Navier-Stokes equations of compressible flow in exterior domains – existence of weak solutions

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1 Introduction

We prove the global existence of weak solutions to the Navier-Stokes equations for compressible, barotropic flow in a domain exterior to a compact obstacle (with nonzero density at infinity). Our equations can be written in the form:

$$\frac{\partial \varrho}{\partial t} + \text{div }(\varrho \vec{u}) = 0, \tag{1.1}$$

$$\frac{\partial \varrho \vec{u}}{\partial t} + \text{div }(\varrho \vec{u} \otimes \vec{u}) + \nabla p(\varrho) = \mu \Delta \vec{u} + (\lambda + \mu) \nabla (\text{div } \vec{u}), \tag{1.2}$$

where the density $\varrho = \varrho(t,x)$ and the velocity $\vec{u} = [u^1(t,x), u^2(t,x), u^3(t,x)]$ are functions of the time $t \in (0,T)$ and the spatial coordinate $x \in \Omega$ where $\Omega \subset \mathbb{R}^3$ is a domain exterior to the compact obstacle and $p(\varrho)$ is the pressure. The viscosity coefficients μ and λ satisfy

$$\mu > 0, \quad \lambda + \mu \geq 0.$$

We prescribe the initial conditions for the density and the momentum:

$$\varrho(0) = \varrho_0, \quad (\varrho u^i)(0) = q^i, \quad i = 1,2,3; \tag{1.3}$$

together with the no-slip boundary conditions for the velocity:

$$u^i|_{\partial\Omega} = 0, \quad i = 1,2,3. \tag{1.4}$$

We also prescribe the conditions at infinity:

$$\lim_{|x|\to\infty} \varrho(t,x) = \overline{\varrho}, \qquad \lim_{|x|\to\infty} \vec{u}(t,x) = 0, \tag{1.5}$$

where $\overline{\varrho} > 0$ is a given constant. Let us suppose that the pressure satisfies the assumptions:

$$p \in \mathrm{C}^1[0,\infty), \qquad \int_0^1 \frac{p'(s)}{s}\,\mathrm{d}s < \infty, \quad \text{and} \tag{1.6}$$

$$\text{there exist } c_1 > 0, c_2 > 0 \text{ such that } c_1 z^{\gamma-1} \leq p'(z) \leq c_2 z^{\gamma-1} \ \text{ for } \gamma > \frac{3}{2}. \tag{1.7}$$

The above assumptions hold for example for isentropic flow where $p(z) = cz^\gamma$.

2 Apriori estimates and function spaces

Condition (1.5) implies that our quantity ϱ cannot belong to L^1. First of all, we formally find the apriori estimates [4]. Let ϱ be a solution of the system (1.1)-(1.5) and let us define the auxiliary function $H : [0,\infty) \to [0,\infty)$ by the formula

$$H(z) = \int_0^z \int_0^s \frac{p'(\sigma)}{\sigma}\, \mathrm{d}\sigma\, \mathrm{d}s.$$

Then, equation (1.1) is equivalent to the identity

$$\frac{\partial}{\partial t}\Big\{H(\varrho) - H(\overline{\varrho}) - H'(\overline{\varrho})(\varrho - \overline{\varrho})\Big\} + \mathrm{div}\ [\varrho\vec{u}(H'(\varrho) - H'(\overline{\varrho}))] = \vec{u}\cdot\nabla p(\varrho). \tag{2.1}$$

Multiplying equation (1.2) by $\vec{u}$, we obtain (using also (1.1))

$$\frac{\partial}{\partial t}\varrho\frac{|\vec{u}|^2}{2} + \mathrm{div}\ \left(\varrho\vec{u}\frac{|\vec{u}|^2}{2}\right) - \vec{u}\cdot\mu\Delta\vec{u} - \vec{u}\cdot(\lambda+\mu)\nabla\mathrm{div}\ \vec{u} + \vec{u}\cdot\nabla p(\varrho) = 0,$$

consequently, by (2.1), we have

$$\frac{\partial}{\partial t}\left(\varrho\frac{|\vec{u}|^2}{2} + H(\varrho) - H(\overline{\varrho}) - H'(\overline{\varrho})(\varrho - \overline{\varrho})\right) + \mathrm{div}\ \left[\varrho\vec{u}\frac{|\vec{u}|^2}{2} + \varrho\vec{u}\big(H'(\varrho) - H'(\overline{\varrho})\big)\right] -$$

$$-\vec{u}\cdot\mu\Delta\vec{u} - \vec{u}\cdot(\lambda+\mu)\nabla\mathrm{div}\ \vec{u} = 0.$$

Integrating over Ω and using the boundary conditions (1.4), (1.5) and the equation (1.1) again, we have the energy identity in the form

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\int_\Omega \varrho\frac{|\vec{u}|^2}{2} + H(\varrho) - H(\overline{\varrho}) - H'(\overline{\varrho})(\varrho - \overline{\varrho})\,\mathrm{d}x + \int_\Omega \mu\,|\nabla\vec{u}|^2 + (\lambda+\mu)\,|\mathrm{div}\ \vec{u}|^2\ \mathrm{d}x = 0. \tag{2.2}$$

Now, let us define the function $G : [0,\infty) \to [0,\infty)$ by the formula

$$G(z) = H(z) - H(\overline{\varrho}) - H'(\overline{\varrho})(z - \overline{\varrho}), \tag{2.3}$$

then we can formally deduce from (2.2), for all $t \geq 0$,

$$\int_\Omega \varrho\frac{|\vec{u}|^2}{2} + G(\varrho)\,\mathrm{d}x + \int_0^t\int_\Omega \mu\,|\nabla\vec{u}|^2 + (\lambda+\mu)\,|\mathrm{div}\ \vec{u}|^2\ \mathrm{d}x\,\mathrm{d}t \leq \int_\Omega \varrho_0\frac{|\vec{q}|^2}{2} + G(\varrho_0)\,\mathrm{d}x.$$

Motivated by the previous formula, it seems reasonable to find the relation between the quantity $\int_\Omega G(\varrho)\,\mathrm{d}x$ and norms in some L^p-spaces. Following [4], Appendix A, we can introduce the space $\mathrm{L}_2^\gamma(\Omega)$ given by the definition: Let $\delta > 0$ is a fixed number. Then

$$\mathrm{L}_2^\gamma(\Omega) = \Big\{h \in \mathrm{L}^1_{loc}(\Omega)\,|\, h\,\chi_{|h|\leq\delta} \in \mathrm{L}^2(\Omega),\ h\,\chi_{|h|>\delta} \in \mathrm{L}^\gamma(\Omega)\Big\}. \tag{2.4}$$

This definition is independent of δ. Let $\psi(x)$ is any convex function on $[0,\infty)$ which is equal to c_1x^2 for x small and to c_2x^γ for x large where c_1 and c_2 are positive constants. Then $\mathrm{L}_2^\gamma(\Omega) = \{h \in \mathrm{L}^1_{loc}(\Omega) \,|\, \psi(h) \in \mathrm{L}^1(\Omega)\}$, therefore $\mathrm{L}_2^\gamma(\Omega)$ is an Orlicz space and we can define the Luxembourg norm in $\mathrm{L}_2^\gamma(\Omega)$.
We have the following lemma:

Lemma 1: The space $\mathrm{L}_2^\gamma(\Omega)$ is a separable, reflexive Banach space. Moreover, $G(\varrho) \in \mathrm{L}^1(\Omega)$ if and only if $\varrho - \overline{\varrho} \in \mathrm{L}_2^\gamma(\Omega)$.

Proof: See [4], Lemma 5.3 and Appendix A.

3 Finite energy weak solution

Motivated by the previous section, we introduce the concept of finite energy weak solutions (see [2] and [4] for details) of the problem (1.1) – (1.5).

Definition: *We shall say that* ϱ, $\vec{u}$ *is a finite energy weak solution of the problem* (1.1), (1.2), (1.4) *and* (1.5) *on* $(0,T)\times\Omega$ *if the following four conditions are satisfied:*

- $\varrho \geq 0,\ \varrho - \overline{\varrho} \in \mathrm{L}^\infty(0,T;\mathrm{L}_2^\gamma(\Omega)),\ u^i \in \mathrm{L}^2(0,T;\mathrm{W}_0^{1,2}(\Omega)),\ i = 1,2;$
- *the energy* $E(t) = E[\varrho,\vec{u}](t) = \int_\Omega \frac{1}{2}\varrho\,|\vec{u}|^2 + G(\varrho)\,dx$ *satisfies the energy inequality*

$$E(t) + \int_0^t \int_\Omega \mu\,|\nabla\vec{u}|^2 + (\lambda+\mu)\,|\mathrm{div}\ \vec{u}|^2\ dx\,dt \leq E(0) \quad \textit{for a.e. } t \in (0,T); \tag{3.1}$$

- *the equations* (1.1), (1.2) *are satisfied in* $D'((0,T)\times\Omega)$*; moreover,* (1.1) *holds in* $D'((0,T)\times\mathbb{R}^3)$ *provided* ϱ, $\vec{u}$ *were prolonged to be zero on* $\mathbb{R}^3 - \Omega$*;*
- *the equation* (1.1) *is satisfied in the sense of renormalized solutions, it means that*

$$b(\varrho)_t + \mathrm{div}\ (b(\varrho)\vec{u}) + (b'(\varrho)\varrho - b(\varrho))\mathrm{div}\ \vec{u} = 0 \tag{3.2}$$

  holds in $D'((0,T)\times\Omega)$ *for any* $b \in \mathrm{C}^1(\mathbb{R})$ *such that*

$$b'(z) \equiv 0 \ \text{ for all } z \in \mathbb{R} \text{ large enough, say, } z \geq M \tag{3.3}$$

  where the constant M *may vary for different functions* b*.*

In the following, we shall introduce that the initial data ϱ_0, q^i, $i = 1,2$, satisfy compatibility conditions of the form:

$$\varrho_0 - \overline{\varrho} \in \mathrm{L}_2^\gamma(\Omega),\ \varrho_0 \geq 0,\ q^i(x) = 0 \text{ whenever } \varrho_0(x) = 0,\ \frac{|q^i|^2}{\varrho_0} \in \mathrm{L}^1(\Omega),\ i = 1,2,3. \tag{3.4}$$

Our main result reads as follows:

Theorem 1: *Let* $A \subset \mathbb{R}^3$ *is a bounded, open domain of the class* $C^{2+\nu}$, $\nu > 0$. *Let* $\Omega = A^C = \mathbb{R}^3 - A$. *Let the data* ϱ_0, q^i *satisfy the compatibility conditions* (3.4) *and the pressure satisfies* (1.6) *and* (1.7).

Then given $T > 0$ arbitrary, there exists a finite energy weak solution ϱ, $\vec{u}$ of the problem (1.1), (1.2), (1.4), (1.5) *satisfying the initial conditions* (1.3).

Let $n > 0$ is sufficiently large and let us denote the ball of the diameter n by B_n, i.e.

$$B_n = \{x \in \mathbb{R}^3 \,:\, \|x\| \leq n\}.$$

Then, we can obtain the existence result for the Navier-Stokes equation for compressible fluid in the bounded domain $B_n \cap \Omega$ using the approzimation scheme introduced in [2] or [4]. The presented proof of Theorem 1 will be done by passing to the limit for $n \to \infty$.

4 Approximation

The starting point of our proof will be the following lemma. It is the existence result for bounded domains proven in [2].

Lemma 2: *Let $\Omega \subset \mathbb{R}^3$ is a domain exterior to the compact obstacle of the class $C^{2+\nu}$, $\nu > 0$. Let the data ϱ_0, q^i satisfy the compatibility conditions* (3.4). *Let $n > 0$ is sufficiently large such that*

$$(\mathbb{R}^3 - \Omega) \subset B_{n-1} = \{x \in \mathbb{R}^3 \,:\, \|x\| \leq n-1\}$$

and let us define the domain

$$\Omega_n = \Omega \cap B_n = \Omega \cap \{x \in \mathbb{R}^3 \,:\, \|x\| \leq n\}.$$

Let $T > 0$. Then there exist functions ϱ_n, $\vec{u}_n$ such that

- $\varrho_n \geq 0$, $\varrho_n - \overline{\varrho} \in \mathrm{L}^\infty(0,T;\mathrm{L}^\gamma(\Omega_n))$, $u_n^i \in \mathrm{L}^2(0,T;\mathrm{W}_0^{1,2}(\Omega_n))$, $i = 1,2$;
- *the energy $E_n(t) = E[\varrho_n, \vec{u}_n](t) = \int_{\Omega_n} \frac{1}{2}\varrho_n |\vec{u}_n|^2 + H(\varrho_n)\, dx$ satisfies the energy inequality*

$$\frac{\mathrm{d}\, E_n}{\mathrm{d}\, t} + \int_{\Omega_n} \mu \, |\nabla \vec{u}_n|^2 + (\lambda + \mu)\, |\mathrm{div}\ \vec{u}_n|^2 \ dx \leq 0 \quad in\ D'(0,T); \tag{4.1}$$

- *the equations* (1.1) *and* (1.2) *are satisfied in $D'((0,T)\times\Omega_n)$; moreover,* (1.1) *holds in $D'((0,T)\times\mathbb{R}^3)$ provided ϱ_n, $\vec{u}_n$ were prolonged to be zero on $\mathbb{R}^3 - \Omega_n$;*
- *the equation (1.1) is satisfied in the sense of renormalized solutions, it means that*

$$b(\varrho_n)_t + \mathrm{div}\ (b(\varrho_n)\vec{u}_n) + (b'(\varrho_n)\varrho_n - b(\varrho_n))\mathrm{div}\ \vec{u}_n = 0$$

  holds in $D'((0,T)\times\mathbb{R}^3)$ for any $b \in \mathrm{C}^1(\mathbb{R})$ such that (3.3) *holds (provided ϱ_n, $\vec{u}_n$ are set zero outside Ω_n);*
- $\varrho_n(0) = \chi_{\Omega_n}\varrho_0, \quad (\varrho u_n^i)(0) = \chi_{\Omega_n} q^i, \quad i = 1,2,3.$

Proof: See [2], Theorem 1.1.

Let $n > 0$ is sufficiently large. Then we can use the previous Lemma 2 to obtain a functions ϱ_n, $\vec{u}_n$ defined on the set $(0,T) \times \Omega_n$. We will prolonge this function to the set $(0,T) \times \Omega$ by the formula

$$\varrho_n = \overline{\varrho} \quad \text{in } (0,T) \times \mathbb{R}^3 - B_n, \qquad\qquad \vec{u}_n = 0 \quad \text{in } (0,T) \times \mathbb{R}^3 - B_n. \tag{4.2}$$

Now, we can recall the basic estimates for a finite energy weak solutions ϱ_n, $\vec{u}_n$ that can be deduced from the continuity equation and from the energy inequality.

Lemma 3 : *Let the pressure satisfies the hypotheses* (1.6) *and* (1.7). *Let* ϱ_n, $\vec{u}_n$ *be a finite energy weak solution of the problem* (1.1) – (1.5) *on* $(0,T) \times \Omega_n$ *obtained by Lemma* 2. *Let us prolonge the functions* ϱ_n *and* $\vec{u}_n$ *by the formula* (4.2). *Then*

$$\int_\Omega \varrho_n(t) - \overline{\varrho}\, dx = \int_{\Omega_n} \varrho_n(0) - \overline{\varrho}\, dx \quad \textit{for any} \quad t \in [0,T]. \tag{4.3}$$

Moreover,

$$\operatorname*{ess\,sup}_{t\in[0,T]} \left(\|\varrho_n(t) - \overline{\varrho}\|_{L_2^\gamma(\Omega)} + \|\sqrt{\varrho_n}\vec{u}_n\|_{L^2(\Omega)^3} \right) + \int_0^T \|\nabla \vec{u}_n(t)\|^2_{L^2(\Omega)}\, dt \le cE_0, \tag{4.4}$$

where c *is a constant and*

$$E_0 = \int_\Omega \varrho_0 \frac{|\vec{q}|^2}{2} + G(\varrho_0)\, dx.$$

Proof: The property (4.3) is a consequence of the continuity equation (1.1), the proof can be found in [3], Proposition 2.1.

Now, we can rewrite (4.3) to the form

$$\int_\Omega -H'(\overline{\varrho})\,(\varrho_n(t) - \overline{\varrho})\, \mathrm{d}x = \int_\Omega -H'(\overline{\varrho})\,(\varrho_n(0) - \overline{\varrho})\, \mathrm{d}x \quad \text{for any} \quad t \in [0,T]. \tag{4.5}$$

By virtue of the energy inequality (4.1), we have

$$\int_{\Omega_n} \frac{1}{2}\varrho_n\, |\vec{u}_n|^2 + H(\varrho_n)\, \mathrm{d}x\,(t) + \int_0^t \int_{\Omega_n} \mu\, |\nabla \vec{u}_n|^2 + (\lambda+\mu)\, |\mathrm{div}\ \vec{u}_n|^2\, \mathrm{d}x\, \mathrm{d}\tau \le$$

$$\le \int_{\Omega_n} \varrho_0 \frac{|\vec{q}|^2}{2} + H(\varrho_0)\, \mathrm{d}x \tag{4.6}$$

for a.e. $t \in [0,T]$.

Consequently, adding (4.5) and (4.6) and using (4.2), we have

$$\operatorname*{ess\,sup}_{t\in[0,T]} \int_\Omega \frac{1}{2}\varrho_n\, |\vec{u}_n|^2 + G(\varrho_n)\, \mathrm{d}x + \int_0^T \int_\Omega \mu\, |\nabla \vec{u}_n|^2 + (\lambda+\mu)\, |\mathrm{div}\ \vec{u}_n|^2\, \mathrm{d}x\, \mathrm{d}t \le$$

$$\int_{\Omega_n} \varrho_0 \frac{|\vec{q}|^2}{2} + H(\varrho_0) - H(\overline{\varrho}) - H'(\overline{\varrho})\,(\varrho_n(0) - \overline{\varrho})\,\mathrm{d}x \le$$

$$\le \int_{\Omega} \varrho_0 \frac{|\vec{q}|^2}{2} + G(\varrho_0)\,\mathrm{d}x = E_0, \tag{4.7}$$

which gives (4.4).

Q.E.D.

Next estimate has a local character and can be formally proved by testing equation (1.2) by the quantity $\triangle^{-1}\partial_{x_i}\varrho^\theta$ – see [4] for details.

Lemma 4: *Let the pressure p satisfy the hypotheses* (1.6) *and* (1.7). *Let ϱ_n, $\vec{u}_n$ be a finite energy weak solution of the problem* (1.1) – (1.5) *on* $(0,T)\times\Omega_n$ *obtained by Lemma* 2. *Let us prolonge the functions ϱ_n and $\vec{u}_n$ by the formula* (4.2). *Let the data ϱ_0, q^i satisfy the compatibility conditions (3.4). Let $B \subset \overline{B} \subset \Omega$ be a given ball. Then there exist $\theta > 0$ and a constant c depending on ϱ_0, q_i, B and T such that*

$$\int_0^T \int_B p(\varrho_n)\varrho_n^\theta \, dx\, dt \le c(B) \tag{4.8}$$

and

$$\int_0^T \int_B G(\varrho_n)\varrho_n^\theta \, dx\, dt \le c(B). \tag{4.9}$$

Proof: The estimate (4.8) can be found e.g. in [2], Proposition 2.3. The estimate (4.9) is an consequence of (4.8) and (4.4).

Q.E.D.

5 Weak convergence

By virtue of the Lemma 2, we can find the sequence ϱ_n, $\vec{u}_n$ of finite energy weak solutions of the problem (1.1) – (1.5) on $(0,T)\times\Omega_n$. Let us prolonge the functions ϱ_n and $\vec{u}_n$ by the formula (4.2). Because of (4.4), we have

$$\operatorname*{ess\,sup}_{t\in[0,T]} \|\varrho_n(t) - \overline{\varrho}\|_{L_2^\gamma(\Omega)}.$$

Thus there exists $w \in \mathrm{L}^\infty(0,T;\mathrm{L}_2^\gamma(\Omega))$ and the subsequence of ϱ_n such that

$$\varrho_n - \overline{\varrho} \to w \quad \text{weakly star in } \mathrm{L}^\infty(0,T;\mathrm{L}_2^\gamma(\Omega))$$

passing to the subsequence as the case may be. Let us denote $\varrho = w + \overline{\varrho}$. Then we have

$$\varrho_n - \overline{\varrho} \to \varrho - \overline{\varrho} \quad \text{weakly star in } \mathrm{L}^\infty(0,T;\mathrm{L}_2^\gamma(\Omega)). \tag{5.1}$$

Similarly, using (4.4) again, we have

$$\vec{u}_n \to \vec{u} \quad \text{weakly in } \mathrm{L}^2(0,T;\mathrm{D}_0^{1,2}(\Omega)) \tag{5.2}$$

where the space $\mathrm{D}_0^{1,2}(\Omega)$ is a completion of $D(\Omega)$ with respect to the norm

$$\|v\|_{D_0^{1,2}(\Omega)} = \sqrt{\int_\Omega |\nabla v|^2 \,\mathrm{d}x}.$$

We shall prove that ϱ and $\vec{u}$ are finite energy weak solutions of the problem (1.1), (1.2), (1.4), (1.5) satisfying the initial conditions (1.3).

As a first step of our proof of Theorem 1, we shall prove the following lemma:

Lemma 5: *The limit functions ϱ and $\vec{u}$ given by* (5.1) *and* (5.2) *satisfy the equation (1.1) in $D'((0,T)\times\mathbb{R}^3)$ provided ϱ, $\vec{u}$ were prolonged to be zero on $\mathbb{R}^3-\Omega$. Moreover*

$$\varrho_n \to \varrho \quad in\, C([0,T];\mathrm{L}^\gamma_{weak}(B)) \text{ for any ball } B\subset\mathbb{R}^3, \tag{5.3}$$

$$\varrho_n u_n \to \varrho u \quad in\, C([0,T];\mathrm{L}^q_{weak}(B)) \text{ for any ball } B\subset\Omega, \quad where\, q=\frac{2\gamma}{\gamma+1} \tag{5.4}$$

$$p(\varrho_n) \to \overline{p(\varrho)} \text{ weakly in } \mathrm{L}^1((0,T)\times B) \tag{5.5}$$

and

$$\frac{\partial\varrho\vec{u}}{\partial t} + \mathrm{div}\,(\varrho\vec{u}\otimes\vec{u}) + \nabla\overline{p(\varrho)} = \mu\Delta\vec{u} + (\lambda+\mu)\nabla(\mathrm{div}\,\vec{u}) \tag{5.6}$$

in $D'((0,T)\times\Omega)$. Here, the bar stands for an L^1*-weak limit.*

Remark: Here, the convergence with respect to the weak topology in (5.3) means

$$t\to\int_\Omega \varrho_n(t) g\,\mathrm{d}x \text{ converges uniformly to } t\to\int_\Omega \varrho(t) g\,\mathrm{d}x$$

for any $g\in L^{\gamma'}(Omega)$ where $1/\gamma+1/\gamma'=1$. Similarly, we understand the convergence with respect to the weak topology in (5.4).

Proof: Let $\phi\in D((0,T)\times\mathbb{R}^3)$. Our functions ϱ_n and $\vec{u}_n$ were prolonged by (4.2). Moreover, let us prolonge functions ϱ_n and $\vec{u}_n$ by zero on $\mathbb{R}^3-\Omega$.

By Lemma 2, the continuity equation holds in $D'((0,T)\times\mathbb{R}^3)$ for the functions $\varrho_n\chi_{\Omega_n}$ and $\vec{u}_n$. It gives

$$\int_0^T\int_\Omega \varrho_n\chi_{\Omega_n}\phi_t\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega \varrho_n\chi_{\Omega_n}\vec{u}_n\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}t = 0. \tag{5.7}$$

As $\vec{u}_n$ were prolonged 0 outside Ω_n, we have the equation

$$\int_0^t\int_{\Omega-\Omega_n} \overline{\varrho}\,\phi_t\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\Omega-\Omega_n} \overline{\varrho}\,\vec{u}_n\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}t = 0. \tag{5.8}$$

Adding (5.7) and (5.8), we have

$$\int_0^T \int_\Omega \varrho_n \phi_t \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_\Omega \varrho_n \vec{u}_n \cdot \nabla\phi \,\mathrm{d}x\,\mathrm{d}t = 0 \quad \text{for all } \phi \in D((0,T)\times\mathbb{R}^3). \tag{5.9}$$

Let $B \subset \mathbb{R}^3$. Then, by virtue of (4.4) and Hölder inequality, we have that ϱ_n are bounded in $\mathrm{L}^\infty(0,T;\mathrm{L}^\gamma(B))$ and

$$\varrho_n \vec{u}_n \ \text{ are bounded in } \ \mathrm{L}^\infty(0,T;\mathrm{L}^q(B)) \ \text{ where } q = \frac{2\gamma}{\gamma+1}. \tag{5.10}$$

Therefore (5.9) implies

$$\frac{\partial \varrho_n}{\partial t} \ \text{ are bounded in } \ \mathrm{L}^\infty(0,T;\mathrm{W}^{-1,q}(B)).$$

And we can use the Arzela-Ascoli theorem to deduce (5.3) (compare with [1], Lemma 3.5). Consequently, using (5.2), (5.3), (5.10) and Sobolev embedding theorem, we have

$$\varrho_n \vec{u}_n \to \varrho \vec{u} \ \text{ in } \mathrm{L}^2(0,T;\mathrm{L}^q(B)).$$

As the ball B was arbitrary, we can pass to the limit in (5.9) for $n \to \infty$ to deduce that the limit functions ϱ and $\vec{u}$ given by (5.1) and (5.2) satisfy the equation (1.1) in $D'((0,T)\times\mathbb{R}^3)$ provided ϱ, $\vec{u}$ were prolonged to be zero on $\mathbb{R}^3 - \Omega$.

Similarly, by virtue of (1.2), (5.10) and Arzela-Ascoli theorem, we obtain (5.4). Consequently, using (5.2), (5.4) and Sobolev embedding theorem, we have

$$\varrho_n \vec{u}_n \otimes \vec{u}_n \to \varrho \vec{u} \otimes \vec{u} \ \text{ in } D'((0,T)\times B) \tag{5.11}$$

for all balls $B \subset \overline{B} \subset \Omega$. Finally, Lemma 4 gives (5.5) and we can pass to the limit for $n \to \infty$ in (1.2) to deduce (5.6).

Q.E.D.

6 Strong convergence of the density

In view of the above results, ϱ and $\vec{u}$ satisfy (1.2) as soon as we show $\overline{p(\varrho)} = p(\varrho)$ in (5.6). To this end, we first prove the strong convergence of the densities ϱ_n. We start with the following lemma about the effective viscous pressure:

Lemma 6 : *Let the pressure p satisfy the hypotheses* (1.6) *and* (1.7)*. Let* ϱ_n, $\vec{u}_n$ *be a finite energy weak solution of the problem* (1.1) – (1.5) *on* $(0,T)\times\Omega_n$ *obtained in Lemma* 2*. Let us prolonge the functions* ϱ_n *and* $\vec{u}_n$ *by the formula* (4.2)*. Let*

$$\varrho, \vec{u} \text{ and } \overline{p(\varrho)} \text{ are weak limits obtained in } (5.1), (5.2) \text{ and } (5.5).$$

Then

$$\lim_{n\to\infty}\int_0^T\int_\Omega \psi\phi\big(p(\varrho_n)-(\lambda+2\mu)\mathrm{div}\ \vec{u}_n\big)b(\varrho_n)\,dx\,dt =$$

$$=\int_0^T\int_\Omega \psi\phi\big(\overline{p(\varrho)}-(\lambda+2\mu)\mathrm{div}\ \vec{u}\big)\overline{b(\varrho)}\,dx\,dt$$

for any $b\in \mathrm{C}^1(\mathbb{R})$ *and any* $\psi=\psi(t)\in D(0,T)$ *and* $\phi=\phi(x)\in D(\Omega)$. *Here again, the bar stands for an* L^1*-weak limit.*

Proof: This is a standard result about the effective viscous flux, i.e. about the quantity $p(\varrho)-(\lambda+2\mu)$. The result has local character, therefore its proof is the same as if ϱ_n and $\vec{u}_n$ were the solutions on a fixed bounded spatial domain - see [4], Chapter 5, or [2], Lemma 4.2.

Q.E.D.

By virtue of the Lemma 6, we can prove the following two important lemmas. Both results have of local character, therefore, its proof is the same as if ϱ_n and $\vec{u}_n$ were solutions solutions on a bounded fixed spatial domain (see [2]).

Lemma 7: *Let* ϱ_n, $\vec{u}_n$ *be a finite energy weak solution of the problem* (1.1) – (1.5) *on* $(0,T)\times\Omega_n$ *obtained in Lemma 2. Let* ϱ *is the weak limit obtained in* (5.1). *Let us introduce the family of cut-off functions* T_k *by*

$$T_k(z)=k\,T\left(\frac{z}{k}\right)\ \textit{for}\ z\in\mathbb{R},\ k=1,2,3,\ldots$$

where $T\in \mathrm{C}^\infty(\mathbb{R})$ *is chosen so that*

$$T(z)=z \text{ for } z\le 1,\quad T(z)=2 \text{ for } z\ge 3,\quad T \text{ concave.}$$

Let $B\subset\mathbb{R}^3$ *is a bounded ball. Then there exists a constant c independent of k such that*

$$\limsup_{n\to\infty}\int_0^T\int_B |T_k(\varrho_n)-T_k(\varrho)|^{\gamma+1}\ dx\,dt\le c$$

for any $k\ge 1$.

Proof: See [2], Lemma 4.3 or [3], Lemma 4.2.

Q.E.D.

Lemma 8: *The limit functions* ϱ, $\vec{u}$ *solve (1.2) in the sense of renormalized solutions, i.e. (3.2) holds in* $D'((0,T)\times\mathbb{R}^3)$ *for any* $b\in\mathrm{C}^1(\mathbb{R})$ *satisfying (3.3) provided* ϱ, $\vec{u}$ *are set zero outside* Ω.

Proof: See [2], Lemma 4.4 or [3], Proposition 4.1.

Q.E.D.

We are going to complete the proof of

$$\overline{p(\varrho)}=p(\varrho). \tag{6.1}$$

To this end, we introduce a family of functions $L_k \in \mathrm{C}^1(\mathbb{R})$:

$$L_k(z) = \begin{cases} z\log(z) & \text{for } 0 \le z < k, \\ z\log(k) + z\int_k^z \frac{T_k(s)}{s^2}\,\mathrm{d}s & \text{for } z \ge k. \end{cases} \tag{6.2}$$

Seeing that L_k can be written as

$$L_k(z) = \beta_k z + b_k(z) \tag{6.3}$$

where b_k satisfy (3.3), we can use the fact that ϱ_n, $\vec{u}_n$ are renormalized solutions of (1.1) to deduce

$$\partial_t L_k(\varrho_n) + \mathrm{div}\ (L_k(\varrho_n)\vec{u}_n) + T_k(\varrho_n)\mathrm{div}\ \vec{u}_n = 0 \quad \text{in } D'((0,T)\times\mathbb{R}^3). \tag{6.4}$$

Similarly, by virtue of Lemma 8 and (6.3),

$$\partial_t L_k(\varrho) + \mathrm{div}\ (L_k(\varrho)\vec{u}) + T_k(\varrho)\mathrm{div}\ \vec{u} = 0 \quad \text{in } D'((0,T)\times\mathbb{R}^3). \tag{6.5}$$

Now, we can estimate (for k sufficently large, say, $k > \overline{\varrho} + c$, where c is a constant which can be easily compute from the definition of L_k)

$$\chi_{\{|L_k(\varrho_n)-\overline{\varrho}\log(\overline{\varrho})|\le 1\}}\big|L_k(\varrho_n) - \overline{\varrho}\log(\overline{\varrho})\big| \le$$

$$\le \chi_{\{|\varrho_n-\overline{\varrho}|\le c_1\}}\cdot\big|L_k(\varrho_n) - \overline{\varrho}\log(\overline{\varrho})\big| + \chi_{\{0<c_1<|\varrho_n-\overline{\varrho}|\le c_3\}},\cdot\big|L_k(\varrho_n) - \overline{\varrho}\log(\overline{\varrho})\big| \tag{6.6}$$

where c_1, c_2 and c_3 are constants. Note, that the constant second term in (6.6) appears only when $\overline{\varrho}$ is small and the equation $z\log(z) = \overline{\varrho}\log\overline{\varrho}$ has two solutions. Consequently, using meanvalue theorem, we have

$$\|\,\chi_{\{|L_k(\varrho_n)-\overline{\varrho}\log(\overline{\varrho})|\le 1\}}\big|L_k(\varrho_n) - \overline{\varrho}\log(\overline{\varrho})\big|\,\|_{L^\infty(0,T;L^2(\Omega))} \le$$

$$\|\,\chi_{\{|\varrho_n-\overline{\varrho}|\le c_1\}}\cdot\big|L_k(\varrho_n) - \overline{\varrho}\log(\overline{\varrho})\big|\,\|_{L^\infty(0,T;L^2(\Omega))} +$$

$$+\|\,\chi_{\{0<c_1<|\varrho_n-\overline{\varrho}|\le c_3\}},\cdot\big|L_k(\varrho_n) - \overline{\varrho}\log(\overline{\varrho})\big|\,\|_{L^\infty(0,T;L^2(\Omega))} \le$$

$$\le c\|\,\chi_{\{|\varrho_n-\overline{\varrho}|\le c_1\}}\cdot\big|\varrho_n - \overline{\varrho}\big|\,\|_{L^\infty(0,T;L^2(\Omega))} + c\ \mu\{0 < c_1 < |\varrho_n - \overline{\varrho}| \le c_3\} \tag{6.7}$$

where $\mu\{\cdot\}$ denotes the Lebesgue measure. By virtue of (4.4), we see that the right hand side in (6.7) is bounded. Using (4.4) again, we obtain that (for each k) the sequence

$$\{L_k(\varrho_n) - \overline{\varrho}\log(\overline{\varrho})\}_n \ \text{ is bounded in}\,\mathrm{L}^\infty(0,T;\mathrm{L}_2^\gamma(\Omega)). \tag{6.8}$$

Thus, we have

$$L_k(\varrho_n) - \overline{\varrho}\log(\overline{\varrho}) \xrightarrow{n\to\infty} \overline{L_k(\varrho) - \overline{\varrho}\log(\overline{\varrho})} \ \text{ weakly star in } \mathrm{L}^\infty(0,T;\mathrm{L}_2^\gamma(\Omega)).$$

In view of (6.4) and abstract Arzela-Ascoli theorem, we also have

$$L_k(\varrho_n) - \overline{\varrho}\log(\overline{\varrho}) \xrightarrow{n\to\infty} \overline{L_k(\varrho) - \overline{\varrho}\log(\overline{\varrho})} \text{ in } C([0,T]; \mathrm{L}^{\gamma}_{weak}(B)) \tag{6.9}$$

for any bounded ball $B \subset \Omega$. Taking the difference of (6.4) and (6.5) and integrating with respect to t, we get

$$\int_\Omega (L_k(\varrho_n) - L_k(\varrho))(t)\phi \,\mathrm{d}x =$$

$$+ \int_0^t \int_\Omega (L_k(\varrho_n)\vec{u}_n - L_k(\varrho)\vec{u}) \cdot \nabla\phi + (T_k(\varrho)\mathrm{div}\ \vec{u} - T_k(\varrho_n)\mathrm{div}\ \vec{u}_n)\phi \,\mathrm{d}x \,\mathrm{d}t$$

for any $\phi \in D(\Omega)$. Passing to the limit for $n \to \infty$ and making use of (5.2) and (6.9), one obtains

$$\int_\Omega (\overline{L_k(\varrho) - \overline{\varrho}\log(\overline{\varrho})} - L_k(\varrho) + \overline{\varrho}\log(\overline{\varrho}))(t)\phi \,\mathrm{d}x = \tag{6.10}$$

$$= \int_0^t \int_\Omega \left(\overline{L_k(\varrho) - \overline{\varrho}\log(\overline{\varrho})} - L_k(\varrho) + \overline{\varrho}\log(\overline{\varrho}) + \right) \vec{u} \cdot \nabla\phi \,\mathrm{d}x \,\mathrm{d}t +$$

$$+ \lim_{n\to\infty} \int_0^t \int_\Omega (T_k(\varrho)\mathrm{div}\ \vec{u} - T_k(\varrho_n)\mathrm{div}\ \vec{u}_n)\phi \,\mathrm{d}x \,\mathrm{d}t$$

for any $\phi \in D(\Omega)$. As the velocity components u^i, $i = 1, 2$ belong to $\mathrm{L}^2(0,T; \mathrm{W}^{1,2}_0(\Omega))$, it holds

$$\frac{|\vec{u}|}{\mathrm{dist}[x, \partial\Omega]} \in \mathrm{L}^2(0,T; \mathrm{L}^2(\Omega)). \tag{6.11}$$

Now, let r be a fixed integer and let us consider a sequence of functions $\phi_m \in D(\Omega)$ such that

$$0 \le \phi_m \le 1,\ \phi_m(x) = 1 \text{ for all } x \in \Omega_r \text{ such that } \mathrm{dist}[x, \partial\Omega] \ge \frac{1}{m} \quad \text{and}$$

$$|\nabla\phi_m(x)| \le 2m \quad \text{for all } x \in \Omega \quad \text{and} \quad \phi_m(x) = 0 \text{ outside } \Omega_r.$$

Taking functions ϕ_m as test functions in (6.10), passing to the limit for $m \to \infty$ and making use of (6.11), one derives

$$\int_{\Omega_r} \left(\overline{L_k(\varrho) - \overline{\varrho}\log(\overline{\varrho})} - L_k(\varrho) + \overline{\varrho}\log(\overline{\varrho})\right)(t) \,\mathrm{d}x =$$

$$= \int_0^t \int_{\Omega_r} T_k(\varrho)\mathrm{div}\ \vec{u} \,\mathrm{d}x \,\mathrm{d}t - \lim_{n\to\infty} \int_0^t \int_{\Omega_r} T_k(\varrho_n)\mathrm{div}\ \vec{u}_n \,\mathrm{d}x \,\mathrm{d}t. \tag{6.12}$$

Observe that the term $\overline{L_k(\varrho) - \overline{\varrho}\log(\overline{\varrho})} - L_k(\varrho) + \overline{\varrho}\log(\overline{\varrho})$ is bounded in view of (6.3).

At this stage, the main idea is to let $k \to \infty$ in (6.12). By virtue of (4.4) and (6.6) (we can use the estimate (6.6) because $L_k(z) = z\log(z)$ for $z < k$), we can assume

$$\varrho_n \log(\varrho_n) - \overline{\varrho}\log(\overline{\varrho}) \to \overline{\varrho\log(\varrho) - \overline{\varrho}\log(\overline{\varrho})} \text{ weakly star in } \mathrm{L}^\infty(0,T; \mathrm{L}^\alpha_2(\Omega))$$

for all $\alpha \in (1,\gamma)$. We also have

$$\overline{L_k(\varrho) - \overline{\varrho}\log(\overline{\varrho})} \to \overline{\varrho\log(\varrho) - \overline{\varrho}\log(\overline{\varrho})} \text{ in } \mathrm{L}^\infty(0,T;\mathrm{L}_2^\alpha(\Omega)) \quad \text{for any } \alpha \in (1,\gamma), \tag{6.13}$$

since, making of use (4.4) and (6.2),

$$\|\overline{L_k(\varrho) - \overline{\varrho}\log(\overline{\varrho})} - \overline{\varrho\log(\varrho) - \overline{\varrho}\log(\overline{\varrho})}\|_{L^\infty(0,T;L_2^\alpha(\Omega))} \le$$

$$\le \liminf_{n\to\infty} \operatorname*{ess\,sup}_{t\in[0,T]} \|L_k(\varrho_n) - \varrho_n\log(\varrho_n)\|_{L_2^\alpha} \le$$

$$\le c\, k^{\gamma-\alpha} \sup_n \operatorname*{ess\,sup}_{t\in[0,T]} \|\chi_{\{|\varrho_n - \overline{\varrho}|\ge 1\}} \cdot \varrho_n(t)\|_{L^\gamma} \to 0 \text{ as } k\to\infty.$$

Similarly, we obtain

$$L_k(\varrho) - \overline{\varrho}\log(\overline{\varrho}) \to \varrho\log(\varrho) - \overline{\varrho}\log(\overline{\varrho}) \text{ in } \mathrm{L}^\infty(0,T;\mathrm{L}_2^\alpha(\Omega)) \quad \text{for any } \alpha\in(1,\gamma). \tag{6.14}$$

Finally, by virtue of Lemma 6 and the monotonicity of the pressure, we can estimate the right hand side of (6.12):

$$\int_0^t\int_{\Omega_r} T_k(\varrho)\mathrm{div}\,\vec{u}\,\mathrm{d}x\,\mathrm{d}t - \lim_{\delta\to0+}\int_0^t\int_{\Omega_r} T_k(\varrho_n)\mathrm{div}\,\vec{u}_n\,\mathrm{d}x\,\mathrm{d}t \le$$

$$\le \int_0^t\int_{\Omega_r} (T_k(\varrho) - \overline{T_k(\varrho)})\mathrm{div}\,\vec{u}\,\mathrm{d}x\,\mathrm{d}t. \tag{6.15}$$

By virtue of Lemma 7 and (5.2), the right-hand side of (6.15) tends to zero as $k\to\infty$. Now, we can pass to the limit for $k\to\infty$ in (6.12) to conclude

$$\int_{\Omega_r} \overline{\varrho\log(\varrho) - \overline{\varrho}\log(\overline{\varrho})} - \varrho\log(\varrho) + \overline{\varrho}\log(\overline{\varrho})\,\mathrm{d}x\,(t) = 0 \text{ for a.e. } t\in[0,T]. \tag{6.16}$$

Because of the convexity of the function $z \to z\log z - \overline{\varrho}\log(\overline{\varrho})$, we have

$$\overline{\varrho\log(\varrho) - \overline{\varrho}\log(\overline{\varrho})} \ge \varrho\log(\varrho) - \overline{\varrho}\log(\overline{\varrho}) \text{ a.e. in } (0,T)\times\Omega_r$$

which, combined with (6.16) and with the fact that r is arbitrary, gives

$$\overline{\varrho\log(\varrho) - \overline{\varrho}\log(\overline{\varrho})} = \varrho\log(\varrho) - \overline{\varrho}\log(\overline{\varrho}) \text{ a.e. in } (0,T)\times\Omega. \tag{6.17}$$

By virtue of (6.2), we can assume (cf. with (6.6))

$$\left(\frac{\varrho+\varrho_n}{2}\right)\log\left(\frac{\varrho+\varrho_n}{2}\right) - \overline{\varrho}\log(\overline{\varrho}) \to w \text{ weakly star in } \mathrm{L}^\infty(0,T;\mathrm{L}_2^\alpha(\Omega))$$

for all $\alpha\in(1,\gamma)$, where, in view of convexity, $w \ge \varrho\log\varrho - \overline{\varrho}\log(\overline{\varrho})$. Thus, using convexity and (6.17),

$$0 \le h_n = \frac{1}{2}\varrho_n\log(\varrho_n) + \frac{1}{2}\varrho\log(\varrho) - \left(\frac{\varrho+\varrho_n}{2}\right)\log\left(\frac{\varrho+\varrho_n}{2}\right) \xrightarrow{w^*} \varrho\log(\varrho) - \overline{\varrho}\log(\overline{\varrho}) - w$$

weakly star in $\mathrm{L}^\infty(0,T;\mathrm{L}_2^\gamma(\Omega))$. As $\varrho\log(\varrho) - \overline{\varrho}\log(\overline{\varrho}) - w \le 0$, we have weak star convergence $h_n \to 0$ which together with $h_n \ge 0$ yields even strong convergence $h_n \to 0$ in $\mathrm{L}^1_{loc}((0,T)\times\Omega)$. Consequently, we have also strong convergence

$$\varrho_n - \overline{\varrho} \to \varrho - \overline{\varrho} \quad \text{in } \mathrm{L}^1_{loc}((0,T)\times\Omega).$$

In particular, it implies (6.1).

7 The energy inequality

Our proof of Theorem 1 will be finished provided that we will prove the energy inequality. To this end, let us consider the positive test function $\psi \in D(0,T)$. Multiplying (4.6) by ψ and integrating over t in the interval $[0,T]$, we obtain

$$\int_0^T \psi \int_{\Omega_n} \frac{1}{2}\varrho_n\,|\vec{u}_n|^2 + H(\varrho_n)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \psi \int_0^t \int_{\Omega_n} \mu\,|\nabla\vec{u}_n|^2 + (\lambda+\mu)\,|\mathrm{div}\ \vec{u}_n|^2\ \mathrm{d}x\,\mathrm{d}\tau\,\mathrm{d}t \le$$

$$\le \int_0^T \psi \int_{\Omega_n} \varrho_0\frac{|\vec{q}|^2}{2} + H(\varrho_0)\,\mathrm{d}x\,\mathrm{d}t. \tag{7.1}$$

Consequently, adding (4.5) and (7.1) and using (4.2), we have

$$\int_0^T \psi \int_{\Omega} \frac{1}{2}\varrho_n\,|\vec{u}_n|^2 + G(\varrho_n)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \psi \int_0^t \int_{\Omega} \mu\,|\nabla\vec{u}_n|^2 + (\lambda+\mu)\,|\mathrm{div}\ \vec{u}_n|^2\ \mathrm{d}x\,\mathrm{d}x\,\mathrm{d}\tau\,\mathrm{d}t \le$$

$$\le \int_0^T \psi \int_{\Omega_n} \varrho_0\frac{|\vec{q}|^2}{2} + G(\varrho_0)\,\mathrm{d}x\,\mathrm{d}t = \int_0^T \psi(t)E_0\,\mathrm{d}t. \tag{7.2}$$

Let B is a bounded ball, then we can estimate the left-hand side of (7.2) from below:

$$\int_0^T \psi \int_{\Omega} \frac{1}{2}\varrho_n\,|\vec{u}_n|^2 + G(\varrho_n)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \psi \int_0^t \int_{\Omega} \mu\,|\nabla\vec{u}_n|^2 + (\lambda+\mu)\,|\mathrm{div}\ \vec{u}_n|^2\ \mathrm{d}x\,\mathrm{d}\tau\,\mathrm{d}t \ge$$

$$\ge \int_0^T \psi \int_{\Omega\cap B} \frac{1}{2}\varrho_n\,|\vec{u}_n|^2 + G(\varrho_n)\,\mathrm{d}x\,\mathrm{d}t +$$

$$+ \int_0^T \psi \int_0^t \int_{\Omega\cap B} \mu\,|\nabla\vec{u}_n|^2 + (\lambda+\mu)\,|\mathrm{div}\ \vec{u}_n|^2\ \mathrm{d}x\,\mathrm{d}\tau\,\mathrm{d}t \tag{7.3}$$

Now, using (4.9), (5.1), (5.2), (5.11) and (5.4), we can pass to the limit for $n \to \infty$ in the inequalities (7.3) and (7.2) to obtain:

$$\int_0^T \psi \int_{\Omega\cap B} \frac{1}{2}\varrho\,|\vec{u}|^2 + G(\varrho)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \psi \int_0^t \int_{\Omega\cap B} \mu\,|\nabla\vec{u}|^2 + (\lambda+\mu)\,|\mathrm{div}\ \vec{u}|^2\ \mathrm{d}x\,\mathrm{d}x\,\mathrm{d}\tau\,\mathrm{d}t \le$$

$$\leq \int_0^T \psi(t) E_0 \, \mathrm{d}t.$$

As B and ψ are arbitrary, it implies our energy inequality (4.1) and the proof of Theorem 1 is finished.

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