{z - c}, \quad a, b, c \in \mathbb{C}, \ |c - C| > \epsilon,$$

we may assume that C is a subset of a bounded disk in $\mathbb{C}$.

4.2

There is a point on C, which we assume to be 0 (by a choice of a and b), where C is well approximated by a straight line $\mathbb{R}$, by a choice of a, that is for any small enough positive real ϵ, for each $t \in (-\epsilon, +\epsilon)$

$$f(t) = t + ig(t), \ g(t) \in \mathbb{R}, \ |t^{-1}g(t)| < \epsilon.$$

It follows that the Jacobian matrix of the map

$$f + if : \mathbb{R}^2 \to \mathbb{C}^2$$

is non-degenerate around $(0, 0)$.

4.3

Now the definable set $B = C + iC$ is inside a disk, say of radius R, and contains an open disk around 0 of radius, say r.

4.4

We now define a binary relation $u << w$ on $\mathbb{C}$:

$$u << w \text{ iff } B \subseteq \frac{w}{u}B.$$

Obviously,

$$|\frac{w}{u}| > \frac{R}{r} \Rightarrow u << w \tag{6}$$

and

$$u << w \Rightarrow |u| < |w|. \tag{7}$$

4.5

Now we introduce an $L_{\omega_1,\omega}$-definable relation

$$u \prec w \Leftrightarrow \bigvee_{n\in\mathbb{N}} u^n << w^n.$$

We claim that $u \prec w$ is equivalent to $|u| < |w|$.
Indeed, if $|u| < |w|$ then for some n

$$|\frac{w}{u}|^n > \frac{R}{r}$$

and hence, by (6), $u^n << w^n$, and so $u \prec w$.
Conversely if, for some n, $u^n << w^n$ then by (7) $|u^n| < |w^n|$ and hence $|u| < |w|$.

4.6

The reals are $L_{\omega_1,\omega}$-definable in $(\mathbb{C}, +, \cdot, C)$.
Indeed, we have from above that the relation $|u| \leq |w|$ is $L_{\omega_1,\omega}$-definable in the structure. We now show how to define in the first order way, without parameters, the reals. It is enough to notice that

$$\{u \in \mathbb{C} : \ \forall v \ |v| = |u| \rightarrow |u+1| \geq |v+1|\} = \{u \in \mathbb{R} : u > 0\}. \tag{8}$$

□

4.7

Suppose now that

$$(\mathbb{C}, +, \cdot, C) \preccurlyeq_{\omega_1,\omega} (F, +, \cdot, C(F)).$$

Then, the $L_{\omega_1,\omega}$-formula in (8) still defines the positive formal reals, and F is the complexes over the formal reals. But also it is an $L_{\omega_1,\omega}$-property that the formal reals are Archimedean.
Hence, $F = \mathbb{C}$. *There is no proper $L_{\omega_1,\omega}$-elementary extensions.*

We also have, since the order of length continuum is definable in the structure

Corollary 4.1 *The structure $(\mathbb{C}, +, \cdot, C)$ has no uncountably categorical $L_{\omega_1,\omega}$-description, moreover it is not a member of an excellent class.*

Appendix 2

5 A superstable field with two subgroups

Let $\mathcal{C}$ be the class of two-sorted structures $\mathbf{V} = (V, \mathrm{ex}, F)$ with F a field and V a $\mathbb{Q}$-vector space, with two distinguished subgroups $A, A^\dagger \leq V$, and a homomorphism ex $: V \to F^*$ between the sorts satisfying the following axioms

$$\begin{array}{l} A \text{ and } A^\dagger \text{ are subgroups of } V \text{ of the form} \\ A_0 + \Delta,\ A_0^\dagger + \Delta^\dagger,\ A_0, A_0^\dagger \text{divisible and} \\ \Delta, \Delta^\dagger \text{elementarily equivalent to the abelian group } \mathbb{Z} \end{array} \tag{9}$$

$$\begin{array}{l} A + A^\dagger = V, \\ \Delta + \Delta^\dagger = A \cap A^\dagger \end{array} \tag{10}$$

$$\ker \mathrm{ex} \cap A = \{0\} = \ker \mathrm{ex} \cap A^\dagger; \tag{11}$$

$$\begin{array}{l} \text{the predimension condition, for every finite } X \subseteq V \\ \text{containing } d \in \Delta \text{ and } d^\dagger \in \Delta^\dagger, \\ \delta(X) = 2\mathrm{tr.d.}(\mathrm{ex}\, X) - (\mathrm{lin.d.}(X \cap A) + \mathrm{lin.d.}(X \cap A^\dagger)) \geq -2 \end{array} \tag{12}$$

It follows from (12) that at some finite $\mathbf{D} \subseteq V$ containing $d, d^\dagger$ the minimum $\delta(\mathbf{D}) \geq -2$ of $\delta(X)$ is reached and then

$$\delta(X/D) \geq 0 \ \text{ for all finite} X \subseteq V, \tag{13}$$

where $\delta(X/\mathbf{D}) = \delta(X \cup \mathbf{D}) - \delta(\mathbf{D})$. Obviously, the theory of $\mathbf{V}$ depends on the type of $\mathbf{D}$.

We also are going to consider the following conditions.

Definition Given $\mathbf{V} \in \mathcal{C}$ and an irreducible algebraic variety $W \subseteq F^{2n}$ in variables $x_1, \ldots, x_n, y_1, \ldots, y_n$ defined over a finite C we say that W is ***GH*-free** if, for $G = \mathrm{ex}\,(A)$, $H = \mathrm{ex}\,(A^\dagger)$, and a generic point $(g_1, \ldots, g_n, h_1, \ldots, h_n)$ in W

$$gp(g_1, \ldots, g_n) \cap \mathrm{acl}(C) = \{1\} = gp(h_1, \ldots, h_n) \cap \mathrm{acl}(C),$$

where $gp(X)$ stands for a multiplicative subgroup of F generated by X.

We also say that W is GH**-normal** if for every multiplicatively independent $\hat{g}_{i_1}, \ldots, \hat{g}_{i_m} \in gp(g_1, \ldots, g_n)$ and multiplicatively independent $\hat{h}_{i_1}, \ldots, \hat{h}_{i_l} \in gp(h_1, \ldots, h_n)$

$$\text{tr.d.}_C(\hat{g}_{i_1}, \ldots, \hat{g}_{i_m},\ \hat{h}_{i_1}, \ldots, \hat{h}_{i_l}) \geq \frac{m+l}{2}.$$

Any GH-free and GH-normal $W \subseteq F^{2n}$ has a generic point
$(g_1, \ldots, g_n, h_1, \ldots, h_n) \in W$ (14)
such that $g_1, \ldots, g_n \in G$ and $h_1, \ldots, h_n \in H$

We denote $\Gamma) = \mathrm{ex}\,(\Delta)$ and $\Gamma^\dagger = \mathrm{ex}\,(\Delta^\dagger)$.

As usual we write, for $X \subseteq Y \subseteq V$, $\mathbf{V} \in \mathcal{C}$, $\delta(Y/X) = \delta(Y) - \delta(X)$ and

$$X \leq Y \ \text{ iff } \ \delta(X'/X) \geq 0 \text{ for every } X \subseteq X' \subseteq Y.$$

Theorem 1 *There exists an existentially closed, with resepect to $\leq$, structure $\mathbf{V}$ in the class $\mathcal{C}$. The class of existentially closed structures $\mathcal{C}^*$ is first order axiomatizable by a theory $T(\mathcal{C}^*)$ and any its subsclass with a fixed quantifier-free type* tp$\mathbf{D}$ *of the minimal $\mathbf{D}$ is completely axiomatizable by the theory $T(\mathcal{C}^*) \cup \mathrm{tp}\mathbf{D}$, superstable with*

$$U(V) = U(F) = \omega{\times}2,\ \ U(A) = U(A^\dagger) = \omega,\ \ U(\Delta) = U(\Delta^\dagger) = 1,\ \ U(\Gamma) = U(\Gamma^\dagger) = 1.$$

More precisely, the first order theory $T(\mathcal{C}^*)$ is given by axioms (9)-(14).

Proof The construction of an existentially closed structure is given by the usual Hrushovski-Fráisse amalgamation very similar to that in [P]. It then satisfies (9)-(14). The fact that (12)-(14) is first order follows from the same argument as in [P] (uses Ax's theorem in interpretation of [2]. The superstability and rank computation is straightforward.$\square$

6 Interpretation in the complexes

Let $\alpha \in \mathbb{R}$ such that $\alpha/\pi \notin \mathbb{Q}$, $\quad \epsilon = 1 + i,\ \ \bar{\epsilon} = 1 - i,$

$$A = \epsilon\mathbb{R} + i\alpha\mathbb{Z}, \quad A^\dagger = \bar{\epsilon}\mathbb{R} + i\alpha\mathbb{Z},$$

$$\Delta = \alpha\mathbb{Z}, \quad \Delta^\dagger = i\alpha\mathbb{Z}.$$

Obviously, $A + A^\dagger = \mathbb{C}$ and $\Delta + \Delta^\dagger = A \cap A^\dagger = \epsilon\mathbb{Z} + \bar{\epsilon}\mathbb{Z}$ and (9)-(11) is satisfied.

Lemma 6.1 *Assume Schanuel's conjecture. Then, for any finite $S, T \subseteq \mathbb{R}$*

$$2\mathrm{tr.d.}\exp(\epsilon S \cup \bar{\epsilon}T) \geq \mathrm{lin.d.}S + \mathrm{lin.d.}T.$$

Proof We assume that $T \cap S = U,\ \ T = T' \cup U,\ \ S = S' \cup U$ and S', T' and U are mutually linearly independent.
Let

$$C = \epsilon U \cup \bar{\epsilon}U, \quad D = (\epsilon S \cup \bar{\epsilon}T) \setminus C.$$

Then, we have the following properties of C, D and their complex conjugates

$$C = \bar{C}; \tag{15}$$

$$\mathrm{lin.d.}(CD\bar{D}) = |C| + 2|D| = 2|U| + 2|D|; \tag{16}$$

$$\mathrm{acl}(C) = \mathrm{acl}(U); \quad \mathrm{acl}(CD) = \mathrm{acl}(C\bar{D}) = \mathrm{acl}(U \cup D); \tag{17}$$

$$\mathrm{tr.d.}\exp(CD) = \mathrm{tr.d.}\exp(C\bar{D}). \tag{18}$$

We use Schanuel's conjecture in the form

$$\partial(X) = \mathrm{tr.d.}(X \cup \exp X) - \mathrm{lin.d.}X \geq 0 \text{ for any finite } X \subseteq \mathbb{C}.$$

Notice that

$$\partial(X/Y) = \partial(XY) - \partial(Y) = \mathrm{tr.d.}(X \exp X / Y \exp Y) - \mathrm{lin.d.}(X/Y),$$

where XY here and below stands for $X \cup Y$.

Claim 1.

$$2\partial(CD) \geq \partial(CD\bar{D}) + \partial(C) + \mathrm{tr.d.}(CD\exp CD) - \mathrm{tr.d.}(C\exp(CD)).$$

Indeed,

$$2\partial(CD) - \partial(CD\bar{D}) - \partial(C) =$$

$$= 2\text{tr.d.}(CD \exp CD) - 2\text{lin.d.}(CD) - \text{tr.d.}(CD\bar{D} \exp(CD\bar{D})) + \text{lin.d.}(CD\bar{D}) - \partial(C) =$$

$$= 2\text{tr.d.}(CD \exp CD) - \text{tr.d.}(CD\bar{D} \exp(CD\bar{D})) - |C| - \partial(C) =$$

$$= 2\text{tr.d.}(CD \exp CD) - \text{tr.d.}(CD \exp(CD\bar{D})) - \text{tr.d.}(C \exp C) =$$

$$= \text{tr.d.}(CD \exp CD) - \text{tr.d.}(\exp \bar{D}/CD \exp(CD)) - \text{tr.d.}(C \exp C) \geq$$

$$\geq \text{tr.d.}(CD \exp CD) - \text{tr.d.}(C \exp(CD)) \geq 0$$

Claim proved.

Claim 2.

$$2\text{tr.d.} \exp(CD) - \text{lin.d.}(CD) \geq 0.$$

Indeed,

$$2\text{tr.d.} \exp(CD) + 2\text{tr.d.}(CD/\exp(CD)) - 2\text{lin.d.}(CD) \geq 2\partial(CD)$$

thus

$$2\text{tr.d.} \exp(CD) - \text{lin.d.}(CD) \geq 2\partial(CD) + \text{lin.d.}(CD) - 2\text{tr.d.}(CD/\exp(CD))$$

and by Claim 1

$$2\partial(CD) \geq \text{tr.d.}(CD \exp CD) - \text{tr.d.}(C \exp(CD)),$$

hence

$$2\text{tr.d.} \exp(CD) - \text{lin.d.}(CD) \geq$$

$$\geq \text{lin.d.}(CD) - 2\text{tr.d.}(CD/\exp(CD)) + \text{tr.d.}(CD \exp CD) - \text{tr.d.}(C \exp(CD)) \geq$$

$$\geq \text{lin.d.}(CD) - \text{tr.d.}(CD/\exp(CD)) + \text{tr.d.}(\exp CD) - \text{tr.d.}(C \exp(CD)) \geq$$

$$\geq \text{lin.d.}(CD) - (\text{tr.d.}(CD/\exp(CD)) + \text{tr.d.}(C/\exp(CD))) \geq |C| + |D| - (|U| + |D| + |U|) = 0.$$

Lemma 6.2 *Assume Schanuel's conjecture. Then, for any finite $W \subseteq A \cup A^\dagger$ such that $\{\alpha, i\alpha\} \subseteq W$,*

$$\delta(W) = 2\text{tr.d.} \exp(W) - (\text{lin.d.span}_{\mathbb{Q}}(W) \cap A + \text{lin.d.span}_{\mathbb{Q}}(W) \cap A^\dagger) \geq -2.$$

Proof We may assume that $\mathrm{span}_{\mathbb{Q}}(W) \cap A$ is generated by a finite set of the form $Y \cup \{i\alpha\}$, $\epsilon\alpha \in Y \subseteq \epsilon\mathbb{R}$, and $\mathrm{span}_{\mathbb{Q}}(W) \cap A^{\dagger}$ by $Z \cup \{i\alpha\}$, with $\bar{\epsilon}\alpha \in Z \subseteq \epsilon\mathbb{R}$. Notice that by definitions

$$\{\alpha, i\alpha\} \subseteq Y \cup \{i\alpha\} \text{ and } \{\alpha, i\alpha\} \subseteq Z \cup \{i\alpha\}.$$

By Lemma 6.1

$$2\text{tr.d.}(\exp(Y \cup Z)) \geq \text{lin.d.}Y + \text{lin.d.}Z. \tag{19}$$

By definitions

$$\text{tr.d.}(\exp(Y \cup Z)) = \text{tr.d.}(\exp(Y \cup Z \cup \{\alpha, i\alpha\}) = \text{tr.d.}\exp W$$

and

$$\text{lin.d.span}_{\mathbb{Q}}(W) \cap A = \text{lin.d.}Y + 1 \text{ and lin.d.span}_{\mathbb{Q}}(W) \cap A^{\dagger} = \text{lin.d.}Z + 1.$$

Hence the inequality of Lemma follows.□

Corollary 6.1 *The structure* $\mathbf{V}(\mathbb{C}, \mathbb{C})$ *with* $V = \mathbb{C}$, $F = \mathbb{C}$, $\mathrm{ex} = \exp$ *and* $A, A^{\dagger}$ *as above satisfies axioms (9) - (12) with a* $\mathbf{D}$ *dependimg on the choice of* α.

We will denote $\mathbf{V}_{\alpha}$ the structure above.

Proposition 6.1 $\mathbf{V}_{\alpha}$ *satisfies axiom (14).*

Proof This is basically the same proof as the proof of the existential closedness axiom in [ZBF].□

Thus we get

Theorem 2 $\mathbf{V}_{\alpha}$ *is a model of the superstable theory* T_{α}.

Problem Prove that some model $\mathbf{V}$ of T_{α} can be compactified to analytic Zariski.