This is a long preprint version of the paper published as Radek Erban, "On the existence of solutions to the Navier-Stokes equations of a two dimensional compressible flow", *Mathematical Methods in the Applied Sciences,* Volume 26, Issue 6, 489-517 (2003). The reader can find here detailed proofs of useful theorems about Orlicz spaces and other results which had to be skipped in the published version.

On the existence of solutions to the Navier-Stokes equations of two-dimensional compressible flow

Radek Erban

1 Introduction

The Navier-Stokes equations for compressible, barotropic flow in 2 space dimensions can be written in the form:

$$\frac{\partial \varrho}{\partial t} + \text{div } (\varrho \vec{u}) = 0, \tag{1.1}$$

$$\frac{\partial \varrho \vec{u}}{\partial t} + \text{div } (\varrho \vec{u} \otimes \vec{u}) + \nabla p(\varrho) = \mu \Delta \vec{u} + (\lambda + \mu) \nabla (\text{div } \vec{u}) + \varrho \vec{f}, \tag{1.2}$$

where the density $\varrho = \varrho(t,x)$ and the velocity $\vec{u} = [u^1(t,x), u^2(t,x)]$ are functions of the time $t \in (0,T)$ and the spatial coordinate $x \in \Omega$ where $\Omega \subset \mathbb{R}^2$ is a bounded regular domain, $\vec{f} = [f^1(t,x), f^2(t,x)]$ is a given external force. The viscosity coefficients μ and λ satisfy

$$\mu > 0, \quad \lambda + \frac{2}{3}\mu \geq 0.$$

We prescribe the initial conditions for the density and the momenta:

$$\varrho(0) = \varrho_0, \quad (\varrho u^i)(0) = q^i, \quad i = 1,2; \tag{1.3}$$

together with the no-slip boundary conditions for the velocity:

$$u^i|_{\partial\Omega} = 0, \quad i = 1,2. \tag{1.4}$$

The problem (1.1) – (1.4) was studied for isentropic fluid (it means $p(z) = az^\gamma$, $a > 0$, $\gamma > 1$) in 3D by [6] and [2]. You can find in [2] the global existence result for $\gamma > 3/2$ in three-dimensional case.

We will deal in the following that

$$p(z) = z \log^d(1+z), \qquad d > 1,$$

moreover, to simplify the presentation, we shall assume that $\vec{f} = 0$ in what follows, thus, the equation (1.2) is of the form

$$\frac{\partial \varrho \vec{u}}{\partial t} + \text{div } (\varrho \vec{u} \otimes \vec{u}) + \nabla \left(\varrho \log^d (1+\varrho) \right) = \mu \Delta \vec{u} + (\lambda + \mu) \nabla (\text{div } \vec{u}). \tag{1.5}$$

Following [6], we introduce the concept of finite energy weak solutions of the problem (1.1), (1.5), (1.4) (let us note that the definition of the function space $\mathrm{L}_{y \log^{d+1}(y)}(\Omega)$ can be found in the next section).

Definition 1.1: We shall say that ϱ, $\vec{u}$ is finite energy weak solution of the problem (1.1), (1.5) and (1.4) if the following four conditions are satisfied:

- $\varrho \geq 0$, $\varrho \in \mathrm{L}^\infty(0,T; \mathrm{L}_{y \log^{d+1}(y)}(\Omega))$, $u^i \in \mathrm{L}^2(0,T; \mathrm{W}_0^{1,2}(\Omega))$, $i = 1, 2$;
- the energy

$$E(t) = E[\varrho, \vec{u}](t) = \int_\Omega \frac{1}{2} \varrho \left| \vec{u} \right|^2 + \frac{a}{d+1} \varrho \log^{d+1}(1+\varrho) + \int_0^\varrho \int_0^\sigma a \frac{\log^d(1+s)}{s(1+s)} \, \mathrm{d}s \, \mathrm{d}\sigma \, \mathrm{d}x$$

satisfies the following energy inequality:

$$E(t) + \int_0^t \int_\Omega \mu \left| \nabla \vec{u} \right|^2 + (\lambda + \mu) \left| \text{div } \vec{u} \right|^2 \, \mathrm{d}x \leq E(0) \text{ for a.e. } t \in (0,T);$$

- the equations (1.1), (1.5) are satisfied in $D'((0,T) \times \Omega)$; moreover, (1.1) holds in $D'((0,T) \times \mathbb{R}^2)$ provided ϱ, $\vec{u}$ were prolonged to be zero on $\mathbb{R}^2 - \Omega$;
- the equation (1.1) is satisfied in the sense of renormalized solutions, it means that

$$b(\varrho)_t + \text{div } (b(\varrho)\vec{u}) + (b'(\varrho)\varrho - b(\varrho)) \text{div } \vec{u} = 0$$

holds in $D'((0,T) \times \Omega)$ for any $b \in C^1(\mathbb{R})$ such that

$$b'(z) \equiv 0 \text{ for all } z \in \mathbb{R} \text{ large enough, say, } z \geq M$$

where the constant M may vary for different functions b.

In the following, we shall suppose that the initial data ϱ_0, q^i, $i = 1, 2$ satisfy compatibility conditions of the form:

$$\varrho_0 \in \mathrm{L}_{y \log^{d+1}(y)}(\Omega), \; \varrho_0 \geq 0, \; q^i(x) = 0 \text{ whenever } \varrho_0(x) = 0, \; \frac{|q^i|^2}{\varrho_0} \in \mathrm{L}^1(\Omega), \; i = 1, 2. \tag{1.6}$$

Our main result reads as follows:

Theorem 1: *Assume $\Omega \subset \mathbb{R}^2$ is a bounded domain of the class $C^{2+\upsilon}$, $\upsilon > 0$. Let the data ϱ_0, q^i satisfy the compatibility conditions (1.6) and let $d > 1$.*

Then given $T > 0$ arbitrary, there exists a finite energy weak solution ϱ, $\vec{u}$ of the problem (1.1), (1.5), (1.4) satisfying the initial conditions (1.3).

At first, in the next two sections, we shall create useful tools for the proof of Theorem 1.

2 Excursion to the theory of Orlicz spaces

In this section, we recall several definitions and well known results concerning Orlicz spaces (for details see [3]). We shall assume that Ω is a bounded domain.

Definition 2.1: We shall say that ϕ is a Young function if ϕ is continuous, increasing and convex in $[0,\infty)$ and if

$$\lim_{t\to 0+}\frac{\phi(t)}{t}=0, \quad \text{and} \quad \lim_{t\to\infty}\frac{\phi(t)}{t}=\infty.$$

Let ϕ be a Young function. Then the complementary function ψ is defined by the formula

$$\psi(z)=\sup_{y\in\mathbb{R}_0^+}(yz-\phi(y)) \ \text{ for } z\in[0,\infty).$$

Definition 2.2: Let ϕ be a Young function and let u be a measurable function defined on Ω. The number

$$\|u\|_\phi=\inf\left\{k>0\,;\int_\Omega\phi\left(\frac{|u(x)|}{k}\right)\,\mathrm{d}x\le 1\right\}$$

is called the Luxembourg norm of u. The Orlicz space $\mathrm{L}_\phi(\Omega)$ is a space of all measurable functions with finite Luxembourg norm.

Remark: The Orlicz space $\mathrm{L}_\phi(\Omega)$ is a Banach space. We can define an equivalent norm in $\mathrm{L}_\phi(\Omega)$ by the formula:

$$|u|_\phi=\sup_v\left|\int_\Omega u(x)v(x)\,\mathrm{d}x\right|, \tag{2.1}$$

where the supremum is taken over all Lebesgue-measurable functions v defined almost everywhere on Ω such that $\int_\Omega\psi(|v(x)|)\,\mathrm{d}x\le 1$.

Definition 2.3: A Young function ϕ is said to satisfy the Δ_2-condition (we shall write briefly $\phi\in\Delta_2$) if there exist $k>0$ and $T\ge 0$ such that

$$\phi(2t)\le k\phi(t) \quad \text{for all} \quad t\ge T.$$

Remark: Let us suppose that $\phi\in\Delta_2$. Then the Orlicz space $\mathrm{L}_\phi(\Omega)$ is separable and the dual space can be represented (using standard isomorphism) in the form $\mathrm{L}_\psi(\Omega)=(\mathrm{L}_\phi(\Omega))^*$, where ψ is the complementary function.

Definition 2.4: Let us denote by $B(\Omega)$ the set of all bounded measurable functions defined on Ω. The space $\mathrm{E}_\phi(\Omega)$ is defined to be the closure of $B(\Omega)$ with respect to the Luxembourg norm $\|\cdot\|_\phi$.

Remark: The space $\mathrm{E}_\phi(\Omega)$ is separable and $\mathrm{L}_\psi(\Omega) = (\mathrm{E}_\phi(\Omega))^*$, where ψ is the complementary function.

Definition 2.5: Let us denote by $\mathrm{L}_{y\log^d(y)}(\Omega)$ the Orlicz space generated by the Young function

$$\phi(y) = y\log^d(1+y), \qquad \text{for } d > 0.$$

We shall write briefly $\mathrm{L}_{y\log(y)}(\Omega)$ for $d = 1$.

Definition 2.6: Let us denote by $\mathrm{L}_{e(\beta)}(\Omega)$ the Orlicz space generated by the Young function

$$\phi(y) = \exp(y^\beta) - 1, \qquad \text{for } \beta > 1,$$

and by the Young function

$$\phi(y) = \begin{cases} \exp(y^\beta), & y \geq \left(\frac{2}{\beta}\right)^{\frac{1}{\beta}}, \\ \left(\frac{\beta}{2}\right)^{\frac{2}{\beta}} \exp\left(\frac{2}{\beta}\right) y^2, & 0 \leq y \leq \left(\frac{2}{\beta}\right)^{\frac{1}{\beta}}, \end{cases} \qquad \text{for } \beta \in (0,1].$$

Remark: The space $\mathrm{E}_\phi(\Omega)$ corresponding to space $\mathrm{L}_{y\log^d(y)}(\Omega)$, resp. $\mathrm{L}_{e(\beta)}(\Omega)$, will be denoted by $\mathrm{E}_{y\log^d(y)}(\Omega)$, resp $\mathrm{E}_{e(\beta)}(\Omega)$. It holds $\mathrm{L}_{y\log^d(y)}(\Omega) = \mathrm{E}_{y\log^d(y)}(\Omega)$,

$$\left(\mathrm{E}_{e(\beta)}(\Omega)\right)^* = \mathrm{L}_{y\log^{1/\beta}(y)}(\Omega) \quad \text{and} \quad \left(\mathrm{L}_{y\log^d(y)}(\Omega)\right)^* = \mathrm{L}_{e(1/d)}(\Omega).$$

Definition 2.7: Let ϕ be a Young function. Let us denote by

$$\mathrm{W\,L}_\phi(\Omega) \quad \text{and} \quad \mathrm{W\,E}_\phi(\Omega)$$

the set of all functions $u \in \mathrm{L}_\phi(\Omega)$, and $u \in \mathrm{E}_\phi(\Omega)$, respectively, such that all distributional derivatives $\partial_{x_i} u$, $i = 1, 2$, are elements of the spaces $\mathrm{L}_\phi(\Omega)$, and $\mathrm{E}_\phi(\Omega)$, respectively.

Remark: $\mathrm{W\,L}_\phi(\Omega)$ and $\mathrm{W\,E}_\phi(\Omega)$ are Banach spaces with norm

$$\|u\|_{WL_\phi} = \|u\|_\phi + \|\partial_{x_1} u\|_\phi + \|\partial_{x_2} u\|_\phi.$$

The space $\mathrm{W\,E}_\phi(\Omega)$ is separable. If Ω be a domain with Lipschitz boundary then

$$\overline{C^\infty(\overline{\Omega})} = \mathrm{W\,E}_\phi(\Omega), \tag{2.2}$$

where the closure is taken with respect to the norm of the space $\mathrm{W\,E}_\phi(\Omega)$. The spaces $\mathrm{W\,L}_\phi(\Omega)$ and $\mathrm{W\,E}_\phi(\Omega)$ are called Sobolev-Orlicz spaces.

3 Auxiliary results

In this section, we shall prove several useful lemmas.

Lemma 3.1: *Let Ω be a bounded domain, let b be a mapping from $\mathbb{R}^2$ to $\mathbb{R}$ and*

$$|\xi|^{|\alpha|}|D^\alpha b(\xi)| \le C < \infty \quad (\text{for } |\alpha| \le 2,\ \alpha \text{ is multiindex}).$$

Then for $d > 0$ there exists a constant D_d such that for all $g \in \mathrm{L}_{y\log^{(d+1)}(y)}(\Omega)$

$$\|\mathcal{F}^{-1}b * g\|_{y\log^d(y)} \le D_d \|g\|_{y\log^{(d+1)}(y)}, \tag{3.1}$$

where g is prolonged to be 0 on $\mathbb{R}^2 - \Omega$ and $\mathcal{F}$ denotes the Fourier transformation.

Proof: Lemma 3.1 has the same assumptions as the standard Mikhlin multiplier theorem, see [1], Theorem 6.1.6. As in the [1], we can show that $\mathcal{F}^{-1}b*$ is bounded mapping as

$$\mathcal{F}^{-1}b* : \mathrm{L}^2(\mathbb{R}^2) \to \mathrm{L}^2(\mathbb{R}^2), \qquad \mathcal{F}^{-1}b* : \mathrm{L}^1(\mathbb{R}^2) \to \mathrm{L}^{1\infty}(\mathbb{R}^2), \tag{3.2}$$

where $\mathrm{L}^{1\infty}(\mathbb{R}^2)$ is one of the Lorentz spaces (we have $u \in \mathrm{L}^{1\infty}(\mathbb{R}^2)$ if and only if $\sup_\sigma \sigma m(\sigma, u) < \infty$ where $m(\sigma, u) = \mu\{x : |u(x)| > \sigma\}$).

Let us suppose that $g \in \mathrm{L}_{y\log^{d+1}(y)}(\Omega)$ and $\|g\|_{y\log^{(d+1)}(y)} = 1$. Let $\sigma > 0$ and define $g = g_1 + g_2$, where

$$g_2(x) = \begin{cases} g(x), & \text{if } |g(x)| \le \sigma, \\ \sigma, & \text{otherwise,} \end{cases} \tag{3.3}$$

and $g_1 = g - g_2$. Then clearly $g_i \in \mathrm{L}^i(\Omega)$, $i = 1, 2$. Let $h_i = \mathcal{F}^{-1}b * g_i$, $i = 1, 2$, and let $h = h_1 + h_2$. Let us define the distribution functions of g (resp. g_i, h and h_i in the similar way) by the formula

$$m(\sigma, g) = \mu\{x : |g(x)| > \sigma\}. \tag{3.4}$$

By virtue of (3.2), we have

$$m(\sigma, h) \le m\left(\frac{\sigma}{2}, h_1\right) + m\left(\frac{\sigma}{2}, h_2\right) \le \frac{C_1}{\sigma}\|g_1\|_{L^1} + \frac{C_2}{\sigma^2}\|g_2\|^2_{L^2}, \tag{3.5}$$

for all $\sigma > 0$, where C_1 and C_2 do not depend on σ. Let us remind that the functions g_i, $i = 1, 2$, depend on σ.

Now, we shall estimate the Luxembourg norm of $h = \mathcal{F}^{-1}b * g$ in the space $\mathrm{L}_{y\log^d(y)}(\Omega)$. For brevity, we shall write $\Phi_d(y) = y\log^d(1+y)$. Let $k \ge 1$. Then

$$\int_\Omega \Phi_d\left(\frac{|h(x)|}{k}\right) \mathrm{d}x \le \frac{1}{k}\int_\Omega \Phi_d(|h(x)|)\,\mathrm{d}x = \frac{1}{k}\int_0^\infty m(\tau, h)\frac{\mathrm{d}\,\Phi_d}{\mathrm{d}\,y}(\tau)\,\mathrm{d}\tau =$$

$$= \frac{1}{k}\int_1^\infty m(\tau, h)\frac{\mathrm{d}\,\Phi_d}{\mathrm{d}\,y}(\tau)\,\mathrm{d}\tau + \frac{1}{k}\int_0^1 m(\tau, h)\frac{\mathrm{d}\,\Phi_d}{\mathrm{d}\,y}(\tau)\,\mathrm{d}\tau \le$$

$$\leq \frac{1}{k}\int_1^\infty m(\tau,h)\frac{\mathrm{d}\,\Phi_d}{\mathrm{d}\,y}(\tau)\,\mathrm{d}\tau + \frac{1}{k}\mu(\Omega)\Phi_d(1), \tag{3.6}$$

Because of (3.5), we can estimate the first term on the right-hand side of (3.6) in the following way

$$\frac{1}{k}\int_1^\infty m(\tau,h)\frac{\mathrm{d}\,\Phi_d}{\mathrm{d}\,y}(\tau)\,\mathrm{d}\tau \leq$$

$$\leq \frac{C_1}{k}\int_1^\infty \frac{1}{\tau}\|g_1\|_{L^1}\frac{\mathrm{d}\,\Phi_d}{\mathrm{d}\,y}(\tau)\,\mathrm{d}\tau + \frac{C_2}{k}\int_1^\infty \frac{1}{\tau^2}\|g_2\|^2_{L^2}\frac{\mathrm{d}\,\Phi_d}{\mathrm{d}\,y}(\tau)\,\mathrm{d}\tau \leq$$

$$\leq \frac{C_1}{k}\int_1^\infty \frac{1}{\tau}\int_0^\infty m(t,g_1)\,\mathrm{d}t\,\frac{\mathrm{d}\,\Phi_d}{\mathrm{d}\,y}(\tau)\,\mathrm{d}\tau +$$

$$+\frac{C_2}{k}\int_1^\infty \frac{1}{\tau^2}\int_0^\infty 2t\cdot m(t,g_2)\,\mathrm{d}t\,\frac{\mathrm{d}\,\Phi_d}{\mathrm{d}\,y}(\tau)\,\mathrm{d}\tau. \tag{3.7}$$

By virtue of the definition of the functions g_i, $i=1,2$, we can put in the inner integrals $m(t,g_2)=m(t,g)\chi_{[t\leq\tau]}$ and $m(t,g_1)=m(t+\tau,g)$. Thus (3.7) implies

$$\frac{1}{k}\int_1^\infty m(\tau,h)\frac{\mathrm{d}\,\Phi_d}{\mathrm{d}\,y}(\tau)\,\mathrm{d}\tau \leq$$

$$\leq \frac{C_1}{k}\int_1^\infty \frac{1}{\tau}\int_\tau^\infty m(t,g)\,\mathrm{d}t\,\frac{\mathrm{d}\,\Phi_d}{\mathrm{d}\,y}(\tau)\,\mathrm{d}\tau + \frac{C_2}{k}\int_1^\infty \frac{1}{\tau^2}\int_0^\tau 2t\cdot m(t,g)\,\mathrm{d}t\,\frac{\mathrm{d}\,\Phi_d}{\mathrm{d}\,y}(\tau)\,\mathrm{d}\tau. \tag{3.8}$$

Let us estimate the first integral on the right side of (3.8). Changing the order of integration and using the explicit formula for $\Phi_d(y)=y\log^d(1+y)$, this gives

$$\int_1^\infty \frac{1}{\tau}\int_\tau^\infty m(t,g)\,\mathrm{d}t\,\frac{\mathrm{d}\,\Phi_d}{\mathrm{d}\,y}(\tau)\,\mathrm{d}\tau = \int_1^\infty m(t,g)\int_1^t \frac{1}{\tau}\frac{\mathrm{d}\,\Phi_d}{\mathrm{d}\,y}(\tau)\,\mathrm{d}\tau\,\mathrm{d}t =$$

$$= \int_1^\infty m(t,g)\int_1^t \frac{\log^d(1+\tau)}{\tau} + \frac{d\log^{d-1}(1+\tau)}{1+\tau}\,\mathrm{d}\tau\,\mathrm{d}t \leq$$

$$\leq \int_1^\infty m(t,g)\int_1^t 2\frac{\log^d(1+\tau)}{1+\tau} + \frac{d\log^{d-1}(1+\tau)}{1+\tau}\,\mathrm{d}\tau\,\mathrm{d}t =$$

$$= \int_1^\infty m(t,g)\,2\,\frac{\log^{d+1}(1+t)}{d+1} + \log^d(1+t)\,\mathrm{d}t =$$

$$= \int_\Omega \chi_{[1\leq|g(x)|]}\int_1^{|g(x)|} 2\,\frac{\log^{d+1}(1+t)}{d+1} + \log^d(1+t)\,\mathrm{d}t\,\mathrm{d}x \leq$$

$$\leq \int_\Omega \chi_{[1\leq|g(x)|]}\int_1^{|g(x)|} 2\,\frac{\log^{d+1}(1+t)}{d+1} + 2\,\frac{t\log^d(1+t)}{1+t}\,\mathrm{d}t\,\mathrm{d}x \leq$$

$$\leq \int_\Omega 2\frac{|g(x)|\log^{d+1}(1+|g(x)|)}{d+1}\,\mathrm{d}x \leq \frac{2}{d+1}\int_\Omega \Phi_{(d+1)}(|g(x)|)\,\mathrm{d}x \leq \frac{2}{d+1}, \tag{3.9}$$

where we use the assumption $\|g\|_{y\log^{(d+1)}(y)} = 1$. Now, we shall estimate the second integral in (3.8). We shall use the similar arguments as in the estimation of the first integral in (3.8).

$$\int_1^\infty \frac{1}{\tau^2}\int_0^\tau 2t\cdot m(t,g)\,\mathrm{d}t\,\frac{\mathrm{d}\,\Phi_d}{\mathrm{d}\,y}(\tau)\,\mathrm{d}\tau = \int_0^\infty 2t\cdot m(t,g)\int_{\max(t,1)}^\infty \frac{1}{\tau^2}\frac{\mathrm{d}\,\Phi_d}{\mathrm{d}\,y}(\tau)\,\mathrm{d}\tau\,\mathrm{d}t =$$

$$= \int_0^\infty 2t\cdot m(t,g)\int_{\max(t,1)}^\infty \frac{\log^d(1+\tau)}{\tau^2} + \frac{d\log^{d-1}(1+\tau)}{(1+\tau)\tau}\,\mathrm{d}\tau\,\mathrm{d}t \le$$

$$\le \int_0^\infty 2t\cdot m(t,g)\int_{\max(t,1)}^\infty 4\frac{\log^d(1+\tau)}{(1+\tau)^2} + 2\frac{d\log^{d-1}(1+\tau)}{(1+\tau)^2}\,\mathrm{d}\tau\,\mathrm{d}t.$$

Using $[d]-$times integration per partes in the inner integral, we can estimate (the constants C_d and $\tilde{C}_d$ depend on d)

$$\int_0^\infty 2t\cdot m(t,g)\int_{\max(t,1)}^\infty 4\frac{\log^d(1+\tau)}{(1+\tau)^2} + 2\frac{d\log^{d-1}(1+\tau)}{(1+\tau)^2}\,\mathrm{d}\tau\,\mathrm{d}t \le$$

$$\le C_d\int_0^\infty t\cdot m(t,g)\left(\frac{1}{1+\max(t,1)} + \sum_{i=0}^{[d]}\frac{\log^{d-i}(1+\max(t,1))}{1+\max(t,1)}\right)\mathrm{d}t \le$$

$$\le C_d\int_0^\infty m(t,g)\left(1+\sum_{i=0}^{[d]}\log^{d-i}(1+\max(t,1))\right)\mathrm{d}t \le$$

$$\le \tilde{C}_d\int_\Omega \Phi_d(|g(x)|)\,\mathrm{d}x \le \tilde{C}_d. \tag{3.10}$$

Substituting (3.9) and (3.10) in (3.8) we get

$$\frac{1}{k}\int_1^\infty m(\tau,h)\frac{\mathrm{d}\,\Phi_d}{\mathrm{d}\,y}(\tau)\,\mathrm{d}\tau \le \frac{1}{k}\hat{C}_d,$$

where the constant $\hat{C}_d$ depends only on d. Finally, substituting the last inequality in (3.6) we have

$$\int_\Omega \Phi_d\left(\frac{|h(x)|}{k}\right)\,\mathrm{d}x \le \frac{1}{k}D_d.$$

Thus

$$\|\mathcal{F}^{-1}b*g\|_{y\log^d(y)} \le D_d \qquad \text{for } \|g\|_{y\log^{(d+1)}(y)} = 1$$

and Lemma 3.1 has been proved.

Q.E.D.

Lemma 3.2: *Let Ω be a bounded domain, let ϕ, ϕ_1 and ϕ_2 be Young functions and let there exist numbers $c>0$, $k\geq 0$, $y_0\geq 0$, $z_0\geq 0$ such that for all $y\geq y_0$, $z\geq z_0$*

$$\phi\left(\frac{y\cdot z}{c}\right)\leq \phi_1(y)+\phi_2(z)+k,$$

then there is a number m such that for all measurable f_1 and f_2

$$\|f_1f_2\|_{\phi}\leq m\|f_1\|_{\phi_1}\|f_2\|_{\phi_2}.$$

Proof:
Let us suppose without loss of generality that $\|f_1\|_{\phi_1}=1$, $\|f_2\|_{\phi_2}=1$. Let $d=2+\mu(\Omega)\phi_1(y_0)+\mu(\Omega)\phi_2(z_0)+\mu(\Omega)k$ and $A=\{\,x\in\Omega:|f_1(x)|\geq y_0\,\}$, $B=\{\,x\in\Omega:|f_2(x)|\geq z_0\,\}$. We put $m=c\cdot d$. Then

$$\int_\Omega \phi\left(\left|\frac{f_1(x)f_2(x)}{m}\right|\right)\,\mathrm{d}x\leq \frac{1}{d}\int_\Omega \phi\left(\left|\frac{f_1(x)f_2(x)}{c}\right|\right)\,\mathrm{d}x\leq$$

$$\leq\frac{1}{d}\left[\int_{A\cap B}\phi\left(\frac{|f_1(x)f_2(x)|}{c}\right)\,\mathrm{d}x+\int_{A-B}\phi\left(\frac{|f_1(x)|\,z_0}{c}\right)\,\mathrm{d}x+\right.$$

$$\left.+\int_{B-A}\phi\left(\frac{y_0\,|f_2(x)|}{c}\right)\,\mathrm{d}x+\int_{(\Omega-A)\cap(\Omega-B)}\phi\left(\frac{y_0z_0}{c}\right)\,\mathrm{d}x\right]\leq$$

$$\leq\frac{1}{d}\left[\int_A\phi_1\left(|f_1(x)|\right)\,\mathrm{d}x+\int_B\phi_2\left(|f_2(x)|\right)\,\mathrm{d}x+\int_\Omega\phi_1\left(y_0\right)\,\mathrm{d}x+\int_\Omega\phi_2\left(z_0\right)\,\mathrm{d}x+\mu(\Omega)k\right]\leq$$

$$\leq\frac{1}{d}\left[1+1+\mu(\Omega)\phi_1(y_0)+\mu(\Omega)\phi_2(z_0)+\mu(\Omega)k\right]=1.$$

By virtue of the definition of Luxembourg norm, we see

$$\|f_1f_2\|_{\phi}\leq m.$$

Q.E.D.

Lemma 3.3: *Let $\beta>0$, $d>0$ and $\gamma=d-\frac{1}{\beta}\geq 0$. Let Ω be a bounded domain. Then there exists a constant a such that*

$$\|f_1f_2\|_{y\log^{\gamma}(y)}\leq a\|f_1\|_{y\log^{d}(y)}\|f_2\|_{e(\beta)},\qquad \text{if } \gamma>0,$$

and

$$\|f_1f_2\|_{L^1(\Omega)}\leq a\|f_1\|_{y\log^{d}(y)}\|f_2\|_{e(\beta)},\qquad \text{if } \gamma=0.$$

Proof: Let us denote $c = 2^\gamma (d+1)^{1/\beta}$. By virtue of Lemma 3.2, it is sufficent to prove that there exists a constant y_0 such that for all $y \geq y_0$, $z \geq 1$

$$\frac{y \cdot z}{c} \log^\gamma \left(\frac{y \cdot z}{c}\right) \leq y \log^d y + \exp z^\beta. \tag{3.11}$$

At this stage, we shall distinguish two cases:
(1) Let

$$y \leq \frac{\exp\left(z^\beta\right)}{z^{d\beta}}. \tag{3.12}$$

Then, substituting (3.12) to the left side of the inequality (3.11), we get

$$\frac{y \cdot z}{c} \log^\gamma \left(\frac{y \cdot z}{c}\right) \leq \frac{1}{c} \exp\left(z^\beta\right) z^{(1-d\beta)} \log^\gamma \left(\frac{1}{c} \exp\left(z^\beta\right) z^{(1-d\beta)}\right) \leq$$

$$\leq \frac{1}{c} \exp\left(z^\beta\right) z^{(-\gamma\beta)} \log^\gamma \left(\frac{1}{c} \exp\left(z^\beta\right) z^{(-\gamma\beta)}\right) \leq$$

$$\leq \frac{1}{c} \exp\left(z^\beta\right) z^{(-\gamma\beta)} \log^\gamma \left(\exp\left(z^\beta\right)\right) = \frac{1}{c} \exp\left(z^\beta\right) < \exp\left(z^\beta\right).$$

Thus, the inequality (3.11) is fulfilled under the assumption (3.12).
(2) Let us suppose that

$$\frac{\exp\left(z^\beta\right)}{z^{d\beta}} \leq y. \tag{3.13}$$

Let us define the auxiliary function $\xi : (0,\infty) \to (0,\infty)$ by the formula

$$\xi(t) = \frac{\exp\left(t^\beta\right)}{t^{d\beta}}.$$

Then ξ is continuous and increasing on the interval $(d^{1/\beta}, \infty)$. Let us suppose $y > d^{1/\beta}$, then (3.13) is equivalent to the condition $z \leq \xi^{-1}(y)$. Substituting this inequality in the left side of (3.11) we get (we use the obvious estimate $\xi^{-1}(y) \leq cy$ for sufficiently large y)

$$\frac{y \cdot z}{c} \log^\gamma \left(\frac{y \cdot z}{c}\right) \leq \frac{1}{c} y\, \xi^{-1}(y) \log^\gamma \left(y \frac{\xi^{-1}(y)}{c}\right) \leq$$

$$\leq \frac{1}{c} y\, \xi^{-1}(y) \log^\gamma \left(y^2\right) \leq \frac{2^\gamma}{c} y\, \xi^{-1}(y) \log^\gamma y.$$

Thus, (3.11) will be a consequence of the following inequality

$$\xi^{-1}(y) \leq \frac{c}{2^\gamma} \log^{d-\gamma} y,$$

which is equivalent to the inequality

$$y \leq \xi((d+1)^{1/\beta} \log^{1/\beta} y) = \frac{\exp\left((d+1)\log y\right)}{(d+1)^d \log^d y} = \frac{y^{d+1}}{(d+1)^d \log^d y} \iff (d+1)\log y \leq y.$$

The last inequality is obviously fulfilled for sufficiently large y. Thus Lemma 3.3 has been proved.

Q.E.D.

Lemma 3.4: *Let $d > 0$ and Ω be a bounded domain. Then there exists a constant a such that*

$$\|f_1 f_2\|_{y\log^d(y)} \leq a\|f_1\|_{y^2\log^{2d}(y)}\|f_2\|_{L^2},$$

where $\|\cdot\|_{y^2\log^{2d}(y)}$ is the Luxembourg norm in the Orlicz space generated by the Young function $\phi(y) = y^2\log^{2d}(1+y)$.

Proof: Let us denote $c = (2+d)^d$. By virtue of Lemma 3.2, it is sufficent to prove that there exists a constant z_0 such that for all $y \geq 1$, $z \geq z_0$

$$\frac{y\cdot z}{c}\log^d\left(\frac{y\cdot z}{c}\right) \leq y^2\log^{2d} y + z^2. \tag{3.14}$$

At this stage, we shall distinguish two cases:
(1) Let

$$z \leq y\log^d y. \tag{3.15}$$

Then, substituting (3.15) to the left side of the inequality (3.14), we get

$$\frac{y\cdot z}{c}\log^d\left(\frac{y\cdot z}{c}\right) \leq \frac{1}{c}y^2\log^d y\log^d\left(\frac{y^2\log^d y}{c}\right) \leq$$

$$\leq y^2\log^d y\log^d\left(\frac{y^2\log^d y}{(2+d)^d}\right)^{\frac{1}{d+2}} \leq y^2\log^{2d} y.$$

Thus, the inequality (3.14) is fulfilled under the assumption (3.15).
(2) Let us suppose that

$$y\log^d y \leq z. \tag{3.16}$$

Let us define the auxiliary function $\xi : (1,\infty) \to (0,\infty)$ by the formula

$$\xi(t) = t\log^d t.$$

Then ξ is continuous and increasing function, thus, (3.16) is equivalent to the condition $y \leq \xi^{-1}(z)$. Substituting this inequality in the left side of (3.14) we get (we use the obvious estimate $\xi^{-1}(z) \leq cz$ for sufficiently large z)

$$\frac{y\cdot z}{c}\log^d\left(\frac{y\cdot z}{c}\right) \leq \frac{1}{c}\xi^{-1}(z)z\log^d\left(z\frac{\xi^{-1}(z)}{c}\right) \leq \frac{1}{c}\xi^{-1}(z)z\log^d\left(z^2\right) \leq \frac{2^d}{c}\xi^{-1}(z)z\log^d z.$$

Thus, (3.14) will be a consequence of the following inequality

$$\xi^{-1}(z) \leq \frac{c}{2^d}\frac{z}{\log^d z},$$

which is equivalent to the inequality

$$z \le \xi\left(\frac{c}{2^d}\frac{z}{\log^d z}\right) = \left(\frac{d+2}{2}\right)^d \frac{z}{\log^d z}\log^d\left(\left(\frac{d+2}{2}\right)^d \frac{z}{\log^d z}\right) \iff$$

$$\iff z^{\frac{2}{d+2}} \le \frac{z}{\log^d z}\left(\frac{d+2}{2}\right)^d \iff \log^{d+2} z \le z\left(\frac{d+2}{d}\right)^{d+2}.$$

The last inequality is obviously fulfilled for sufficiently large z. Thus Lemma 3.4 has been proved.

Q.E.D.

Lemma 3.5: *Let Ω be a bounded domain, $d > 0$ and $\beta > 2/d$. Then there exists a constant a such that*

$$\|f_1 f_2\|_{L^2} \le a\|f_1\|_{y^2\log^d(y)}\|f_2\|_{e(\beta)},$$

where $\|\cdot\|_{y^2\log^d(y)}$ is the Luxembourg norm in the Orlicz space generated by the Young function $\phi(y) = y^2\log^d(1+y)$.

Proof: By virtue of Lemma 3.2, it is sufficent to prove that there exist constants y_0, z_0 and c such that for all $y \ge y_0$, $z \ge z_0$

$$\left(\frac{y\cdot z}{c}\right)^2 \le y^2\log^d y + \exp z^\beta. \tag{3.17}$$

At this stage, we shall distinguish two cases:
(1) Let

$$y \le \frac{\exp\left(z^{2/d}\right)}{z}. \tag{3.18}$$

Then, substituting (3.18) to the left side of the inequality (3.17), we get

$$\left(\frac{y\cdot z}{c}\right)^2 \le \frac{1}{c^2}\exp\left(2z^{2/d}\right) \le \exp\left(z^\beta\right)$$

where the last estimate holds for sufficiently large z provided $\beta > 2/d$ and $c > 1$. Thus, the inequality (3.17) is fulfilled under the assumption (3.18).
(2) Let us suppose that

$$\frac{\exp\left(z^{2/d}\right)}{z} \le y. \tag{3.19}$$

Let us define the auxiliary function $\xi : (0,\infty) \to (0,\infty)$ by the formula

$$\xi(t) = \frac{\exp\left(t^{2/d}\right)}{t}.$$

Then ξ is continuous and increasing on the interval $(1+d,\infty)$. Let us suppose $y>1+d$, then (3.19) is equivalent to the condition $z\leq\xi^{-1}(y)$. Substituting this inequality in the left side of (3.17) we get

$$\left(\frac{y\cdot z}{c}\right)^2\leq\frac{1}{c^2}y^2\left(\xi^{-1}(y)\right)^2.$$

Thus, (3.17) will be a consequence of the following inequality

$$\xi^{-1}(y)\leq c\log^{d/2}y,$$

which is equivalent to the inequality

$$y\leq\xi\left(c\log^{d/2}y\right)=\frac{\exp\left(c^{2/d}\log y\right)}{c\log^{d/2}y}=\frac{y^{\left(c^{2/d}\right)}}{c\log^{d/2}y}.$$

The last inequality is obviously fulfilled for sufficiently large y provided $c>1$. Thus Lemma 3.5 has been proved.

Q.E.D.

Lemma 3.6: *Let Ω be a bounded domain, $\alpha>0$, $\beta>0$, $\gamma>0$ and*

$$(\alpha-\gamma)(\beta-\gamma)\geq\gamma^2. \tag{3.20}$$

Then there exists a constant a such that

$$\|f_1f_2\|_{e(\gamma)}\leq a\|f_1\|_{e(\alpha)}\|f_2\|_{e(\beta)}.$$

Proof: By virtue of Lemma 3.2, it is sufficent to prove that for all $y\geq 1$, $z\geq 1$

$$\exp\left((y\cdot z)^\gamma\right)\leq\exp\left(y^\alpha\right)+\exp\left(z^\beta\right). \tag{3.21}$$

At this stage, we shall distinguish two cases:
(1) Let

$$y\leq z^{\frac{\beta-\gamma}{\gamma}}. \tag{3.22}$$

Then, substituting (3.22) to the left side of the inequality (3.21), we get

$$\exp\left((y\cdot z)^\gamma\right)\leq\exp\left(\left(z^{\frac{\beta-\gamma}{\gamma}}\cdot z\right)^\gamma\right)=\exp\left(z^\beta\right).$$

Thus, the inequality (3.21) is fulfilled under the assumption (3.22).
(2) Let us suppose that

$$z^{\frac{\beta-\gamma}{\gamma}}\leq y. \tag{3.23}$$

By virtue of (3.20), we have

$$z\leq y^{\frac{\alpha-\gamma}{\gamma}}$$

which gives

$$\exp\left((y \cdot z)^{\gamma}\right) \leq \exp\left(\left(y \cdot y^{\frac{\alpha-\gamma}{\gamma}}\right)^{\gamma}\right) = \exp\left(y^{\alpha}\right)$$

Thus, the inequality (3.21) has been proved.

Q.E.D.

Lemma 3.7: *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with the Lipschitz boundary. Then*

$$\mathrm{W}^{1,2}(\Omega) \hookrightarrow \mathrm{L}_{e(2)}(\Omega),$$

and

$$\mathrm{W}^{1,2}(\Omega) \hookrightarrow\hookrightarrow \mathrm{L}_{e(\beta)}(\Omega), \qquad \text{for } \beta < 2.$$

Proof: See [7] and [3] (Lemma 7.4.1).

Lemma 3.8: *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with the Lipschitz boundary. Let h_n be a sequence of functions such that*

$$h_n \to h \text{ weakly star in } \mathrm{L}^{\infty}(0,T;\mathrm{L}_{y\log^d(y)}(\Omega)) \quad \text{with } d > \frac{1}{2}. \tag{3.24}$$

Let $\mathcal{Z} \subset \mathrm{E}_{e(1/d)}(\Omega)$ be dense in $\mathrm{E}_{e(1/d)}(\Omega)$ and assume that the functions

$$t \to \int_{\Omega} h_n(t)\psi\,dx \ \text{ are uniformly bounded and uniformly continuous on } [0,T]$$

for any $\psi \in \mathcal{Z}$. Then

$$h_n \to h \text{ in } C([0,T];\mathrm{L}^{*-weak}_{y\log^d(y)}(\Omega)) \tag{3.25}$$

and

$$h_n \to h \text{ in } C([0,T];\mathrm{W}^{-1,2}(\Omega)). \tag{3.26}$$

Remark: Here, the convergence with respect to the weak star topology in (3.25) means that

$$t \to \int_{\Omega} h_n(t) g\,\mathrm{d}x \text{ converges uniformly to } t \to \int_{\Omega} h(t) g\,\mathrm{d}x$$

for any $g \in \mathrm{E}_{e(1/d)}(\Omega)$.

Proof: The set $\mathcal{Z}$ is dense in $\mathrm{E}_{e(1/d)}(\Omega)$ which is a separable space. Thus, there exists a sequence $\psi_j \in \mathcal{Z}$, $j = 1, 2, \ldots$, $\|\psi_j\|_{e(1/d)} = 1$,

$$\mathrm{E}_{e(1/d)}(\Omega) = \overline{\text{span } \{\psi_j\}_{j=1}^{\infty}}^{E_{e(1/d)}(\Omega)}.$$

By virtue of (3.24) and Hahn-Banach theorem, we can change the values of function h_n on the set of zero measure in $(0,T)$ so that we can suppose the existence of a constant m independent on n such that

$$\sup_{t\in[0,T]} \|h_n(t)\|_{y\log^d(y)} \le m \quad \text{and} \quad \sup_{t\in[0,T]} \|h(t)\|_{y\log^d(y)} \le m.$$

Let $\psi \in A = \text{span } \{\psi_j\}_{j=1}^{\infty}$. Then, we can use Arzela-Ascoli theorem to deduce that

$$\int_\Omega h_n(t)\psi \,\mathrm{d}x \rightrightarrows \int_\Omega h(t)\psi \,\mathrm{d}x. \tag{3.27}$$

Let $g \in \mathrm{E}_{e(1/d)}(\Omega)$ and let $\varepsilon > 0$. Then there exists $\psi_\varepsilon \in A$ such that

$$\|\psi_\varepsilon - g\|_{e(1/d)} \le \frac{\varepsilon}{2m}.$$

Thus

$$\sup_{t\in[0,T]} \int_\Omega (h_n(t,x) - h(t,x))g \,\mathrm{d}x \le \sup_{t\in[0,T]} \int_\Omega (h_n(t,x) - h(t,x))(g - \psi_\varepsilon) \,\mathrm{d}x +$$

$$+ \sup_{t\in[0,T]} \int_\Omega (h_n(t,x) - h(t,x))\psi_\varepsilon \,\mathrm{d}x \le \varepsilon + \sup_{t\in[0,T]} \int_\Omega (h_n(t,x) - h(t,x))\psi_\varepsilon \,\mathrm{d}x,$$

which, combined with (3.27), gives (3.25).

Finally, (3.26) is equivalent to the statement

$$\sup_{t\in[0,T]} \sup_{\|\psi\|_{W_0^{1,2}}=1} \int_\Omega (h_n(t) - h(t))\psi \,\mathrm{d}x \to 0.$$

Because of Lemma 3.7, we have the compact imbedding

$$\mathrm{W}^{1,2}(\Omega) \hookrightarrow\hookrightarrow \mathrm{L}_{e(1/d)}(\Omega),$$

thus, there exists a function $\zeta(t) \in \mathrm{L}_{e(1/d)}(\Omega)$,

$$\|\zeta(t)\|_{e(1/d)} \le c, \tag{3.28}$$

such that

$$\sup_{t\in[0,T]} \sup_{\|\psi\|_{W_0^{1,2}}=1} \int_\Omega (h_n(t) - h(t))\psi \,\mathrm{d}x = \sup_{t\in[0,T]} \int_\Omega (h_n(t) - h(t))\zeta(t) \,\mathrm{d}x.$$

Thus

$$\sup_{t\in[0,T]} \sup_{\|\psi\|_{W_0^{1,2}}=1} \int_\Omega (h_n(t) - h(t))\psi \,\mathrm{d}x \le \sup_{t\in[0,T]} \sup_{s\in[0,T]} \int_\Omega (h_n(t) - h(t))\zeta(s) \,\mathrm{d}x. \tag{3.29}$$

Using the estimate (3.28) and $(\mathrm{L}_{y\log^d(y)})^* = \mathrm{L}_{e(1/d)}$, we deduce that there exists a function $\xi \in \mathrm{L}_{e(1/d)}(\Omega)$ such that

$$\sup_{s\in[0,T]} \int_\Omega (h_n(t) - h(t))\zeta(s)\,\mathrm{d}x = \int_\Omega (h_n(t) - h(t))\xi\,\mathrm{d}x$$

which, combined with (3.29) and (3.25), gives (3.26).

Q.E.D.

Lemma 3.9: *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with the Lipschitz boundary. Let h_n be a sequence of functions such that*

$$h_n \to h \text{ weakly star in } \mathrm{L}^\infty(0,T;\mathrm{L}_{e(\beta)}(\Omega)) \quad \text{with } \beta > 0. \tag{3.30}$$

Let the functions

$$t \to \int_\Omega h_n(t)\psi\,dx \text{ are uniformly bounded and uniformly continuous on } [0,T]$$

for any $\psi \in D(\Omega)$. Then

$$h_n \to h \text{ in } C([0,T];\mathrm{L}_{e(\beta)}^{*-weak}(\Omega)) \qquad (3.31).$$

Proof: The set $D(\Omega)$ is dense in $\mathrm{E}_{y\log^{1/\beta}(y)}(\Omega)$ which is a separable space. Thus, there exists a sequence $\psi_j \in D(\Omega)$, $j = 1, 2, \ldots$, $\|\psi_j\|_{y\log^{1/\beta}(y)} = 1$,

$$\mathrm{E}_{y\log^{1/\beta}(y)}(\Omega) = \overline{\text{span } \{\psi_j\}_{j=1}^\infty}^{L_{y\log^{1/\beta}(y)}(\Omega)}.$$

By virtue of (3.30) and Hahn-Banach theorem, we can change the values of function h_n on the set of zero measure in $(0,T)$ so that we can suppose the existence of a constant m independent on n such that

$$\sup_{t\in[0,T]} \|h_n(t)\|_{e(\beta)} \le m \quad \text{and} \quad \sup_{t\in[0,T]} \|h(t)\|_{e(\beta)} \le m.$$

Let $\psi \in A = \text{span } \{\psi_j\}_{j=1}^\infty$. Then, we can use Arzela-Ascoli theorem to deduce that

$$\int_\Omega h_n(t)\psi\,\mathrm{d}x \rightrightarrows \int_\Omega h(t)\psi\,\mathrm{d}x. \tag{3.32}$$

Let $g \in \mathrm{E}_{y\log^{1/\beta}(y)}(\Omega)$ and let $\varepsilon > 0$. Then there exists $\psi_\varepsilon \in A$ such that

$$\|\psi_\varepsilon - g\|_{y\log^{1/\beta}(y)} \le \frac{\varepsilon}{2m}.$$

Thus

$$\sup_{t\in[0,T]}\int_\Omega (h_n(t,x)-h(t,x))g\,\mathrm{d}x \le \sup_{t\in[0,T]}\int_\Omega (h_n(t,x)-h(t,x))(g-\psi_\varepsilon)\,\mathrm{d}x +$$

$$+\sup_{t\in[0,T]}\int_\Omega (h_n(t,x)-h(t,x))\psi_\varepsilon\,\mathrm{d}x \le \varepsilon + \sup_{t\in[0,T]}\int_\Omega (h_n(t,x)-h(t,x))\psi_\varepsilon\,\mathrm{d}x,$$

which, combined with (3.32), gives (3.31).

Q.E.D.

Lemma 3.10: *Let Ω be a bounded domain, let b be a mapping from $\mathbb{R}^2$ to $\mathbb{R}$ and*

$$|\xi|^{|\alpha|}|D^\alpha b(\xi)| \le C < \infty \quad (\textit{for } |\alpha|\le 2,\ \alpha \textit{ is multiindex}).$$

Then for $\beta > 0$ there exists a constant D_β such that for all $g \in \mathrm{L}_{e(\beta)}(\Omega)$

$$\|\mathcal{F}^{-1}b * g\|_{e\left(\frac{\beta}{\beta+1}\right)} \le D_\beta \|g\|_{e(\beta)}, \tag{3.33}$$

where g is prolonged to be 0 on $\mathbb{R}^2-\Omega$ and $\mathcal{F}$ denotes the Fourier transformation.

Proof: Let $g \in \mathrm{L}_{e(\beta)}(\Omega)$, $h \in \mathrm{L}_{y\log^{\frac{\beta+1}{\beta}}(y)}(\Omega)$ and let us prolonge g, h to be 0 on $\mathbb{R}^2-\Omega$. Then, by virtue of the Lemma 3.1,

$$\left|\int_\Omega \left(\mathcal{F}^{-1}b * g\right)\cdot h\,\mathrm{d}x\right| = \left|\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \mathcal{F}^{-1}b(x-y)h(x)\,\mathrm{d}x\, g(y)\,\mathrm{d}y\right| \le$$

$$\le D_\beta \|h\|_{y\log^{\frac{\beta+1}{\beta}}(y)} \|g\|_{e(\beta)},$$

which, combined with the definition of the Orlicz norm (2.1), gives (3.33).

Q.E.D.

Lemma 3.11: *Let v_n, w_n be two sequences,*

$$v_n \to v \quad \textit{weakly in } \mathrm{L}^p(\mathbb{R}^2), \quad w_n \to w \quad \textit{weakly in } \mathrm{L}^q(\mathbb{R}^2)$$

where $1/p+1/q=1/r<1$.

Then

$$v_n\mathcal{R}_{i,j}[w_n] - w_n\mathcal{R}_{i,j}[v_n] \to v\mathcal{R}_{i,j}[w] - w\mathcal{R}_{i,j}[v] \quad \textit{weakly in } L^r(\mathbb{R}^2)$$

where $\mathcal{R}_{i,j}$ are defined by their Fourier symbol

$$\mathcal{R}_{i,j}[u] = \mathcal{F}^{-1}\left(\frac{\xi_i\xi_j}{|\xi|^2}\right) * u, \quad i,j=1,2.$$

Proof: See [2], Lemma 3.4.

Lemma 3.12: *Let* v_n, w_n *be two sequences,*

$$v_n(x) = w_n(x) = 0 \quad \textit{in } \mathbb{R}^2 - \Omega,$$

$$v_n \to v \quad \textit{weakly star in } \mathrm{L}_{e(\beta)}(\Omega), \quad w_n \to w \quad \textit{weakly star in } \mathrm{L}_{y\log^d(y)}(\Omega).$$

Let

$$\gamma = d - \frac{1}{\beta} - 1 > 0.$$

Then

$$v_n\mathcal{R}_{i,j}[w_n] - w_n\mathcal{R}_{i,j}[v_n] \to v\mathcal{R}_{i,j}[w] - w\mathcal{R}_{i,j}[v] \quad \textit{weakly star in } \mathrm{L}_{y\log^\gamma(y)}(\Omega)$$

where $\mathcal{R}_{i,j}$ *are defined by their Fourier symbol*

$$\mathcal{R}_{i,j}[u] = \mathcal{F}^{-1}\left(\frac{\xi_i\xi_j}{|\xi|^2}\right) * u, \quad i,j = 1,2. \tag{3.34}$$

Proof: By virtue of Lemma 3.1 and Lemma 3.10, we obtain the estimates

$$\|\mathcal{R}_{i,j}[v]\|_{y\log^p(y)} \le c(p)\|v\|_{y\log^{(p+1)}(y)}, \tag{3.35}$$

$$\|\mathcal{R}_{i,j}[v]\|_{e\left(\frac{p}{p+1}\right)} \le c(p)\|v\|_{e(p)}, \qquad \text{for } p > 0. \tag{3.36}$$

Let us introduce the cut-off operator

$$Q_k(z) = \begin{cases} -k & \text{for } z \in (-\infty, -k], \\ z & \text{for } z \in (-k, k), \\ k & \text{for } z \in [k, \infty). \end{cases}$$

thus, using (3.35), we see

$$\|\mathcal{R}_{i,j}[Q_k(w_n) - w_n]\|_{y\log^{\frac{1}{\beta}}(y)} \le c\|Q_k(w_n) - w_n\|_{y\log^{\frac{\beta+1}{\beta}}(y)} =$$

$$= c\|\chi_{[w_n \ge k]}\left(Q_k(w_n) - w_n\right)\|_{y\log^{\frac{\beta+1}{\beta}}(y)}. \tag{3.37}$$

As w_n are bounded in $\mathrm{L}_{y\log^d(y)}(\Omega)$ uniformly in n, we see that there exists a constant c independent of n such that

$$\int_\Omega w_n \,\mathrm{d}x \le c, \quad \text{for all } n \in \mathbb{N},$$

thus, there exists a function $r : (0,\infty) \to (0,\infty)$ such that

$$\lim_{k\to\infty} r(k) = 0, \quad \mu\big\{x \in \Omega \,:\, w_n(x) \geq k\big\} \leq r(k), \ \text{ for all } n \in \mathbb{N}. \tag{3.38}$$

Now, we can use Lemma 3.3 to deduce

$$\|\chi_{[w_n\geq k]}\,(Q_k(w_n) - w_n)\,\|_{y\log^{\frac{\beta+1}{\beta}}(y)} \leq c\|Q_k(w_n) - w_n\|_{y\log^d(y)}\|\chi_{[w_n\geq k]}\|_{e(\frac{1}{\gamma})} \leq$$

$$\leq c\sup_n \|w_n\|_{y\log^d(y)}\|\chi_{[w_n\geq k]}\|_{e(\frac{1}{\gamma})} \leq c\|\chi_{[w_n\geq k]}\|_{e(\frac{1}{\gamma})} \tag{3.39}$$

By virtue of the definition of the Luxembourg norm and using (3.38), we easily observe

$$\|\chi_{[w_n\geq k]}\|_{e(\frac{1}{\gamma})} \leq \left(\phi^{-1}\left(\frac{1}{r(k)}\right)\right)^{-1} \tag{3.40}$$

where ϕ is the Young function corresponding to the space $L_{e(\frac{1}{\gamma})}(\Omega)$. Thus, the right hand side in (3.40) tends to 0 for $k \to \infty$ uniformly in n. Summing up the results (3.37) – (3.40), there exists a function $q : (0,\infty) \to (0,\infty)$ such that

$$\lim_{k\to\infty} q(k) = 0$$

and

$$\|\mathcal{R}_{i,j}[Q_k(w_n) - w(n)]\|_{y\log^{\frac{1}{\beta}}(y)} \leq c\|Q_k(w_n) - w(n)\|_{y\log^{\frac{\beta+1}{\beta}}(y)} \leq q(k). \tag{3.41}$$

Consequently,

$$\|\mathcal{R}_{i,j}[\overline{Q_k(w)} - w]\|_{y\log^{\frac{1}{\beta}}(y)} \leq c\|\overline{Q_k(w)} - w\|_{y\log^{\frac{\beta+1}{\beta}}(y)} \leq q(k) \tag{3.42}$$

where $\overline{Q_k(w)}$ stands for a weak limit of $Q_k(w_n)$.

Finally, we write

$$\Big[v_n\mathcal{R}_{i,j}[w_n] - w_n\mathcal{R}_{i,j}[v_n]\Big] - \Big[v\mathcal{R}_{i,j}[w] - w\mathcal{R}_{i,j}[v]\Big] =$$

$$= \left(\Big[v_n\mathcal{R}_{i,j}[Q_k(w_n)] - Q_k(w_n)\mathcal{R}_{i,j}[v_n]\Big] - \Big[v\mathcal{R}_{i,j}[\overline{Q_k(w)}] - \overline{Q_k(w)}\mathcal{R}_{i,j}[v]\Big]\right) +$$

$$+ \Big[v_n\mathcal{R}_{i,j}[w_n - Q_k(w_n)] - (w_n - Q_k(w_n))\mathcal{R}_{i,j}[v_n]\Big] +$$

$$+ \Big[v\mathcal{R}_{i,j}[\overline{Q_k(w)} - w] - (\overline{Q_k(w)} - w)\mathcal{R}_{i,j}[v]\Big]$$

where the firts term on the right-hand side converges to zero in $D'(\Omega)$ because of Lemma 3.11 and the rest is uniformly small for large k in view of the estimates (3.41) and (3.42) together with (3.35) and (3.36). Thus Lemma 3.12 have been proved.

Q.E.D.

Lemma 3.13: *Let $z > y \geq 0$. Then*

$$z \log^d(1+z) - y \log^d(1+y) \geq \frac{1}{2}(z-y)\log^d(1+z-y).$$

Proof: Let us distinguish two cases:
(1) Let $2y \geq z$. Then, by virtue of the mean value theorem, we get

$$\frac{z \log^d(1+z) - y \log^d(1+y)}{z-y} \geq \log^d(1+y) \geq \log^d(1+z-y).$$

(2) Let $z \geq 2y$. Then

$$z \log^d(1+z) - y \log^d(1+y) \geq z \log^d(1+z) - \frac{z}{2}\log^d(1+z) =$$

$$= \frac{1}{2} z \log^d(1+z) \geq \frac{1}{2}(z-y)\log^d(1+z-y).$$

Q.E.D.

Lemma 3.14: *Let $d > 0$, $\beta > 2/d$, $y_0^d > 0$ and Ω be a bounded domain. Let Φ_d be a Young function satisfying $\Phi_d(y) = y^2 \log^{-d}(y)$ for $y \geq y_0^d$. Then there exists a constant a such that*

$$\|f_1 f_2\|_{\Phi_d} \leq \|f_1\|_{e(\beta)} \|f_2\|_{L^2}.$$

Proof: By virtue of Lemma 3.2, it is sufficent to prove that there exist constants y_0, z_0 such that for all $y \geq y_0$, $z \geq z_0$

$$\left(\frac{y \cdot z}{2}\right)^2 \log^{-d}\left(\frac{y \cdot z}{2}\right) \leq \exp\left(y^\beta\right) + z^2. \tag{3.43}$$

At this stage, we shall distinguish two cases:
(1) Let

$$z \leq \exp\left(y^{2/d}\right). \tag{3.44}$$

Then, substituting (3.44) to the left side of the inequality (3.43), we get for sufficiently large y

$$\left(\frac{y \cdot z}{2}\right)^2 \log^{-d}\left(\frac{y \cdot z}{2}\right) \leq \exp\left(2y^{2/d}\right)\frac{y^2}{4}\log^{-d}\left(\exp\left(y^{2/d}\right)\right) =$$

$$= \exp\left(2y^{2/d}\right)\frac{1}{4} \leq \exp\left(y^\beta\right),$$

provided $\beta > 2/d$. Thus, the inequality (3.43) is fulfilled under the assumption (3.44).

(2) Let us suppose that

$$\exp\left(y^{2/d}\right) \leq z. \tag{3.45}$$

Let us define the auxiliary function $\xi : (0,\infty) \to (0,\infty)$ by the formula

$$\xi(t) = \exp\left(t^{2/d}\right).$$

Then ξ is continuous and increasing function, thus, (3.45) is equivalent to the condition $y \leq \xi^{-1}(z)$. Substituting this inequality in the left side of (3.43) we get (we use the obvious estimate $\xi^{-1}(z) \geq 4$ for sufficiently large z)

$$\left(\frac{y \cdot z}{2}\right)^2 \log^{-d}\left(\frac{y \cdot z}{2}\right) \leq \frac{1}{4}\left(\xi^{-1}(z) z\right)^2 \log^{-d}\left(z\frac{\xi^{-1}(z)}{4}\right) \leq \frac{1}{4}\left(\xi^{-1}(z)\right)^2 z^2 \log^{-d} z.$$

Thus, (3.43) will be a consequence of the following inequality

$$\xi^{-1}(z) \leq 2\log^{d/2} z,$$

which is equivalent to the inequality

$$z \leq \xi\left(2\log^{d/2} z\right) = \exp\left(2^{2/d}\log z\right) = z^{2^{2/d}}.$$

The last inequality is obviously fulfilled for sufficiently large z. Thus Lemma 3.14 has been proved.

Q.E.D.

Lemma 3.15: *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and let $d > 0$. Let Φ_γ, $\gamma > 0$, be a Young function satisfying $\Phi_\gamma(y) = y^2 \log^{-\gamma}(y)$ for $y \geq y_0^\gamma$ and*

$$\int_0^1 \frac{\Phi_\gamma^{-1}(s)}{s^{3/2}}\, ds < \infty. \tag{3.46}$$

Then there exists a constant D_d such that for all $g \in \mathrm{L}_{\Phi_d}(\Omega)$

$$\|\mathcal{A}_i[g]\|_{e\left(\frac{2}{d+3}\right)} \leq D_d \|g\|_{\Phi_d}, \tag{3.47}$$

where $\mathcal{A}_i$, $i = 1,2$, are defined by their Fourier symbol

$$\mathcal{A}_i[v] = \mathcal{F}^{-1}\left(\frac{-i\xi_i}{|\xi|^2}\right) * v$$

and g is prolonged to be 0 on $\mathbb{R}^2 - \Omega$.

Proof: We shall prove Lemma 3.15 in two steps.

(1) At first, we shall prove the inequality

$$\|\mathcal{A}_i[g]\|_{e\left(\frac{2}{d+3}\right)} \leq c(d)\,\|\mathcal{A}_i[g]\|_{WL_{\Phi_{d+1}}}. \tag{3.48}$$

To this end, we will use imbedding theorems for Sobolev-Orlicz spaces (see [3], Section 7.2). By virtue of (3.46), we can define the Sobolev conjugate Φ^*_{d+1} by the formula

$$\left(\Phi^*_{d+1}\right)^{-1}(y) = \int_0^y \frac{\left(\Phi_{d+1}\right)^{-1}(s)}{s^{3/2}}\,\mathrm{d}s.$$

Then, using [3], Theorem 7.2.3, we get

$$\|\mathcal{A}_i[g]\|_{\Phi^*_{d+1}} \leq c(d)\|\mathcal{A}_i[g]\|_{WL_{\Phi_{d+1}}}. \tag{3.49}$$

Let us define an auxiliary function $\xi : [0,\infty) \to [0,\infty)$ by the formula

$$\xi(z) = \begin{cases} \sqrt{z}\log^{\frac{d+1}{2}} z & \text{for } z \geq z_0, \\ \left(\Phi_{d+1}\right)^{-1}(z) & \text{for } z < z_0, \end{cases}$$

where z_0 is chosen so that

$$\Phi_{d+1}(\Phi^{-1}_{d+1}(z)) = z \leq \Phi_{d+1}(\xi(z)) \quad \text{for all } z \geq 0.$$

Thus, $\Phi^{-1}_{d+1}(z) \leq \xi(z)$ and consequently

$$\left(\Phi^*_{d+1}\right)^{-1}(z) = \int_0^z \frac{\left(\Phi_{d+1}\right)^{-1}(s)}{s^{3/2}}\,\mathrm{d}s \leq \int_0^z \frac{\xi(s)}{s^{3/2}}\,\mathrm{d}s \leq \frac{2}{d+3}\log^{\frac{d+3}{2}} z \quad \text{for } z \geq z_0$$

which implies

$$\exp\left(y^{\frac{2}{d+3}}\right) \leq \Phi^*_{d+1} \quad \text{for } y \geq \log^{\frac{d+3}{2}} z_0,$$

in particular

$$\|\mathcal{A}_i[g]\|_{e\left(\frac{2}{d+3}\right)} \leq c(d)\|\mathcal{A}_i[g]\|_{\Phi^*_{d+1}}. \tag{3.50}$$

Finally, (3.49) and (3.50) yield (3.48).

(2) Now, we shall prove the inequality:

$$\|\mathcal{A}_i[g]\|_{WL_{\Phi_{d+1}}} \leq c(d)\|g\|_{\Phi_d}. \tag{3.51}$$

To this end, let us introduce the operator $\mathcal{R}_{i,j} = \partial_{x_j}\mathcal{A}_i$, $i,j = 1,2$, defined by the formula (3.34). The operator $\mathcal{R}_{i,j}$ satisfies the assumptions of the standard Mikhlin multiplier theorem, see [1], Theorem 6.1.6. Thus, we can show (compare with [1] and the proof of our Lemma 3.1) that $\mathcal{R}_{i,j}$ is bounded mapping as

$$\mathcal{R}_{i,j} : \mathrm{L}^2(\mathbb{R}^2) \to \mathrm{L}^2(\mathbb{R}^2), \qquad \mathcal{R}_{i,j} : \mathrm{L}^1(\mathbb{R}^2) \to \mathrm{L}^{1\infty}(\mathbb{R}^2), \tag{3.52}$$

where $L^{1\infty}(\mathbb{R}^2)$ is one of the Lorentz spaces (we have $u \in L^{1\infty}(\mathbb{R}^2)$ if and only if $\sup_\sigma \sigma m(\sigma, u) < \infty$ where $m(\sigma, u) = \mu\{x : |u(x)| > \sigma\}$).

Now, we are in the same situation as in the proof of Lemma 3.1. Persuing step by step the proof of Lemma 3.1, let us suppose that $g \in L_{\Phi_d}(\Omega)$ and $\|g\|_{\Phi_d} = 1$. Let $\sigma > 0$ and define $g = g_1 + g_2$, where g_2 satisfies (3.3) and $g_1 = g - g_2$. Then clearly $g_k \in L^k(\Omega)$, $k = 1, 2$. Let $h_k = \mathcal{R}_{i,j}[g_k]$, $k = 1, 2$, and let $h = h_1 + h_2$. Let us define the distribution functions of g (resp. g_k, h and h_k in the similar way) by the formula (3.4). By virtue of (3.52), we have (3.5) for all $\sigma > 0$, where C_1 and C_2 do not depend on σ. Let us remind that the functions g_k, $k = 1, 2$, depend on σ.

Now, we shall estimate the Luxembourg norm of $h = \mathcal{R}_{i,j}[g]$ in the space $L_{\Phi_{d+1}}(\Omega)$. Let $k \geq 1$. Let us fix $q \in (1, \infty)$ such that

$$q \geq y_0^{d+1}, \quad \frac{1}{\log^{1/2} q} \leq 1 - \frac{d + 1/2}{\log q} \quad \text{and} \quad \frac{1}{\log q} \leq 1 - \frac{d}{2 \log q}. \tag{3.53}$$

Then, similarly as in the proof of Lemma 3.1, we derive the estimate (compare with (3.6), (3.7) and (3.8))

$$\int_\Omega \Phi_{d+1}\left(\frac{|h(x)|}{k}\right) \mathrm{d}x \leq \frac{C_1}{k} \int_q^\infty \frac{1}{\tau} \int_\tau^\infty m(t, g) \,\mathrm{d}t \,\frac{\mathrm{d}\,\Phi_{d+1}}{\mathrm{d}\,y}(\tau) \,\mathrm{d}\tau +$$
$$+ \frac{C_2}{k} \int_q^\infty \frac{1}{\tau^2} \int_0^\tau 2t \cdot m(t, g) \,\mathrm{d}t \,\frac{\mathrm{d}\,\Phi_{d+1}}{\mathrm{d}\,y}(\tau) \,\mathrm{d}\tau + \frac{1}{k}\mu(\Omega)\Phi_{d+1}(q). \tag{3.54}$$

Let us estimate the first integral on the right side of (3.54). Changing the order of integration, using (3.53) and using the explicit formula for $\Phi_{d+1}(y) = y^2 \log^{-d-1}(y)$, $y \geq q$, this gives

$$\int_q^\infty \frac{1}{\tau} \int_\tau^\infty m(t, g) \,\mathrm{d}t \,\frac{\mathrm{d}\,\Phi_{d+1}}{\mathrm{d}\,y}(\tau) \,\mathrm{d}\tau = \int_q^\infty m(t, g) \int_q^t \frac{1}{\tau} \frac{\mathrm{d}\,\Phi_{d+1}}{\mathrm{d}\,y}(\tau) \,\mathrm{d}\tau \,\mathrm{d}t =$$
$$= \int_q^\infty m(t, g) \int_q^t \frac{2}{\log^{d+1}(\tau)} - \frac{d+1}{\log^{d+2}(\tau)} \,\mathrm{d}\tau \,\mathrm{d}t =$$
$$= \int_q^\infty m(t, g) \int_q^t \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\tau}{\log^{d+1}(\tau)}\right) + \frac{1}{\log^{d+1}(\tau)} \,\mathrm{d}\tau \,\mathrm{d}t \leq$$
$$\leq \int_q^\infty m(t, g) \int_q^t \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\tau}{\log^{d+1}(\tau)}\right) + \frac{1}{\log^{d+1/2}(\tau)}\left(1 - \frac{d + 1/2}{\log \tau}\right) \mathrm{d}\tau \,\mathrm{d}t =$$
$$= \int_q^\infty m(t, g) \int_q^t \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\tau}{\log^{d+1}(\tau)} + \frac{\tau}{\log^{d+1/2}(\tau)}\right) \mathrm{d}\tau \,\mathrm{d}t \leq$$
$$\leq \int_q^\infty m(t, g) \left(\frac{t}{\log^{d+1}(t)} + \frac{t}{\log^{d+1/2}(t)}\right) \mathrm{d}t =$$

$$= \int_\Omega \chi_{[q \le |g(x)|]} \int_q^{|g(x)|} \frac{t}{\log^{d+1}(t)} + \frac{t}{\log^{d+1/2}(t)} \,\mathrm{d}t\,\mathrm{d}x \le$$

$$\le \int_\Omega \chi_{[q \le |g(x)|]} \int_q^{|g(x)|} \frac{t}{\log^{d}(t)} \left(1 - \frac{d}{2\log t}\right) + \frac{t}{\log^{d}(t)} \left(1 - \frac{d}{2\log t}\right) \mathrm{d}t\,\mathrm{d}x =$$

$$= \int_\Omega \chi_{[q \le |g(x)|]} \int_q^{|g(x)|} \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{t^2}{\log^d(t)}\right) \mathrm{d}t\,\mathrm{d}x \le$$

$$\le \int_\Omega \chi_{[q \le |g(x)|]} \frac{|g(x)|^2}{\log^d(|g(x)|)} \,\mathrm{d}x \le \int_\Omega \Phi_d(|g(x)|)\,\mathrm{d}x \le 1, \tag{3.55}$$

where we use the assumption $\|g\|_{\Phi d} = 1$. Now, we shall estimate the second integral in (3.54). We shall use the similar arguments as in the estimation of the first integral in (3.54).

$$\int_q^\infty \frac{1}{\tau^2} \int_0^\tau 2t \cdot m(t,g)\,\mathrm{d}t \,\frac{\mathrm{d}\,\Phi_{d+1}}{\mathrm{d}\,y}(\tau)\,\mathrm{d}\tau = \int_0^\infty 2t \cdot m(t,g) \int_{\max(t,q)}^\infty \frac{1}{\tau^2} \frac{\mathrm{d}\,\Phi_{d+1}}{\mathrm{d}\,y}(\tau)\,\mathrm{d}\tau\,\mathrm{d}t =$$

$$= \int_0^\infty 2t \cdot m(t,g) \int_{\max(t,q)}^\infty \frac{2}{\tau \log^{d+1}(\tau)} - \frac{d+1}{\tau \log^{d+2}(\tau)} \,\mathrm{d}\tau\,\mathrm{d}t =$$

$$= \int_0^q 2t \cdot m(t,g) \int_q^\infty \frac{\mathrm{d}}{\mathrm{d}\tau} \left(-\frac{2}{d\log^d(\tau)} + \frac{1}{\log^{d+1}(\tau)}\right) \mathrm{d}\tau\,\mathrm{d}t +$$

$$+ \int_q^\infty 2t \cdot m(t,g) \int_t^\infty \frac{\mathrm{d}}{\mathrm{d}\tau} \left(-\frac{2}{d\log^d(\tau)} + \frac{1}{\log^{d+1}(\tau)}\right) \mathrm{d}\tau\,\mathrm{d}t. \tag{3.56}$$

The first integral on the right-hand side of (3.56) is obviously finite, thus we shall estimate the second integral on the right hand side of (3.56):

$$\int_q^\infty 2t \cdot m(t,g) \int_t^\infty \frac{\mathrm{d}}{\mathrm{d}\tau} \left(-\frac{2}{d\log^d(\tau)} + \frac{1}{\log^{d+1}(\tau)}\right) \mathrm{d}\tau\,\mathrm{d}t \le$$

$$\le \int_q^\infty m(t,g) \left(\frac{4t}{d\log^d(t)} - \frac{2t}{\log^{d+1}(t)}\right) \mathrm{d}\tau\,\mathrm{d}t =$$

$$= \int_\Omega \chi_{[q \le |g(x)|]} \int_q^{|g(x)|} \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{2t^2}{d\log^d(t)}\right) \mathrm{d}t\,\mathrm{d}x \le$$

$$\le \int_\Omega \chi_{[q \le |g(x)|]} \frac{2|g(x)|^2}{d\log^d(|g(x)|)} \,\mathrm{d}x \le \frac{2}{d} \int_\Omega \Phi_d(|g(x)|)\,\mathrm{d}x \le \frac{2}{d}. \tag{3.57}$$

Substituting (3.55), (3.56), (3.57) in (3.54), we get

$$\int_\Omega \Phi_{d+1}\left(\frac{|h(x)|}{k}\right) \mathrm{d}x \le \frac{1}{k} c(d)$$

where the constant $c(d)$ depends only on d. Thus

$$\|\mathcal{R}_{i,j}[g]\|_{\Phi_{d+1}} \leq c(d) \qquad \text{for } \|g\|_{\Phi_d} = 1$$

and (3.51) follows.

Finally, combining (3.48) and (3.51), we get (3.47). Thus Lemma 3.15 has been proved.

Q.E.D.

4 The Faedo-Galerkin approximation

Our first goal is to solve the problem:

$$\varrho_t + \text{div } (\varrho \vec{u}) = \varepsilon \Delta \varrho, \tag{4.1}$$

$$(\varrho u^i)_t + \text{div } (\varrho u^i \vec{u}) + a \left(\varrho \log^d (1+\varrho) \right)_{x_i} + \delta \left(\varrho^\beta \right)_{x_i} + \varepsilon \nabla u^i \cdot \nabla \varrho =$$

$$= \mu \Delta u^i + (\lambda + \mu)(\text{div } \vec{u})_{x_i}, \qquad i = 1, 2, \tag{4.2}$$

complemented by the boundary conditions:

$$\nabla \varrho \cdot \vec{n}|_{\partial \Omega} = 0, \tag{4.3}$$

$$\vec{u}|_{\partial \Omega} = 0, \tag{4.4}$$

and modified initial data:

$$\varrho(0) = \varrho_0 \in C^{2+v}(\overline{\Omega}),\ 0 < \underline{\varrho} \leq \varrho_0(x) \leq \overline{\varrho},\ \nabla \varrho_0 \cdot \vec{n}|_{\partial \Omega} = 0, \tag{4.5}$$

$$(\varrho \vec{u})(0) = \vec{q},\ \vec{q} = [q^1, q^2],\ q^i \in C^2(\overline{\Omega}),\ i = 1, 2. \tag{4.6}$$

4.1 The Faedo-Galerkin approximation scheme

The following auxiliary result may be found in [2] (Lemma 2.2):

Lemma 4.1: *Let Ω is a bounded domain of the class C^{2+v}. Let the initial datum ϱ_0 satisfy (4.5). Then there exists a mapping $\mathcal{S} = \mathcal{S}(\vec{u})$,*

$$\mathcal{S} : \left\{ u \in C\left([0,T]; [C^2(\overline{\Omega})]^2 \right) \,:\, \vec{u}|_{\partial \Omega} = 0 \right\} \to C\left([0,T]; C^{2+v}(\overline{\Omega}) \right)$$

enjoying the following properties:

- $\varrho = \mathcal{S}(\vec{u})$ *is the unique classical solution of (4.1), (4.3), (4.5);*

- $$\underline{\varrho} \exp\left(-\int_0^t \|\mathrm{div}\ \vec{u}(s)\|_{L^\infty(\Omega)}\, ds\right) \le \mathcal{S}(\vec{u})(t,x) \le \overline{\varrho} \exp\left(\int_0^t \|\mathrm{div}\ \vec{u}(s)\|_{L^\infty(\Omega)}\, ds\right) \tag{4.7}$$

  for all $t \ge 0$;

- $$\|\mathcal{S}(\vec{u}^1) - \mathcal{S}(\vec{u}^2)\|_{C([0,T];W^{1,2}(\Omega))} \le Tc(\kappa, T)\|\vec{u}^1 - \vec{u}^2\|_{C([0,T];W_0^{1,2}(\Omega))} \tag{4.8}$$

  for any $\vec{u}^1$, $\vec{u}^2$ *belonging to the set*

  $$M_\kappa = \left\{\vec{u} \in C([0,T]; \mathrm{W}_0^{1,2}(\Omega)) \mid \|\vec{u}(t)\|_{L^\infty(\Omega)} + \|\nabla\vec{u}(t)\|_{L^\infty(\Omega)} \le \kappa \text{ for all } t\right\}.$$

The constant $c(\kappa, T)$ *satisfies the condition* $c(\kappa, T_1) \le c(\kappa, T_2)$ *for* $T_1 \le T_2$.

Let ψ_n are the eigenfunctions of the Laplacian:

$$-\Delta\psi_n = \lambda_n\psi_n, \quad \psi_n|_{\partial\Omega} = 0.$$

Now, let us consider a sequence of finite dimensional spaces

$$X_n = \left[\mathrm{span}\ \{\psi\}_{j=1}^n\right]^2, \quad n = 1, 2, \dots$$

The approximate solution $\vec{u} \in C([0,T]; X_n)$ we shall look for are required to satisfy the integral equation

$$\int_\Omega \varrho(t)\vec{u}_n(t) \cdot \vec{\psi}\, \mathrm{d}x - \int_\Omega \vec{q} \cdot \vec{\psi}\, \mathrm{d}x = \tag{4.9}$$

$$= \int_\Omega \int_0^t \left[\nabla\left((\lambda + \mu)\mathrm{div}\ \vec{u}_n - a\varrho\log^d(1+\varrho) - \delta\varrho^\beta\right) - \mathrm{div}\ (\varrho\vec{u}_n \otimes \vec{u}_n)\right] \cdot \vec{\psi}\, \mathrm{d}s\, \mathrm{d}x +$$

$$+ \int_\Omega \int_0^t \left[\mu\Delta\vec{u}_n - \varepsilon\nabla\varrho \cdot \nabla\vec{u}_n\right] \cdot \vec{\psi}\, \mathrm{d}s\, \mathrm{d}x$$

for all $t \in [0,T]$ and any function $\vec{\psi} \in X_n$. Next we introduce a family of operators (let $\eta > 0$)

$$\mathcal{M}[\varrho] : X_n \to X_n^*, \ < \mathcal{M}[\varrho]\vec{v}, \vec{w} >= \int_\Omega \varrho\vec{v} \cdot \vec{w}\, \mathrm{d}x$$

for

$$\varrho \in N_\eta = \left\{\varrho \in \mathrm{L}^1(\Omega) \mid \inf_{x\in\Omega} \varrho \ge \eta > 0\right\}.$$

If ϱ is strictly positive on Ω then the formula $\int_\Omega \varrho\vec{v} \cdot \vec{w}\, \mathrm{d}x$ defines an inner product on X_n. Thus, by virtue of the Riesz Representation Theorem, the operators $\mathcal{M}[\varrho]$ are invertible provided ϱ is strictly positive on Ω, and we have

$$\|\mathcal{M}^{-1}[\varrho]\|_{\mathcal{L}(X_n^*, X_n)} \le \left(\inf_{x\in\Omega} \varrho(x)\right)^{-1}.$$

Moreover, making use of the identities

$$\mathcal{M}^{-1}[\varrho^1] - \mathcal{M}^{-1}[\varrho^2] = \mathcal{M}^{-1}[\varrho^2]\left(\mathcal{M}[\varrho^2] - \mathcal{M}[\varrho^1]\right)\mathcal{M}^{-1}[\varrho^1],$$

and

$$\|\mathcal{M}[\varrho^2] - \mathcal{M}[\varrho^1]\|_{\mathcal{L}(X_n,X_n^*)} = \sup_{\|\vec{v}\|_{X_n}=1} \sup_{\|\vec{w}\|_{X_n}=1} \int_\Omega \left(\varrho^2 - \varrho^1\right)\vec{v}\cdot\vec{w}\,\mathrm{d}x \leq c(n)\|\varrho^2 - \varrho^1\|_{L^1(\Omega)},$$

one can see that the map $\varrho \to \mathcal{M}^{-1}[\varrho]$ mapping N_η into $\mathcal{L}(X_n^*, X_n)$ is well defined and satisfies

$$\|\mathcal{M}^{-1}[\varrho^1] - \mathcal{M}^{-1}[\varrho^2]\|_{\mathcal{L}(X_n^*,X_n)} \leq c(n,\eta)\|\varrho^1 - \varrho^2\|_{L^1(\Omega)} \tag{4.10}$$

for any ϱ^1, ϱ^2 from the set N_η.

Now, the identity (4.9) can be rephrased as follows:

$$\vec{u}_n(t) = \mathcal{M}^{-1}[\varrho(t)]\Big((\vec{q})^* + \mathcal{N}\big[\varrho(t), \vec{u}_n(t)\big]\Big) \tag{4.11}$$

where

$$< (\vec{q})^*, \vec{\psi} >= \int_\Omega \vec{q}_0\cdot\vec{\psi}\,\mathrm{d}x$$

and

$$< \mathcal{N}\big[\varrho(t), \vec{u}_n(t)\big], \vec{\psi} >= \int_\Omega\int_0^t \Big[\nabla\Big((\lambda+\mu)\mathrm{div}\ \vec{u}_n - a\varrho\log^d(1+\varrho) - \delta\varrho^\beta\Big)\Big]\cdot\vec{\psi}\,\mathrm{d}s\,\mathrm{d}x +$$

$$+\int_\Omega\int_0^t \left[\mu\Delta\vec{u}_n - \mathrm{div}\ (\varrho\vec{u}_n\otimes\vec{u}_n) - \varepsilon\nabla\varrho\cdot\nabla\vec{u}_n\right]\cdot\vec{\psi}\,\mathrm{d}s\,\mathrm{d}x$$

for all $\vec{\psi} \in X_n$.

4.2 The approximate solutions

The approximate solutions for the problem (4.1) – (4.6) will be found by means of (4.11) where we take $\varrho = \mathcal{S}(\vec{u}_n)$, the mapping $\mathcal{S}$ being defined in Lemma 4.1. Accordingly, the resulting equation reads:

$$\vec{u}_n(t) = \mathcal{M}^{-1}[\mathcal{S}(\vec{u}_n)(t)]\Big((\vec{q})^* + \mathcal{N}\big[\mathcal{S}(\vec{u}_n)(t), \vec{u}_n(t)\big]\Big) \tag{4.12}$$

where

$$\vec{u}_n \in C([0,T],\, X_n), \quad \vec{u}_n|_{\partial\Omega} = 0.$$

We shall want to solve the integral equation (4.12) by means of Banach fixed point argument. For that reason, let us consider $\vec{v} \in C([0,T], X_n)$, $\vec{w} \in C([0,T], X_n)$, $\vec{\psi} \in X_n$ and let us define the auxiliary function $j : [0,1] \times [0,T] \to \mathbb{R}$ by the formula

$$j(\xi, t) = < \mathcal{N}\big[\mathcal{S}\left(\vec{v} + \xi(\vec{w} - \vec{v})\right)(t), \left(\vec{v} + \xi(\vec{w} - \vec{v})\right)(t)\big], \vec{\psi} > .$$

Thus, $j(\cdot, t)$ is continuously differentiable function and, by virtue of (4.7) and (4.8), we have (let us note that all norms are equivalent on X_n)

$$\left|\frac{\partial j}{\partial \xi}(\xi, t)\right| \le t\, c(n)\, \|\vec{w} - \vec{v}\|_{C([0,T],X_n)} \|\vec{\psi}\|_{X_n} \text{ for all } \xi \in (0,1),\ t \in [0,T].$$

Thus

$$\|\mathcal{N}\big[\mathcal{S}\left(\vec{w}\right)(t), \vec{w}(t)\big] - \mathcal{N}\big[\mathcal{S}\left(\vec{v}\right)(t), \vec{v}(t)\big]\|_{X_n^*} = \sup_{\|\vec{\psi}\|_{X_n}=1} |j(1,t) - j(0,t)| \le$$

$$\le \sup_{\|\vec{\psi}\|_{X_n}=1} \sup_{\xi \in (0,1)} \left|\frac{\partial j}{\partial \xi}(\xi, t)\right| \le t\, c(n)\, \|\vec{w} - \vec{v}\|_{C([0,T],X_n)}. \tag{4.13}$$

Thus, by virtue of the (4.8), (4.10) and (4.13), we get

$$\|\mathcal{M}^{-1}\left[\mathcal{S}(\vec{w})(t)\right]\Big((\vec{q})^* + \mathcal{N}\big[\mathcal{S}(\vec{w})(t), \vec{w}(t)\big]\Big) - \mathcal{M}^{-1}\left[\mathcal{S}(\vec{v})(t)\right]\Big((\vec{q})^* + \mathcal{N}\big[\mathcal{S}(\vec{v})(t), \vec{v}(t)\big]\Big)\|_{X_n} \le$$

$$\le t\, c(n, T, \varrho_0, \vec{q})\, \|\vec{w} - \vec{v}\|_{C([0,T],X_n)},$$

which is sufficent for solving the integral equation (4.12) at least on a short time interval $[0, T(n)]$, $T(n) \le T$. Using Banach fixed point theorem we obtain a local solution ϱ_n, $\vec{u}_n$ of the problem (4.1), (4.9) complemented by the conditions (4.3) and (4.5) for ϱ_n. Now, in order to show that $T(n) = T$ for any n, it is enough to prove that $\|\vec{u}_n(t)\|_{X_n}$ stays bounded on the whole interval $[0, T(n)]$.

To get uniform bounds on $\vec{u}_n(t)$, we derive an energy inequality. Differentiating (4.9) with respect to t, taking $\psi = \vec{u}_n(t)$ and using integration per partes and (4.1), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega \frac{1}{2}\varrho_n|\vec{u}_n(t)|^2 + \frac{\delta}{\beta - 1}\varrho_n^\beta \,\mathrm{d}x\right) + \int_\Omega \mu|\nabla \vec{u}_n|^2 + (\lambda + \mu)|\mathrm{div}\ \vec{u}_n|^2 \,\mathrm{d}x + \tag{4.14}$$

$$+\varepsilon\delta\beta \int_\Omega \varrho_n^{\beta-2}|\nabla \varrho_n|^2 \,\mathrm{d}x + \int_\Omega \nabla\left(a\varrho_n \log^d(1 + \varrho_n)\right) \cdot \vec{u}_n \,\mathrm{d}x = 0 \ \text{ on } \ (0, T(n)).$$

Now, we will deal with the last term on the left side of the equation (4.14). Using integration per partes and (4.1), we obtain

$$\int_\Omega \nabla\left(a\varrho_n \log^d(1 + \varrho_n)\right) \cdot \vec{u}_n \,\mathrm{d}x =$$

$$= \int_\Omega \left(a \log^d(1+\varrho_n) + a\varrho_n \frac{d \log^{d-1}(1+\varrho_n)}{1+\varrho_n} \right) \nabla \varrho_n \cdot \vec{u}_n \, \mathrm{d}x =$$

$$= \int_\Omega \left(a \frac{\log^d(1+\varrho_n)}{1+\varrho_n} \frac{1+\varrho_n}{\varrho_n} + a \frac{d \log^{d-1}(1+\varrho_n)}{1+\varrho_n} \right) \nabla \varrho_n \cdot (\varrho_n \vec{u}_n) \, \mathrm{d}x =$$

$$= \int_\Omega \nabla \left(a \frac{\log^{d+1}(1+\varrho_n)}{d+1} + a \log^d(1+\varrho_n) + \int_0^{\varrho_n} a \frac{\log^d(1+s)}{s(1+s)} \, \mathrm{d}s \right) \cdot (\varrho_n \vec{u}_n) \, \mathrm{d}x =$$

$$= \frac{\mathrm{d}}{\mathrm{d}t} \left(\int_\Omega \frac{a}{d+1} (1+\varrho_n) \log^{d+1}(1+\varrho_n) + R(\varrho_n) \, \mathrm{d}x \right) +$$

$$+ \varepsilon \int_\Omega \left(a \frac{\log^d(1+\varrho_n)}{1+\varrho_n} + a \frac{d \log^{d-1}(1+\varrho_n)}{1+\varrho_n} \right) |\nabla \varrho_n|^2 \, \mathrm{d}x \qquad (4.15)$$

where R is an auxiliary function $R : [0, \infty) \to [0, \infty)$ defined by the formula

$$R(y) = \int_0^y \int_0^\sigma a \frac{\log^d(1+s)}{s(1+s)} \, \mathrm{d}s \, \mathrm{d}\sigma.$$

By virtue of (4.14) and (4.15), we derive an energy inequality

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\int_\Omega \frac{1}{2} \varrho_n |\vec{u}_n(t)|^2 + \frac{\delta}{\beta - 1} \varrho_n^\beta + \frac{a}{d+1} (1+\varrho_n) \log^{d+1}(1+\varrho_n) + R(\varrho_n) \, \mathrm{d}x \right) + \qquad (4.16)$$

$$+ \int_\Omega \mu |\nabla \vec{u}_n|^2 + (\lambda + \mu) |\mathrm{div}\ \vec{u}_n|^2 \, \mathrm{d}x + \frac{4\varepsilon\delta}{\beta} \int_\Omega \left| \nabla (\varrho_n)^{\frac{\beta}{2}} \right|^2 \, \mathrm{d}x \leq 0 \ \text{ on } \ (0, T(n)).$$

Let us denote

$$E_\delta[\varrho_0, \vec{q}] = \int_\Omega \frac{1}{2} \frac{|\vec{q}|}{\varrho_0} + \frac{\delta}{\beta - 1} \varrho_0^\beta + \frac{a}{d+1} (1+\varrho_0) \log^{d+1}(1+\varrho_0) + R(\varrho_0) \, \mathrm{d}x. \qquad (4.17)$$

The first consequence of (4.16) is the estimate

$$\sup_{t \in [0, T(n)]} \int_\Omega \varrho_n(t) |\vec{u}_n(t)|^2 \, \mathrm{d}x \leq 2E_\delta[\varrho_0, \vec{q}].$$

which, combined with (4.7) and the fact that all norms are equivalent on X_n, yields

$$\sup_{t \in [0, T(n)]} \|\vec{u}_n(t)\|_{X_n} \leq c(n, \varrho_0, \vec{q}).$$

The last relation and (4.7) furnish the desired estimates which give the possibility to repeat the above fixed point argument to conclude, after a finite number steps, that $T(n) = T$.

To conclude this part we sum up the consequences of the energy inequality (4.16). The following lemma deals with the properties of the sequence of the approximate solutions ϱ_n, $\vec{u}_n$:

Lemma 4.2: *Assume $\beta \geq 4$. Let ϱ_n, $\vec{u}_n$ be the solution of (4.1), (4.9) on $(0,T) \times \Omega$ constructed above. Then*

$$\sup_{t\in[0,T]} \|\varrho_n(t)\|_{y\log^{(d+1)}(y)} \leq cE_\delta[\varrho_0, \vec{q}], \tag{4.18}$$

$$\delta \sup_{t\in[0,T]} \|\varrho_n(t)\|^\beta_{L^\beta(\Omega)} \leq cE_\delta[\varrho_0, \vec{q}], \tag{4.19}$$

$$\sup_{t\in[0,T]} \|\sqrt{\varrho_n}(t)\vec{u}_n(t)\|^2_{L^2(\Omega)} \leq cE_\delta[\varrho_0, \vec{q}], \tag{4.20}$$

$$\int_0^T \|\vec{u}_n(t)\|^2_{L^2(\Omega)} + \|\nabla\vec{u}_n(t)\|^2_{L^2(\Omega)}\, dt \leq cE_\delta[\varrho_0, \vec{q}], \tag{4.21}$$

$$\varepsilon \int_0^T \|\nabla\varrho_n(t)\|^2_{L^2(\Omega)}\, dt \leq c(\delta, \varrho_0, \vec{q}), \tag{4.22}$$

$$\|\varrho_n\|_{L^{\beta+1}((0,T)\times\Omega)} \leq c(\varepsilon, \delta, \varrho_0, \vec{q}). \tag{4.23}$$

All the above estimates hold independently of n.

Proof: The estimate (4.18) follows immediately from the energy inequality (4.16) and the definition of the Luxembourg norm. The estimates (4.19), (4.20), (4.21) are also the straightforward consequences of the (4.16).

Multiplying (4.1) by ϱ_n and integrating over $[0,T]\times\Omega$, we obtain the estimate

$$\varepsilon \int_0^T \|\nabla\varrho_n(t)\|^2_{L^2(\Omega)}\, \mathrm{d}t \leq \frac{1}{2}\|\varrho_0\|^2_{L^2(\Omega)} - \int_0^T \int_\Omega \mathrm{div}\, (\varrho_n\vec{u}_n)\varrho_n\, \mathrm{d}x\, \mathrm{d}t \leq$$

$$\leq \frac{1}{2}\|\varrho_0\|^2_{L^2(\Omega)} - \frac{1}{2}\int_0^T \int_\Omega \varrho_n^2 \mathrm{div}\, (\vec{u}_n)\, \mathrm{d}x\, \mathrm{d}t \leq$$

$$\leq \frac{1}{2}\|\varrho_0\|^2_{L^2(\Omega)} + \frac{1}{2}\sup_{t\in[0,T]} \|\varrho_n(t)\|^2_{L^4(\Omega)} \int_0^T \left(\int_\Omega |\mathrm{div}\, \vec{u}_n|^2\, \mathrm{d}x\right)^{\frac{1}{2}} \mathrm{d}t$$

which, combined with (4.16) and (4.19) provided $\beta \geq 4$, yields the estimate (4.22).

Now, we shall prove the last estimate (4.23). By virtue of (4.16), we get

$$\sqrt{\frac{\varepsilon\delta}{\beta}}(\varrho_n)^{\frac{\beta}{2}} \text{ is bounded in } \mathrm{L}^2(0,T;\mathrm{W}^{1,2}(\Omega)) \text{ independently of } n.$$

Using the imbedding $\mathrm{W}^{1,2}(\Omega) \hookrightarrow \mathrm{L}^8(\Omega)$ and (4.16) again, we obtain

$$\varrho_n^\beta \text{ is bounded in } \mathrm{L}^1(0,T;\mathrm{L}^4(\Omega)) \text{ and in } \mathrm{L}^\infty(0,T;\mathrm{L}^1(\Omega)).$$

By interpolation theorem, ϱ_n^β is bounded in $\mathrm{L}^{\frac{3}{2}}(0,T;L^2(\Omega))$ independently of n. Consequently, (4.23) follows for sufficiently large β.

Q.E.D.

Remark: The constants on the right hand side of the estimates (4.18) – (4.23) also depend on parameter β. It is not explicitly written because we shall not pass to a limit with β (β will be a sufficiently large constant during a limiting process).

4.3 The existence of the first level approximate solutions

The final task of this section is to employ the estimates obtained in Lemma 4.2 to pass to the limit for $n\to\infty$ in the sequence ϱ_n, $\vec{u}_n$ to obtain a solution of the problem (4.1) – (4.6).

Let us suppose $\beta\geq 4$. By virtue of the estimate (4.22), it holds that ϱ_n is bounded in $\mathrm{L}^2(0,T;\mathrm{W}^{1,2}(\Omega))$. The equation (4.1), together with the estimates (4.19), (4.21) and (4.22), implies that $(\varrho_n)_t$ is bounded in the space $\mathrm{L}^2(0,T;\mathrm{W}^{-1,2}(\Omega))$.

Thus, using the compact imbedding $\mathrm{W}^{1,2}(\Omega)\hookrightarrow\hookrightarrow \mathrm{L}^2(\Omega)$ and using the Aubin-Lions lemma ([4], Chapter 1, Theorem 5.1), we see that there exists a function $\varrho\in\mathrm{L}^2((0,T)\times\Omega)$ such that

$$\varrho_n\to\varrho \quad \text{in } \mathrm{L}^2((0,T)\times\Omega), \tag{4.24}$$

passing to the subsequences as the case may be.

In uniformly convex Banach space, particularly in $\mathrm{L}^p((0,T)\times\Omega)$, $1<p<\infty$, the strong convergence follows from the weak convergence and the convergence of norms. Thus, by virtue of (4.23), (4.24) and the Lebesgue convergence theorem, we obtain a subsequence so that

$$\varrho_n^\beta\to\varrho^\beta,\ \text{in, say, } \mathrm{L}^1((0,T)\times\Omega). \tag{4.25}$$

By virtue of the mean value theorem applied on the function $\phi(y)=y\log^d(1+y)$, there exists a constant c such that

$$\left|y\log^d(1+y)-z\log^d(1+z)\right|\leq c\ (1+|y|+|z|)\,|y-z|\,,\quad \text{for all } y\in(0,\infty),\ z\in(0,\infty) \tag{4.26}$$

which, combined with (4.24), gives

$$\varrho_n\log^d(1+\varrho_n)\to\varrho\log^d(1+\varrho)\ \text{in } \mathrm{L}^1((0,T)\times\Omega). \tag{4.27}$$

Now, our task is to pass to the limit in the products $\varrho_n\vec{u}_n$ and $\varrho_n\vec{u}_n\otimes\vec{u}_n$. By virtue of the (4.21), there is a subsequence so that

$$\vec{u}_n\to\vec{u}\ \text{weakly in } \mathrm{L}^2(0,T;\mathrm{W}_0^{1,2}(\Omega)). \tag{4.28}$$

The estimates (4.18), (4.20) and Lemma 3.4 imply boundedness of the sequence $\varrho_n\vec{u}_n$ in the space $\mathrm{L}^\infty(0,T;\mathrm{L}_{y\log^{d+1}(y)}(\Omega))$, which, combined with (4.24) and (4.28), yields

$$\varrho_n\vec{u}_n\to\varrho\vec{u}\ \text{weakly star in } \mathrm{L}^\infty(0,T;\mathrm{L}_{y\log^{d+1}(y)}(\Omega)).$$

Consequently, we can pass to the limit in the continuity equation to obtain that (4.1) is fulfilled in $D'((0,T)\times\Omega)$.

The following lemma says that the continuity equation (4.1) holds even in the strong sense. The lemma is proved in [2] (see Lemma 2.4).

Lemma 4.3: *There exist $r>1$, $q>2$ such that*

$$\frac{\partial \varrho_n}{\partial t},\ \triangle\varrho_n \text{ are bounded in } \mathrm{L}^r((0,T)\times\Omega),\ \nabla\varrho_n \text{ is bounded in } \mathrm{L}^q(0,T;\mathrm{L}^2(\Omega))$$

independently of n. Consequently, the limit function ϱ belongs to the same class and satisfies the equation (4.1) almost everywhere on $(0,T)\times\Omega$ and the boundary conditions (4.3) in the sense of traces.

Since $\varrho_n\vec{u}_n$ satisfies (4.9), we can use Lemma 3.8 to deduce

$$\varrho_n\vec{u}_n \to \varrho\vec{u} \text{ in } C([0,T];\mathrm{L}^{*-weak}_{y\log^{d+1}(y)}(\Omega))$$

and

$$\varrho_n\vec{u}_n \to \varrho\vec{u} \text{ in } C([0,T];\mathrm{W}^{-1,2}(\Omega)). \tag{4.29}$$

Indeed observe that for any fixed $\vec{\psi}\in\cup_{n=1}^{\infty}X_n$ in (4.9) the term $\int_0^t\int_\Omega(\nabla\varrho_n\cdot\nabla\vec{u}_n)\cdot\vec{\psi}\,\mathrm{d}x\,\mathrm{d}s$ is uniformly continuous in t by virtue of (4.21) and Lemma 4.3. The other terms in (4.9) are uniformly continuous in t by virtue of the estimates from Lemma 4.2. Let us note that the set $\cup_{n=1}^{\infty}X_n$ is dense in $\mathrm{E}_{e(1/(d+1))}(\Omega)$ (it is well-known fact that eigenfunctions of the Laplacian form an orthogonal basis of the space $\mathrm{L}^2(\Omega)$).

Finally, (4.28) and (4.29) yield

$$\varrho_n u_n^i u_n^j \to \varrho u^i u^j \text{ in, say, } \mathrm{L}^1((0,T)\times\Omega),\ i,j=1,2.$$

Now, we shall deal with the behaviour of the term $\nabla\varrho_n\cdot\nabla\vec{u}_n$. To this end, we multiply (4.1) by ϱ_n and integrate by parts to obtain

$$\|\varrho_n(t)\|^2_{L^2(\Omega)} + 2\varepsilon\int_0^t\|\nabla\varrho_n\|^2_{L^2(\Omega)}\,\mathrm{d}t = -\int_0^t\int_\Omega \operatorname{div}\,\vec{u}_n|\varrho_n|^2\,\mathrm{d}x\,\mathrm{d}t + \|\varrho_0\|^2_{L^2(\Omega)}. \tag{4.30}$$

If $\beta\geq 4$, then using (4.19) and (4.24) we get

$$\varrho_n\to\varrho \quad \text{in } \mathrm{L}^4((0,T)\times\Omega). \tag{4.31}$$

Let us define an auxiliary function $b_k:\mathbb{R}\to\mathbb{R}$ such that b_k is even and

$$b_k(z)=\begin{cases} z^2 & \text{for } z\in[0,k),\\ -\frac{k}{3}z^3+(k^2+1)z^2-k^3z+\frac{k^4}{3} & \text{for } z\in\left[k,k+\frac{1}{k}\right),\\ \left(2k+\frac{1}{k}\right)x-k^2-1-\frac{1}{3k^2} & \text{for } z\in\left[k+\frac{1}{k},\infty\right).\end{cases}$$

It is easy to see that $b_k \in C^2(\mathbb{R})$ and

$$\frac{\mathrm{d}^2 b_k}{\mathrm{dz}^2}(z) = \begin{cases} 2 & \text{for } z \in [0,k), \\ -2kz + 2(k^2+1) & \text{for } z \in \left[k, k+\frac{1}{k}\right), \\ 0 & \text{for } z \in \left[k+\frac{1}{k}, \infty\right), \end{cases}$$

consequently (using also $\beta \geq 4$)

$$\frac{\mathrm{d}^2 b_k}{\mathrm{dz}^2}(\varrho) \to 2 \ \text{ in } \mathrm{L}^\infty((0,T)\times\Omega) \quad \text{and} \quad b_k(\varrho) \to \varrho^2 \in C([0,T];\mathrm{L}^2(\Omega)). \tag{4.32}$$

Now we multiply, the limit equation (4.1) by $b'(\varrho)$ and integrate by parts to obtain

$$\int_\Omega b(\varrho)(t)\,\mathrm{d}x + \varepsilon \int_0^t \int_\Omega \frac{\mathrm{d}^2 b_k}{\mathrm{dz}^2}(\varrho)|\nabla\varrho|^2\,\mathrm{d}x\,\mathrm{d}t = \int_0^t \int_\Omega \frac{\mathrm{d}^2 b_k}{\mathrm{dz}^2}(\varrho)\varrho\vec{u}\cdot\nabla\varrho\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega b(\varrho_0)\,\mathrm{d}x,$$

which, combined with (4.32) gives

$$\|\varrho(t)\|^2_{L^2(\Omega)} + 2\varepsilon \int_0^t \|\nabla\varrho\|^2_{L^2(\Omega)}\,\mathrm{d}t = -\int_0^t \int_\Omega \mathrm{div}\ \vec{u}|\varrho|^2\,\mathrm{d}x\,\mathrm{d}t + \|\varrho_0\|^2_{L^2(\Omega)}. \tag{4.33}$$

Thus using (4.30), (4.33) together with (4.28), (4.31) and Lemma 4.3 we conclude

$$\|\nabla\varrho_n\|_{L^2((0,T)\times\Omega)} \to \|\nabla\varrho\|_{L^2((0,T)\times\Omega)} \ \text{ and } \ \|\varrho_n(t)\|_{L^2(\Omega)} \to \|\varrho(t)\|_{L^2(\Omega)} \ \text{ for any } t$$

yielding, in particular, strong convergence of $\nabla\varrho_n$ and, consequently,

$$\nabla\varrho_n \cdot \nabla u_n^i \to \nabla\varrho\cdot\nabla u^i \ \text{ in } D'((0,T)\times\Omega),\ i=1,2.$$

Let us write the results achieved in this section in a more coincise form:

Theorem 2: *Suppose $\beta \geq 4$. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with C^{2+v} boundary. Assume the initial datum ϱ_0, $\vec{q}$ satisfy (4.5), (4.6). Then there exists a weak solution ϱ, $\vec{u}$ of the problem (4.1) – (4.6) such that $\varrho \in \mathrm{L}^{\beta+1}((0,T)\times\Omega)$ and the following estimates hold:*

$$\sup_{t\in[0,T]} \|\varrho(t)\|_{y\log^{(d+1)}(y)} \leq cE_\delta[\varrho_0,\vec{q}], \tag{4.34}$$

$$\delta \sup_{t\in[0,T]} \|\varrho(t)\|^\beta_{L^\beta(\Omega)} \leq cE_\delta[\varrho_0,\vec{q}], \tag{4.35}$$

$$\sup_{t\in[0,T]} \|\sqrt{\varrho}(t)\vec{u}(t)\|^2_{L^2(\Omega)} \leq cE_\delta[\varrho_0,\vec{q}], \tag{4.36}$$

$$\int_0^T \|\vec{u}(t)\|^2_{L^2(\Omega)} + \|\nabla\vec{u}(t)\|^2_{L^2(\Omega)}\,dt \leq cE_\delta[\varrho_0,\vec{q}], \tag{4.37}$$

and

$$\varepsilon \int_0^T \|\nabla \varrho(t)\|^2_{L^2(\Omega)} \, dt \le c(\delta, \varrho_0, \vec{q}). \tag{4.38}$$

Moreover, the energy inequality

$$\frac{d}{dt}\left(\int_\Omega \frac{1}{2} \varrho |\vec{u}(t)|^2 + \frac{\delta}{\beta - 1} \varrho^\beta + \frac{a}{d+1} (1+\varrho) \log^{d+1}(1+\varrho) + R(\varrho) \, dx \right) +$$

$$+ \int_\Omega \mu |\nabla \vec{u}|^2 + (\lambda + \mu) |\mathrm{div}\ \vec{u}|^2 \, dx \le 0$$

holds in $D'(0,T)$. Finally, there exists $r > 1$ such that ϱ_t, $\triangle \varrho \in \mathrm{L}^r((0,T) \times \Omega)$ and the equation (4.1) is satisfied almost everywhere on $(0,T) \times \Omega$.

5 The vanishing viscosity limit

The aim of the present section is to pass to the limit in (4.1), (4.2) letting $\varepsilon \to 0$. Accordingly, the solution of the problem (4.1) – (4.6) obtained in Theorem 2 above will be denoted ϱ_ε, $\vec{u}_\varepsilon$.

5.1 Estimates of the density independent of viscosity

You can find the following lemma in [2] (see section 3.1):

Lemma 5.1: *Let Ω be a bounded domain with C^{2+v} boundary. There exists a linear operator $\mathcal{B} = [\mathcal{B}_1, \mathcal{B}_2]$ enjoying the properties:*

- $$\mathcal{B} : \left\{ f \in \mathrm{L}^p(\Omega) \ : \ \int_\Omega f \, dx = 0 \right\} \to \left[\mathrm{W}^{1,2}_0(\Omega) \right]^2$$

  is a bounded linear operator, i. e.,

  $$\|\mathcal{B}[f]\|_{W_0^{1,p}(\Omega)} \le c(p) \|f\|_{L^p(\Omega)} \text{ for any } 1 < p < \infty; \tag{5.1}$$

- *the function $\vec{v} = \mathcal{B}[f]$ solve the problem:*

  $$\mathrm{div}\ \vec{v} = f \text{ in } \Omega;$$

- *if, moreover, f can be written in the form $f = \mathrm{div}\ \vec{g}$ for a certain $\vec{g} \in [\mathrm{L}^r(\Omega)]^2$, $\vec{g} \cdot \vec{n}|_{\partial\Omega} = 0$, then*

  $$\|\mathcal{B}[f]\|_{L^r(\Omega)} \le c(r) \|\vec{g}\|_{L^r(\Omega)} \tag{5.2}$$

for arbitrary $1 < r < \infty$.

We are going to use the operator $\mathcal{B}$ to improve the estimates of the density component. To this end, consider the quantities

$$\psi(t)\mathcal{B}_i[\varrho_\varepsilon - m_0],\ \psi \in D(0,T),\ 0 \le \psi \le 1,\ m_0 = \frac{1}{|\Omega|}\int_\Omega \varrho_\varepsilon(t)\,\mathrm{d}x$$

as test functions for (4.2). Note that the total mass $m = \int_\Omega \varrho_\varepsilon(t)\,\mathrm{d}x$ is a constant of motion.

Remark: By virtue of (4.35) and (5.1), we see that

$$\mathcal{B}_i[\varrho_\varepsilon - m_0] \text{ is bounded in } \mathrm{L}^\infty(0,T;\mathrm{W}^{1,\beta}(\Omega) \text{ independently of } \varepsilon. \tag{5.3}$$

Because of (5.1) and Lemma 4.3, it holds

$$\varrho_\varepsilon \in \mathrm{W}^{1,r}(0,T;\mathrm{L}^r(\Omega)) \quad \text{and} \quad \mathcal{B}_i[\varrho_\varepsilon - m_0] \in \mathrm{W}^{1,r}(0,T;\mathrm{W}^{1,r}(\Omega)). \tag{5.4}$$

After a little bit lengthy but straghtforward computation, we obtain:

$$\int_0^T \int_\Omega \psi\left(a\varrho_\varepsilon^2 \log^d(1+\varrho_\varepsilon) + \delta\varrho_\varepsilon^{\beta+1}\right)\,\mathrm{d}x\,\mathrm{d}t =$$

$$= m_0 \int_0^T \psi\left(\int_\Omega a\varrho_\varepsilon \log^d(1+\varrho_\varepsilon) + \delta\varrho_\varepsilon^\beta\,\mathrm{d}x\right)\,\mathrm{d}t + (\lambda+\mu)\int_0^T \psi\int_\Omega \varrho_\varepsilon \mathrm{div}\ \vec{u}_\varepsilon\,\mathrm{d}x\,\mathrm{d}t +$$
$$+\mu\int_0^T \psi\int_\Omega \partial_{x_j}u_\varepsilon^i\partial_{x_j}\mathcal{B}_i[\varrho_\varepsilon - m_0]\,\mathrm{d}x\,\mathrm{d}t - \int_0^T \psi_t\int_\Omega \varrho_\varepsilon u_\varepsilon^i\mathcal{B}_i[\varrho_\varepsilon - m_0]\,\mathrm{d}x\,\mathrm{d}t +$$
$$+\varepsilon\int_0^T \psi\int_\Omega \partial_{x_j}u_\varepsilon^i\partial_{x_j}\varrho_\varepsilon\mathcal{B}_i[\varrho_\varepsilon - m_0]\,\mathrm{d}x\,\mathrm{d}t - \int_0^T \psi\varrho_\varepsilon u_\varepsilon^i u_\varepsilon^j\partial_{x_j}\mathcal{B}_i[\varrho_\varepsilon - m_0]\,\mathrm{d}x\,\mathrm{d}t +$$
$$+\varepsilon\int_0^T \psi\int_\Omega \varrho_\varepsilon u_\varepsilon^i\mathcal{B}_i[\triangle\varrho_\varepsilon]\,\mathrm{d}x\,\mathrm{d}t - \int_0^T \psi\int_\Omega \varrho_\varepsilon u_\varepsilon^i\mathcal{B}_i[\mathrm{div}\ (\varrho_\varepsilon\vec{u}_\varepsilon)]\,\mathrm{d}x\,\mathrm{d}t = \sum_{l=1}^8 I_l.$$

where the summation convention is used to simplify notation. Now, we shall estimate the integrals I_l, $l = 1,\dots 8$.

By virtue of (4.35), we get

$$|I_1| = \left|m_0\int_0^T \psi\left(\int_\Omega a\varrho_\varepsilon \log^d(1+\varrho_\varepsilon) + \delta\varrho_\varepsilon^\beta\,\mathrm{d}x\right)\,\mathrm{d}t\right| \le c(\delta,\varrho_0,\vec{q}).$$

Similarly, using (4.35) and (4.37),

$$|I_2| = \left|(\lambda+\mu)\int_0^T \psi\int_\Omega \varrho_\varepsilon \mathrm{div}\ \vec{u}_\varepsilon\,\mathrm{d}x\,\mathrm{d}t\right| \le c(\delta,\varrho_0,\vec{q}).$$

Analogously, we use the Hölder inequality, the estimates (4.35) – (4.38) and (5.3) to deduce consecutively

$$|I_3| = \left|\mu \int_0^T \psi \int_\Omega \partial_{x_j} u^i_\varepsilon \partial_{x_j} \mathcal{B}_i[\varrho_\varepsilon - m_0] \, \mathrm{d}x \, \mathrm{d}t\right| \le$$

$$\le \mu \|\nabla \vec{u}_\varepsilon\|_{L^2((0,T)\times\Omega)} \|\mathcal{B}_i[\varrho_\varepsilon - m_0]\|_{L^2(0,T;W^{1,2}(\Omega))} \le c(\delta, \varrho_0, \vec{q}),$$

$$|I_4| = \left| \int_0^T \psi_t \int_\Omega \varrho_\varepsilon u^i_\varepsilon \mathcal{B}_i[\varrho_\varepsilon - m_0] \, \mathrm{d}x \, \mathrm{d}t\right| \le$$

$$\le c \int_0^T |\psi_t| \|\sqrt{\varrho_\varepsilon}\|_{L^2(\Omega)} \|\sqrt{\varrho_\varepsilon} \vec{u}_\varepsilon\|_{L^2(\Omega)} \|\mathcal{B}_i[\varrho_\varepsilon - m_0]\|_{L^\infty(\Omega)} \, \mathrm{d}t \le c(\delta, \varrho_0, \vec{q}),$$

$$|I_5| = \left|\varepsilon \int_0^T \psi \int_\Omega \partial_{x_j} u^i_\varepsilon \partial_{x_j} \varrho_\varepsilon \mathcal{B}_i[\varrho_\varepsilon - m_0] \, \mathrm{d}x \, \mathrm{d}t\right| \le$$

$$\le \sqrt{\varepsilon} \|\sqrt{\varepsilon} \nabla \varrho_\varepsilon\|_{L^2((0,T)\times\Omega)} \|\nabla \vec{u}_\varepsilon\|_{L^2((0,T)\times\Omega)} \|\mathcal{B}_i[\varrho_\varepsilon - m_0]\|_{L^\infty((0,T)\times\Omega)} \le \sqrt{\varepsilon} c(\delta, \varrho_0, \vec{q})$$

and

$$|I_6| = \left| \int_0^T \psi \varrho_\varepsilon u^i_\varepsilon u^j_\varepsilon \partial_{x_j} \mathcal{B}_i[\varrho_\varepsilon - m_0] \, \mathrm{d}x \, \mathrm{d}t\right| \le$$

$$\le \int_0^T \|\varrho_\varepsilon\|_{L^4(\Omega)} \|\vec{u}_\varepsilon\|^2_{L^4(\Omega)} \|\mathcal{B}_i[\varrho_\varepsilon - m_0]\|_{W^{1,4}(\Omega)} \, \mathrm{d}x \, \mathrm{d}t \le c(\delta, \varrho_0, \vec{q}).$$

Finally, in view of the estimates (4.35), (4.37), (4.38) and (5.2), it holds

$$|I_7| = \left|\varepsilon \int_0^T \psi \int_\Omega \varrho_\varepsilon u^i_\varepsilon \mathcal{B}_i[\triangle \varrho_\varepsilon] \, \mathrm{d}x \, \mathrm{d}t\right| \le$$

$$\le c \int_0^T \|\varrho_\varepsilon\|_{L^4(\Omega)} \|\vec{u}_\varepsilon\|_{L^4(\Omega)} \|\nabla \varrho_\varepsilon\|_{L^2(\Omega)} \, \mathrm{d}t \le c(\delta, \varrho_0, \vec{q})$$

and

$$|I_8| = \left| \int_0^T \psi \int_\Omega \varrho_\varepsilon u^i_\varepsilon \mathcal{B}_i[\mathrm{div}\ (\varrho_\varepsilon \vec{u}_\varepsilon)] \, \mathrm{d}x \, \mathrm{d}t\right| \le \int_0^T \|\varrho_\varepsilon\|^2_{L^4(\Omega)} \|\vec{u}_\varepsilon\|^2_{L^4(\Omega)} \, \mathrm{d}t \le c(\delta, \varrho_0, \vec{q})$$

where we use the imbedding $\mathrm{W}^{1,2}(\Omega) \hookrightarrow \mathrm{L}^4(\Omega)$ and the assumption $\beta \ge 4$.

Summing up the previsous results, we have proved the following statement:

Lemma 5.2: *Let ϱ_ε, $\vec{u}_\varepsilon$ be the sequence of solutions of the problem (4.1) – (4.6) constructed in Theorem 2 provided $0 < \varepsilon \le 1$. Then there exists a constant $c = c(\delta, \varrho_0, \vec{q})$, independent of ε, such that*

$$\|\varrho_\varepsilon\|_{y^2 \log^d(y)} + \|\varrho_\varepsilon\|_{L^{\beta+1}((0,T)\times\Omega)} \le c(\delta, \varrho_0, \vec{q}), \tag{5.5}$$

where $\|\cdot\|_{y^2\log^d(y)}$ *is the Luxembourg norm in the Orlicz space generated by the Young function* $\phi(y)=y^2\log^d(1+y)$.

5.2 The vanishing viscosity limit passage

At this stage, we are ready to pass to the limit for $\varepsilon\to 0$ to get rid of the diffusion term in the equation (4.1) as well as of the $\varepsilon-$quantities occuring in (4.2). Note that the parameter δ is kept fixed throughout this procedure so that we may use the estimates derived above.

To begin, it is easy to deduce from (4.37), (4.38) that

$$\varepsilon\nabla\varrho_\varepsilon\cdot\nabla u^i_\varepsilon\to 0 \text{ in } \mathrm{L}^1((0,T)\times\Omega),\quad i=1,2,$$

and

$$\varepsilon\triangle\varrho_\varepsilon\to 0 \text{ in } \mathrm{L}^2(0,T;\mathrm{W}^{-1,2}(\Omega)).$$

As ϱ_ε satisfies equation (4.1), we infer, by virtue of the estimate (4.34), Lemma 3.8,

$$\varrho_\varepsilon\to\varrho \text{ in } C([0,T];\mathrm{L}^{*-weak}_{y\log^{d+1}(y)}(\Omega)) \text{ and } \varrho_\varepsilon\to\varrho \text{ in } C([0,T];\mathrm{W}^{-1,2}(\Omega)), \tag{5.6}$$

next, the estimate (5.5) yields

$$\varrho_\varepsilon\to\varrho \text{ weakly in } \mathrm{L}^{\beta+1}((0,T)\times\Omega) \tag{5.7}$$

passing to subsequences if necessary. Moreover, because of (4.37),

$$\vec{u}_\varepsilon\to\vec{u} \text{ weakly in } \mathrm{L}^2(0,T;\mathrm{W}^{1,2}_0(\Omega)) \tag{5.8}$$

and, combining (5.6), (5.8) with (4.36), Lemma 3.4 and Lemma 3.8, we infer

$$\varrho_\varepsilon\vec{u}_\varepsilon\to\varrho\vec{u} \text{ in } C([0,T];\mathrm{L}^{*-weak}_{y\log^{d+1}(y)}(\Omega)) \text{ and } \varrho_\varepsilon\vec{u}_\varepsilon\to\varrho\vec{u} \text{ in } C([0,T];\mathrm{W}^{-1,2}(\Omega)). \tag{5.9}$$

Finally, (5.8) and (5.9) yield

$$\varrho_\varepsilon u^i_\varepsilon u^j_\varepsilon\to\varrho u^i u^j \text{ in } \mathrm{L}^1((0,T)\times\Omega),\ i,j=1,2. \tag{5.10}$$

Thus we have proved that the limits ϱ, $\vec{u}$ satisfy the following system of equations:

$$\varrho_t+\mathrm{div}\,(\varrho\vec{u})=0, \tag{5.11}$$

$$(\varrho u^i)_t+\mathrm{div}\,(\varrho u^i\vec{u})+p_{x_i}=\mu\Delta u^i+(\lambda+\mu)(\mathrm{div}\,\vec{u})_{x_i},\qquad i=1,2, \tag{5.12}$$

in $D'((0,T)\times\Omega)$, where (because of (5.5))

$$a\varrho_\varepsilon\log^d(1+\varrho_\varepsilon)+\delta\varrho_\varepsilon^\beta\to p \text{ weakly in } \mathrm{L}^{\frac{\beta+1}{\beta}}((0,T)\times\Omega). \tag{5.13}$$

Moreover, in accordance with (5.6) and (5.9), the limit functions ϱ, $\varrho\vec{u}$ satisfy the initial conditions

$$\varrho(0) = \varrho_0, \quad (\varrho\vec{u})(0) = \vec{q}$$

where ϱ_0, $\vec{q}$ are as in (4.5), (4.6). Thus our ultimate goal is to show that

$$p = a\varrho \log^d(1+\varrho) + \delta\varrho^\beta. \tag{5.14}$$

5.3 The effective viscous flux and its properties

We introduce the quantity $a\varrho\log^d(1+\varrho) + \delta\varrho^\beta - (\lambda + 2\mu)\text{div}\,\vec{u}$ called usually the effective viscous flux. Here we shall prove the result analogous to the result in [2] (see Lemma 3.2).

Lemma 5.3: *Let ϱ_ε, $\vec{u}_\varepsilon$ be the sequence of approximate solutions, the existence of which is guaranteed by Theorem 2, and let ϱ, $\vec{u}$, and p be the limits appearing in (5.6), (5.8), and (5.13) respectively.*

Then

$$\lim_{\varepsilon\to 0_+} \int_0^T \psi \int_\Omega \phi \left(a\varrho_\varepsilon \log^d(1+\varrho_\varepsilon) + \delta\varrho_\varepsilon^\beta - (\lambda+2\mu)\text{div}\,\vec{u}_\varepsilon \right) \varrho_\varepsilon \, dx\, dt =$$

$$= \int_0^T \psi \int_\Omega \phi \Big(p - (\lambda+2\mu)\text{div}\,\vec{u} \Big) \varrho \, dx\, dt \quad \textit{for any } \psi \in D(0,T), \phi \in D(\Omega).$$

Proof: Let us consider the operators

$$\mathcal{A}_i[v] = \triangle^{-1}[\partial_{x_i} v], \ i = 1, 2$$

where $\triangle^{-1}$ stands for the inverse of the Laplace operator on $\mathbb{R}^2$. To be more specific, the Fourier symbol of $\mathcal{A}_i$ is

$$\mathcal{A}_i[v] = \mathcal{F}^{-1}\left(\frac{-i\xi_i}{|\xi|^2} \right) * v.$$

Note, that $\partial_{x_i}\mathcal{A}_i[v] = v$ and, by virtue of the classical Mikhlin multiplier theorem (see [1], Theorem 6.1.6):

$$\|\mathcal{A}_i[v]\|_{W^{1,s}(\Omega)} \le c(s)\|v\|_{L^s(\mathbb{R}^2)}, \ \ 1 < s < \infty, \quad \text{in particular,} \tag{5.15}$$

$$\|\mathcal{A}_i[v]\|_{L^\infty(\Omega)} \le c(s)\|v\|_{L^s(\mathbb{R}^2)} \ \text{ for } s > 2.$$

Similarly, according to Lemma 3.10, we have

$$\|\mathcal{A}_i[v]\|_{WL_{e\left(\frac{\gamma}{\gamma+1}\right)}(\Omega)} \le c(\gamma)\|v\|_{L_{e(\gamma)}(\Omega)}, \quad \text{for all } v \in L_{e(\gamma)}(\Omega), \ \gamma > 0. \tag{5.16}$$

Next, let us define the operators

$$\mathcal{R}_{i,j}[v] = \partial_{x_j}\mathcal{A}_i[v], \text{ thus } \mathcal{R}_{i,j}[v] = \mathcal{F}^{-1}\left(\frac{\xi_i\xi_j}{|\xi|^2}\right) * v, \quad i,j = 1,2,$$

and by virtue of the Mikhlin multiplier theorem, Lemma 3.1 and Lemma 3.10, we obtain

$$\|\mathcal{R}_{i,j}[v]\|_{L^s(\Omega)} \le c(s)\|v\|_{L^s(\Omega)}, \tag{5.17}$$

$$\|\mathcal{R}_{i,j}[v]\|_{y\log^{\gamma}(y)} \le c(\gamma)\|v\|_{y\log^{(\gamma+1)}(y)}, \tag{5.18}$$

$$\|\mathcal{R}_{i,j}[v]\|_{e(\frac{\gamma}{\gamma+1})} \le c(\gamma)\|v\|_{e(\gamma)},\ \gamma > 0. \tag{5.19}$$

Prolonging ϱ_ε to be zero outside Ω we see that ϱ_ε admits the partial derivative with respect to t and

$$\partial_t\varrho_\varepsilon = \begin{cases} \varepsilon\triangle\varrho_\varepsilon - \text{div }(\varrho_\varepsilon\vec{u}_\varepsilon) & \text{in } \Omega, \\ 0 & \text{in } \mathbb{R}^2 - \Omega. \end{cases}$$

Moreover, since $\vec{u}_\varepsilon$ vanishes on $\partial\Omega$, we have

$$\text{div }(\varrho_\varepsilon\vec{u}_\varepsilon) = 0 \text{ on } \mathbb{R}^2 - \Omega,$$

and, since $\chi_\Omega\nabla\varrho_\varepsilon$ vanishes, by virtue of (4.3), on $\partial\Omega$, we obtain

$$\text{div }(\chi_\Omega\nabla\varrho_\varepsilon) = \begin{cases} \triangle\varrho_\varepsilon & \text{in } \Omega, \\ 0 & \text{in } \mathbb{R}^2 - \Omega. \end{cases}$$

Thus, the function ϱ_ε prolonged by zero outside Ω satisfies equation

$$\partial_t\varrho_\varepsilon = \varepsilon\text{div }(\chi_\Omega\nabla\varrho_\varepsilon) - \text{div }(\varrho_\varepsilon\vec{u}_\varepsilon). \tag{5.20}$$

Applying the operator $\mathcal{A}_i$ on the equation (5.20), we get

$$\mathcal{A}_i[\partial_t\varrho_\varepsilon] = \mathcal{A}_i[\varepsilon\text{div }(\chi_\Omega\nabla\varrho_\varepsilon)] - \mathcal{A}_i[\text{div }(\varrho_\varepsilon\vec{u}_\varepsilon)] = \varepsilon\mathcal{R}_{i,j}[\chi_\Omega\partial_{x_j}\varrho_\varepsilon] - \mathcal{R}_{i,j}[\varrho_\varepsilon u_\varepsilon^j]$$

which, combined with (5.4), gives

$$\partial_t\mathcal{A}_i[\varrho_\varepsilon] = \varepsilon\mathcal{R}_{i,j}[\chi_\Omega\partial_{x_j}\varrho_\varepsilon] - \mathcal{R}_{i,j}[\varrho_\varepsilon u_\varepsilon^j]. \tag{5.21}$$

Let us consider functions of the form

$$\varphi_i(t,x) = \psi(t)\phi(x)\mathcal{A}_i[\varrho_\varepsilon],\ i = 1,2, \text{ where } \psi \in D(0,T),\ \phi \in D(\Omega).$$

Now the regularity properties of ϱ_ε, specifically (5.5) and (5.4), justify the choice of φ_i as test functions for the equation (4.2). Thus we arrive, using (5.21) also, at the following formula:

$$\int_0^T \psi\int_\Omega \phi\left(a\varrho_\varepsilon\log^d(1+\varrho_\varepsilon) + \delta\varrho_\varepsilon^\beta - (\lambda+2\mu)\text{div }\vec{u}_\varepsilon\right)\varrho_\varepsilon\,\text{d}x\,\text{d}t = \tag{5.22}$$

$$= \int_0^T \psi \int_\Omega \partial_{x_i}\phi \Big((\lambda+\mu)\mathrm{div}\; (\vec{u}_\varepsilon) - a\varrho_\varepsilon \log^d(1+\varrho_\varepsilon) - \delta\varrho_\varepsilon^\beta \Big) \mathcal{A}_i[\varrho_\varepsilon]\,\mathrm{d}x\,\mathrm{d}t +$$

$$+\,\mu \int_0^T \psi \int_\Omega \left(\partial_{x_j}\phi\right) u_\varepsilon^j \varrho_\varepsilon \,\mathrm{d}x\,\mathrm{d}t - \mu \int_0^T \psi \int_\Omega \left(\partial_{x_j}\phi\right) u_\varepsilon^i \mathcal{R}_{i,j}[\varrho_\varepsilon]\,\mathrm{d}x\,\mathrm{d}t +$$

$$+\,\mu \int_0^T \psi \int_\Omega \left(\partial_{x_j}\phi\right) \partial_{x_j} u_\varepsilon^i \mathcal{A}_i[\varrho_\varepsilon]\,\mathrm{d}x\,\mathrm{d}t - \int_0^T \psi_t \int_\Omega \phi\varrho_\varepsilon u_\varepsilon^i \mathcal{A}_i[\varrho_\varepsilon]\,\mathrm{d}x\,\mathrm{d}t +$$

$$+\,\varepsilon \int_0^T \psi \int_\Omega \phi\partial_{x_j} u_\varepsilon^i\, \partial_{x_j}\varrho_\varepsilon \mathcal{A}_i[\varrho_\varepsilon]\,\mathrm{d}x\,\mathrm{d}t - \varepsilon \int_0^T \psi \int_\Omega \phi\varrho_\varepsilon u_\varepsilon^i \mathcal{R}_{i,j}[\chi_\Omega \partial_{x_j}\varrho_\varepsilon]\,\mathrm{d}x\,\mathrm{d}t -$$

$$- \int_0^T \psi \int_\Omega \left(\partial_{x_j}\phi\right) \varrho_\varepsilon u_\varepsilon^i u_\varepsilon^j \mathcal{A}_i[\varrho_\varepsilon]\,\mathrm{d}x\,\mathrm{d}t +$$

$$+ \int_0^T \psi \int_\Omega \phi u_\varepsilon^i \left(\varrho_\varepsilon \mathcal{R}_{i,j}[\varrho_\varepsilon u_\varepsilon^j] - \varrho_\varepsilon u_\varepsilon^j \mathcal{R}_{i,j}[\varrho_\varepsilon] \right)\,\mathrm{d}x\,\mathrm{d}t,$$

where the summation convention is used again to simplify notation.

As ϱ, $\vec{u}$ are the solution of (5.11) in $D'((0,T)\times\Omega)$ and $\varrho \in \mathrm{L}^2((0,T)\times\Omega)$, $\vec{u} \in \mathrm{L}^2(0,T;\mathrm{W}_0^{1,2}(\Omega))$, then prolonging ϱ, $\vec{u}$ to be zero on $\mathbb{R}^2 - \Omega$, the equation (5.11) holds in $D'((0,T)\times\mathbb{R}^2)$ (see [2], Lemma 3.3 for details).

Now we can repeat the same procedure as above with the limit equation (5.12) making use of the test functions

$$\varphi_i(t,x) = \psi(t)\phi(x)\mathcal{A}_i[\varrho], \quad i=1,2, \text{ where } \psi \in D(0,T),\ \phi \in D(\Omega),$$

ϱ being set zero outside Ω.

Similarly as in (5.22), we can deduce:

$$\int_0^T \psi \int_\Omega \phi\Big(p - (\lambda+2\mu)\mathrm{div}\;\vec{u}\Big)\varrho\,\mathrm{d}x\,\mathrm{d}t = \tag{5.23}$$

$$= \int_0^T \psi \int_\Omega \partial_{x_i}\phi\Big((\lambda+\mu)\mathrm{div}\; (\vec{u}) - p \Big)\mathcal{A}_i[\varrho]\,\mathrm{d}x\,\mathrm{d}t +$$

$$+\,\mu \int_0^T \psi \int_\Omega \left(\partial_{x_j}\phi\right) u^j \varrho\,\mathrm{d}x\,\mathrm{d}t - \mu \int_0^T \psi \int_\Omega \left(\partial_{x_j}\phi\right) u^i \mathcal{R}_{i,j}[\varrho]\,\mathrm{d}x\,\mathrm{d}t +$$

$$+\,\mu \int_0^T \psi \int_\Omega \left(\partial_{x_j}\phi\right) \partial_{x_j} u^i \mathcal{A}_i[\varrho]\,\mathrm{d}x\,\mathrm{d}t - \int_0^T \psi_t \int_\Omega \phi\varrho u^i \mathcal{A}_i[\varrho]\,\mathrm{d}x\,\mathrm{d}t -$$

$$- \int_0^T \psi \int_\Omega \left(\partial_{x_j}\phi\right) \varrho u^i u^j \mathcal{A}_i[\varrho]\,\mathrm{d}x\,\mathrm{d}t +$$

$$+ \int_0^T \psi \int_\Omega \phi u^i \left(\varrho\mathcal{R}_{i,j}[\varrho u^j] - \varrho u^j \mathcal{R}_{i,j}[\varrho] \right)\,\mathrm{d}x\,\mathrm{d}t.$$

Now, we shall prove that the integrals on the right-hand side of (5.22) converge for $\varepsilon \to 0$ to their counterparts in (5.23) or to zero.

To this end, observe (we use the estimates (4.35), (4.37) and (4.38))

$$\varepsilon \left| \int_0^T \psi \int_\Omega \phi \partial_{x_j} u_\varepsilon^i \, \partial_{x_j} \varrho_\varepsilon \mathcal{A}_i[\varrho_\varepsilon] \, \mathrm{d}x \, \mathrm{d}t \right| \leq$$

$$\leq \sqrt{\varepsilon} \|\psi\phi\|_{L^\infty((0,T)\times\Omega)} \|\sqrt{\varepsilon}\nabla \varrho_\varepsilon\|_{L^2((0,T)\times\Omega)} \|\nabla \vec{u}_\varepsilon\|_{L^2((0,T)\times\Omega)} \|\mathcal{A}_i[\varrho_\varepsilon]\|_{L^\infty((0,T)\times\Omega)} \leq$$
$$\leq \sqrt{\varepsilon}\, c(\delta, \varrho_0, \vec{q}),$$

consequently, we conclude that the integral on the left hand side tends to zero for δ fixed and $\varepsilon \to 0$.

Analogously, using (4.35), (4.37), (4.38) and (5.17), we obtain

$$\varepsilon \left| \int_0^T \psi \int_\Omega \phi \varrho_\varepsilon u_\varepsilon^i \mathcal{R}_{i,j}[\chi_\Omega \partial_{x_j} \varrho_\varepsilon] \, \mathrm{d}x \, \mathrm{d}t \right| \leq$$

$$\leq \sqrt{\varepsilon} \|\psi\phi\|_{L^\infty((0,T)\times\Omega)} \|\varrho_\varepsilon\|_{L^\infty(0,T;L^4(\Omega))} \|\vec{u}_\varepsilon\|_{L^2(0,T;L^4(\Omega))} \|\mathcal{R}_{i,j}[\chi_\Omega \sqrt{\varepsilon} \partial_{x_j} \varrho_\varepsilon]\|_{L^2((0,T)\times\Omega)} \leq$$
$$\leq \sqrt{\varepsilon}\, c(\delta, \varrho_0, \vec{q}) \, \|\sqrt{\varepsilon}\nabla \varrho_\varepsilon\|_{L^2((0,T)\times\Omega)} \leq \sqrt{\varepsilon}\, c(\delta, \varrho_0, \vec{q})$$

where the right-hand side tends to zero for δ fixed and $\varepsilon \to 0$.

We have proved that two of the integrals on the right-hand side of (5.22) tends to zero for $\varepsilon \to 0$. Now, we shall prove that the rest of integrals on the right-hand side of (5.22) converge to their counterparts in (5.23).

To begin, one observes easily that (4.35) implies

$$\varrho_\varepsilon \to \varrho \ \text{ in } C([0,T]; \mathrm{L}^\beta_{weak}(\Omega)) \tag{5.24}$$

which, combined with (5.15) and (5.17), gives

$$\mathcal{A}_i[\varrho_\varepsilon] \to \mathcal{A}_i[\varrho] \text{ in } C\left(\overline{(0,T)\times\Omega}\right) \quad \text{and} \quad \mathcal{R}_{i,j}[\varrho_\varepsilon] \to \mathcal{R}_{i,j}[\varrho] \text{ in } C(0,T; \mathrm{L}^\beta_{weak}(\Omega)).$$

Consequently, with the relations (5.6) – (5.10) in mind, it is easy to see that it is enough to show

$$\int_0^T \psi \int_\Omega \phi u_\varepsilon^i \left(\varrho_\varepsilon \mathcal{R}_{i,j}[\varrho_\varepsilon u_\varepsilon^j] - \varrho_\varepsilon u_\varepsilon^j \mathcal{R}_{i,j}[\varrho_\varepsilon] \right) \mathrm{d}x \, \mathrm{d}t \ \to \tag{5.25}$$
$$\to \int_0^T \psi \int_\Omega \phi u^i \left(\varrho \mathcal{R}_{i,j}[\varrho u^j] - \varrho u^j \mathcal{R}_{i,j}[\varrho] \right) \mathrm{d}x \, \mathrm{d}t \quad \text{as } \varepsilon \to 0.$$

To this end, we shall use Lemma 3.11. By virtue of (5.8), (5.24) and (4.36), we have

$$\varrho_\varepsilon \vec{u}_\varepsilon \to \varrho \vec{u} \text{ in } C([0,T]; \mathrm{L}^{\frac{2\beta}{\beta+1}}_{weak}(\Omega))$$

which, combined with (5.24) and Lemma 3.11 gives (for $\beta \geq 4$)

$$\varrho_\varepsilon \mathcal{R}_{i,j}[\varrho_\varepsilon u_\varepsilon^j] - \varrho_\varepsilon u_\varepsilon^j \mathcal{R}_{i,j}[\varrho_\varepsilon] \ \to \ \varrho \mathcal{R}_{i,j}[\varrho u^j] - \varrho u^j \mathcal{R}_{i,j}[\varrho] \text{ strongly in } \mathrm{L}^2(0,T;\mathrm{W}^{-1,2}(\Omega))$$

which, together with (5.8), completes the proof of (5.25). Thus we have proved Lemma 5.3.

5.4 Strong convergence of the density

We conclude this section by showing (5.14) and, consequently, strong convergence of the sequence ϱ_ε.

By virtue of [2], Lemma 3.3, the functions ϱ, $\vec{u}$ solve the continuity equation (5.11) in $D'((0,T)\times\mathbb{R}^2)$ prolonged to be zero outside Ω. Taking a regularizing sequence $\vartheta_m = \vartheta_m(x)$, we obtain

$$\partial_t S_m[\varrho] + \mathrm{div}\ (S_m[\varrho]\vec{u}) = r_m \quad \text{on } (0,T)\times\mathbb{R}^2 \tag{5.26}$$

where $S_m[v] = \vartheta_m * v$ are the standard smoothing operators. Here, by virtue of [5], Lemma 2.3, $r_m \to 0$ in $\mathrm{L}^1((0,T)\times\mathbb{R}^2)$ as $m\to\infty$.

If we multiply (5.26) by $b'(S_m[\varrho])$ and pass to the limit for $m\to\infty$ then we obtain

$$b(\varrho)_t + \mathrm{div}\ (b(\varrho)\vec{u}) + (b'(\varrho)\varrho - b(\varrho))\mathrm{div}\ \vec{u} = 0 \quad \text{in } D'((0,T)\times\mathbb{R}^2) \tag{5.27}$$

for any continuously differentiable function b such that b' is uniformly bounded. Thus ϱ, $\vec{u}$ solve (5.11) in the sense of renormalized solutions.

Multiplying (5.26) by $S_m[\varrho]\log(S_m[\varrho])$ and passing $m\to\infty$, we get (using the similar estimate to (4.26))

$$\big(\varrho\log(\varrho)\big)_t + \mathrm{div}\ (\varrho\log(\varrho)\vec{u}) + \varrho\,\mathrm{div}\ \vec{u} = 0 \text{ in } D'((0,T)\times\mathbb{R}^2). \tag{5.28}$$

Let us consider $\psi\in D(0,T)$, $\phi\in D(\mathbb{R}^2)$, $\phi(x)\equiv 1$ on an open set containing $\overline{\Omega}$ and take the product $\psi(t)\phi(x)$ as a test function in (5.28). We obtain

$$\int_0^T \psi_t \int_\Omega \varrho\log(\varrho)\,\mathrm{d}x\,\mathrm{d}t - \int_0^T \psi \int_\Omega \varrho\,\mathrm{div}\ \vec{u}\,\mathrm{d}x\,\mathrm{d}t = 0,$$

thus, $\int_\Omega \varrho\log(\varrho)\,\mathrm{d}x \in \mathrm{W}^{1,2}(0,T)$ and

$$\int_0^T \int_\Omega \varrho\,\mathrm{div}\ \vec{u}\,\mathrm{d}x\,\mathrm{d}t = \int_\Omega \varrho_0\log(\varrho_0)\,\mathrm{d}x - \int_\Omega \varrho(T)\log(\varrho(T))\,\mathrm{d}x. \tag{5.29}$$

On the other hand, ϱ_ε solves (4.1) a.e. on $(0,T)\times\Omega$, in particular,

$$b(\varrho_\varepsilon)_t + \mathrm{div}\ (b(\varrho_\varepsilon)\vec{u}_\varepsilon) + (b'(\varrho_\varepsilon)\varrho_\varepsilon - b(\varrho_\varepsilon))\mathrm{div}\ \vec{u}_\varepsilon - \varepsilon\triangle b(\varrho_\varepsilon) \leq 0$$

for any b convex and globally Lipschitz on $\mathbb{R}^+$; this gives

$$\int_0^T \int_\Omega (b'(\varrho_\varepsilon)\varrho_\varepsilon - b(\varrho_\varepsilon)) \mathrm{div}\ \vec{u}_\varepsilon \,\mathrm{d}x\,\mathrm{d}t \le \int_\Omega b(\varrho_0)\,\mathrm{d}x - \int_\Omega b(\varrho_\varepsilon(T))\,\mathrm{d}x$$

from which we easily deduce

$$\int_0^T \int_\Omega \varrho_\varepsilon \mathrm{div}\ \vec{u}_\varepsilon \,\mathrm{d}x\,\mathrm{d}t \le \int_\Omega \varrho_0 \log(\varrho_0)\,\mathrm{d}x - \int_\Omega \varrho_\varepsilon(T)\log(\varrho_\varepsilon(T))\,\mathrm{d}x. \tag{5.30}$$

Take two nondecreasing sequences ψ_n, ϕ_n of nonnegative functions such that

$$\psi_n \subset D(0,T), \ \ \psi_n \to 1, \ \ \phi_n \in D(\Omega), \ \ \phi_n \to 1.$$

Let us suppose $m \le n$. Then combining the conclusion of Lemma 5.3 together with (5.7), (5.8), (5.29), (5.30), we obtain

$$\limsup_{\varepsilon\to 0+} \int_0^T \psi_m \int_\Omega \phi_m \left(a\varrho_\varepsilon \log^d(1+\varrho_\varepsilon) + \delta \varrho_\varepsilon^\beta \right) \varrho_\varepsilon \,\mathrm{d}x\,\mathrm{d}t \le$$

$$\le \limsup_{\varepsilon\to 0+} \int_0^T \psi_n \int_\Omega \phi_n \left(a\varrho_\varepsilon \log^d(1+\varrho_\varepsilon) + \delta \varrho_\varepsilon^\beta \right) \varrho_\varepsilon \,\mathrm{d}x\,\mathrm{d}t \le$$

$$\le \lim_{\varepsilon\to 0_+} \int_0^T \psi_n \int_\Omega \phi_n \left(a\varrho_\varepsilon \log^d(1+\varrho_\varepsilon) + \delta \varrho_\varepsilon^\beta - (\lambda+2\mu)\mathrm{div}\ \vec{u}_\varepsilon \right) \varrho_\varepsilon \,\mathrm{d}x\,\mathrm{d}t +$$

$$+ (\lambda+2\mu) \limsup_{\varepsilon\to 0+} \int_0^T \psi_n \int_\Omega \phi_n \varrho_\varepsilon \mathrm{div}\ \vec{u}_\varepsilon \,\mathrm{d}x\,\mathrm{d}t \le$$

$$\le \int_0^T \psi_n \int_\Omega \phi_n \Big(p - (\lambda+2\mu)\mathrm{div}\ \vec{u} \Big) \varrho \,\mathrm{d}x\,\mathrm{d}t +$$

$$(\lambda+2\mu) \limsup_{\varepsilon\to 0+} \int_0^T \int_\Omega \varrho \,|1-\psi_n\phi_n|\, |\mathrm{div}\ \vec{u}_\varepsilon| \,\mathrm{d}x\,\mathrm{d}t +$$

$$+(\lambda+2\mu) \limsup_{\varepsilon\to 0+} \int_0^T \int_\Omega \varrho_\varepsilon \mathrm{div}\ \vec{u}_\varepsilon \,\mathrm{d}x\,\mathrm{d}t \le$$

$$\le \int_0^T \int_\Omega p\varrho \,\mathrm{d}x\,\mathrm{d}t - (\lambda+2\mu) \int_0^T \int_\Omega \varrho\, \mathrm{div}\ \vec{u} \,\mathrm{d}x\,\mathrm{d}t + \eta_1(n) +$$

$$+ (\lambda+2\mu) \|1-\psi_n\phi_n\|_{L^4((0,T)\times\Omega)} \sup_{\varepsilon\in(0,1)} \Big(\|\varrho_\varepsilon\|_{L^4((0,T)\times\Omega)} \|\mathrm{div}\ \vec{u}_\varepsilon\|_{L^2((0,T)\times\Omega)} \Big) +$$

$$+ (\lambda+2\mu) \left[\int_\Omega \varrho_0 \log(\varrho_0)\,\mathrm{d}x - \liminf_{\varepsilon\to 0+} \int_\Omega \varrho_\varepsilon(T)\log(\varrho_\varepsilon(T))\,\mathrm{d}x \right] \le$$

$$\leq \int_0^T \int_\Omega p\varrho \,\mathrm{d}x\,\mathrm{d}t + \eta_1(n) + \eta_2(n) +$$

$$+(\lambda+2\mu)\left[\int_\Omega \varrho(T)\log(\varrho(T))\,\mathrm{d}x - \liminf_{\varepsilon\to 0+}\int_\Omega \varrho_\varepsilon(T)\log(\varrho_\varepsilon(T))\,\mathrm{d}x\right] \leq$$

$$\leq \int_0^T \int_\Omega p\varrho \,\mathrm{d}x\,\mathrm{d}t + \eta_1(n) + \eta_2(n)$$

where

$$\eta_1(n), \eta_2(n) \to 0 \text{ for } n \to \infty.$$

Thus we have proved

$$\limsup_{\varepsilon\to 0+} \int_0^T \psi_m \int_\Omega \phi_m \left(a\varrho_\varepsilon \log^d(1+\varrho_\varepsilon) + \delta\varrho_\varepsilon^\beta\right)\varrho_\varepsilon \,\mathrm{d}x\,\mathrm{d}t \leq \int_0^T \int_\Omega p\varrho\,\mathrm{d}x\,\mathrm{d}t \tag{5.31}$$

for all $m = 1, 2, 3, \ldots$

To conclude the proof of (5.14), we shall use Minty's trick. Since the nonlinearity $P(z) = az\log^d(1+z) + \delta z^\beta$ is monotone, we have

$$\int_0^T \psi_m \int_\Omega \phi_m \big(P(\varrho_\varepsilon) - P(v)\big)(\varrho_\varepsilon - v)\,\mathrm{d}x\,\mathrm{d}t \geq 0$$

for all $v \in \mathrm{L}^{\beta+1}((0,T)\times\Omega)$. By virtue of (5.7), (5.13) and (5.31), we have

$$\int_0^T \int_\Omega p\varrho\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \psi_m\phi_m P(v)v\,\mathrm{d}x\,\mathrm{d}t - \int_0^T \psi_m \int_\Omega \phi_m(pv + P(v)\varrho)\,\mathrm{d}x\,\mathrm{d}t \geq 0.$$

Now, letting $m \to \infty$, we get

$$\int_0^T \int_\Omega \big(p - P(v)\big)(\varrho - v)\,\mathrm{d}x\,\mathrm{d}t \geq 0$$

and the choice $v = \varrho + \omega\phi$, $\omega \to 0$, $\phi \in \mathrm{L}^{\beta+1}((0,T)\times\Omega)$ arbitrary, yields desired conclusion $p = a\varrho\log^d(1+\varrho) + \delta\varrho^\beta$.

Let us summarize the results achieved in this section:

Theorem 3: *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with C^{2+v} boundary and let $\beta \geq 4$. Then, given initial data ϱ_0, $\vec{q}$ as in (4.5), (4.6), there exists a finite energy weak solution ϱ, $\vec{u}$ of the problem*

$$\varrho_t + \mathrm{div}\ (\varrho\vec{u}) = 0, \tag{5.32}$$

$$(\varrho u^i)_t + \mathrm{div}\ (\varrho u^i\vec{u}) + \Big(a\varrho\log^d(1+\varrho) + \delta\varrho^\beta\Big)_{x_i} = \mu\Delta u^i + (\lambda+\mu)(\mathrm{div}\ \vec{u})_{x_i},\ \ i = 1,2, \tag{5.33}$$

$$\vec{u}|_{\partial\Omega} = 0, \tag{5.34}$$

$$\varrho(0) = \varrho_0, \quad (\varrho u^i)(0) = q_i,\ i = 1,2. \tag{5.35}$$

Moreover, $\varrho \in L^{\beta+1}((0,T)\times\Omega)$ and the equation (5.32) holds in the sense of renormalized solutions in $D'((0,T)\times\mathbb{R}^2)$ provided ϱ, $\vec{u}$ were prolonged to be zero on $\mathbb{R}^2 - \Omega$.
Finally, ϱ, $\vec{u}$ satisfy the estimates:

$$\sup_{t\in[0,T]} \|\varrho(t)\|_{y\log^{(d+1)}(y)} \leq cE_\delta[\varrho_0,\vec{q}], \tag{5.36}$$

$$\delta \sup_{t\in[0,T]} \|\varrho(t)\|^\beta_{L^\beta(\Omega)} \leq cE_\delta[\varrho_0,\vec{q}], \tag{5.37}$$

$$\sup_{t\in[0,T]} \|\sqrt{\varrho}(t)\vec{u}(t)\|^2_{L^2(\Omega)} \leq cE_\delta[\varrho_0,\vec{q}], \tag{5.38}$$

$$\|\vec{u}\|_{L^2(0,T;W_0^{1,2}(\Omega))} \leq cE_\delta[\varrho_0,\vec{q}] \tag{5.39}$$

where the constant c is independent of $\delta > 0$.

6 Passing to the limit in the artificial pressure term

Our ultimate goal is to let $\delta \to 0$ in (5.33) and to relax our hypothesis on the initial data ϱ_0, $\vec{q}$, which, up to now, have been supposed to satisfy (4.5) and (4.6).

To begin, consider general initial data satisfying the compatibility conditions (1.6). It is easy to find a sequence $\delta_k \to 0$ such that there exists a sequence of functions $\varrho_{0,\delta_k} \in C^{2+v}(\overline{\Omega})$ and

$$0 < \delta_k \leq \varrho_{0,\delta_k}(x) \leq \delta_k^{-1/\beta}, \quad \nabla\varrho_{0,\delta_k}\cdot\vec{n}|_{\partial\Omega} = 0, \tag{6.1}$$

$$\varrho_{0,\delta_k} \to \varrho_0 \text{ in } \mathrm{L}_{y\log^{d+1}(y)}(\Omega) \text{ as } k\to\infty. \tag{6.2}$$

For simplifying notation, we shall omit the index k in the following and write δ instead of δ_k.

Set

$$\tilde{q}^i_\delta(x) = \begin{Bmatrix} q^i(x)\sqrt{\frac{\varrho_{0,\delta}(x)}{\varrho_0(x)}} & \text{if } \varrho_0(x) > 0, \\ 0 & \text{if } \varrho_0(x) = 0, \end{Bmatrix} \quad i = 1,2.$$

By virtue of (1.6), we have

$$\frac{|\tilde{q}^i_\delta|^2}{\varrho_{0,\delta}} \text{ bounded in } \mathrm{L}^1(\Omega) \text{ independently of } \delta > 0,\ i = 1,2$$

and we can find $h^i_\delta \in D(\Omega)$ such that

$$\left\| \frac{\tilde{q}^i_\delta}{\sqrt{\varrho_{0,\delta}}} - h^i_\delta \right\|_{L^2(\Omega)} < \delta, \quad i = 1,2.$$

Taking $q^i_\delta = h^i_\delta \sqrt{\varrho_{0,\delta}}$, $i = 1,2$, one checks easily that

$$\frac{|q^i_\delta|^2}{\varrho_{0,\delta}} \text{ are bounded in } \mathrm{L}^1(\Omega),\ i = 1,2, \text{ independently of } \delta > 0 \tag{6.3}$$

and

$$q^i_\delta \to q^i \text{ in } \mathrm{L}^1(\Omega) \text{ as } \delta \to 0 \text{ for } i = 1,2. \tag{6.4}$$

From now on, we shall deal with the sequence of approximate solutions ϱ_δ, $\vec{u}_\delta$ of the problem (5.32) – (5.35) with the initial data $\varrho_{0,\delta}$, $\vec{q}_\delta = [q^1_\delta, q^2_\delta]$, the existence of which is guaranteed by Theorem 3.

By virtue of (6.1) and (6.3), we have

$$E_\delta[\varrho_{0,\delta}, \vec{q}_\delta] = \int_\Omega \frac{1}{2}\frac{|\vec{q}_\delta|^2}{\varrho_{0,\delta}} + \frac{\delta}{\beta - 1}\varrho^\beta_{0,\delta} + \frac{a}{d+1}(1+\varrho_{0,\delta})\log^{d+1}(1+\varrho_{0,\delta}) + R(\varrho_{0,\delta})\,\mathrm{d}x < c$$

where the constant c does not depend on δ. Consequently, the estimates (5.36) – (5.39) hold independently of δ.

6.1 On the integrability of the density

We first derive an estimate of the density ϱ_δ to make possible passing to the limit in the term $\delta\varrho^\beta_\delta$ as $\delta \to 0$. We shall prove the following lemma:

Lemma 6.1: *Let $d > 1$ and $0 < \theta < \min\{d-1, 1\}$. Let $\psi \in D(0,T)$ and $\phi \in D(\Omega)$. Then*

$$\int_0^T \int_\Omega \psi\phi\delta\varrho^\beta_\delta \log^\theta(1+\varrho_\delta)\,dx\,dt \le c$$

where the constant c is independent of $\delta > 0$.

Proof: Using renormalized continuity equation (5.27) for $b(y) = \log^\theta(1+y)$, we obtain

$$\Big(\log^\theta(1+\varrho_\delta)\Big)_t + \mathrm{div}\ (\log^\theta(1+\varrho_\delta)\vec{u}_\delta) +$$

$$+ \left(\theta\log^{\theta-1}(1+\varrho_\delta)\frac{\varrho_\delta}{1+\varrho_\delta} - \log^\theta(1+\varrho_\delta)\right)\mathrm{div}\ \vec{u}_\delta = 0 \qquad \text{in } D'((0,T)\times\mathbb{R}^2). \tag{6.5}$$

Let us consider the functions

$$\varphi_i(t,x) = \psi(t)\phi(x)\mathcal{A}_i[\log^\theta(1+\varrho_\delta)],\ i = 1,2, \text{ where } \psi \in D(0,T),\ \phi \in D(\Omega),$$

where, as always, ϱ_δ is prolonged by zero outside Ω.

By virtue of (5.36), we get

$$\sup_{t\in[0,T]} \|\log^\theta(1+\varrho_\delta(t))\|_{e(\frac{1}{\theta})} \le c(\varrho_0, \vec{q}), \tag{6.6}$$

where the constant $c(\varrho_0, \vec{q})$ does not depend on δ. Consequently, by virtue of (5.16)

$$\mathcal{A}_i[\log^\theta(1+\varrho_\delta)] \in C([0,T]; \mathrm{W\,L}_{e(\frac{1}{\theta+1})}(\Omega)),$$

in particular, using the definition of the space $\mathrm{E}_\phi(\Omega)$, we get

$$\mathcal{A}_i[\log^\theta(1+\varrho_\delta)] \in C([0,T]; \mathrm{W\,E}_{e(\beta)}(\Omega)), \ \text{ for } 0<\beta<\frac{1}{\theta+1}, \ \ t\in[0,T].$$

Now, (2.2) justifies the choice of φ_i as test functions for the equation (5.33) to obtain

$$\int_0^T \int_\Omega \psi\phi \left(a\varrho_\delta \log^{d+\theta}(1+\varrho_\delta) + \delta\varrho_\delta^\beta \log^\theta(1+\varrho_\delta)\right) \,\mathrm{d}x\,\mathrm{d}t = \tag{6.7}$$

$$-\int_0^T \psi \int_\Omega \partial_{x_j}\phi\, \varrho_\delta u_\delta^i u_\delta^j \mathcal{A}_i[\log^\theta(1+\varrho_\delta)]\,\mathrm{d}x\,\mathrm{d}t - \int_0^T \psi \int_\Omega \phi\varrho_\delta u_\delta^i u_\delta^j \mathcal{R}_{i,j}[\log^\theta(1+\varrho_\delta)]\,\mathrm{d}x\,\mathrm{d}t +$$

$$+\mu \int_0^T \psi \int_\Omega \partial_{x_j}\phi\, \partial_{x_j} u_\delta^i \mathcal{A}_i[\log^\theta(1+\varrho_\delta)]\,\mathrm{d}x\,\mathrm{d}t +$$

$$+\mu \int_0^T \psi \int_\Omega \phi\, \partial_{x_j} u_\delta^i \mathcal{R}_{i,j}[\log^\theta(1+\varrho_\delta)]\,\mathrm{d}x\,\mathrm{d}t +$$

$$+(\lambda+\mu) \int_0^T \psi \int_\Omega \partial_{x_i}\phi\,\mathrm{div}\ \vec{u}_\delta \mathcal{A}_i[\log^\theta(1+\varrho_\delta)]\,\mathrm{d}x\,\mathrm{d}t +$$

$$+(\lambda+\mu) \int_0^T \psi \int_\Omega \phi\,\mathrm{div}\ (\vec{u}_\delta)\log^\theta(1+\varrho_\delta)\,\mathrm{d}x\,\mathrm{d}t +$$

$$+\int_0^T \psi \int_\Omega \phi\varrho_\delta u_\delta^i \mathcal{R}_{i,j}[\log^\theta(1+\varrho_\delta)u_\delta^j]\,\mathrm{d}x\,\mathrm{d}t - \int_0^T \partial_t\psi \int_\Omega \phi\varrho_\delta u_\delta^i \mathcal{A}_i[\log^\theta(1+\varrho_\delta)]\,\mathrm{d}x\,\mathrm{d}t +$$

$$+\int_0^T \psi \int_\Omega \phi\varrho_\delta u_\delta^i \mathcal{A}_i \left[\left(\theta\log^{\theta-1}(1+\varrho_\delta)\frac{\varrho_\delta}{1+\varrho_\delta} - \log^\theta(1+\varrho_\delta)\right)\mathrm{div}\ \vec{u}_\delta\right]\,\mathrm{d}x\,\mathrm{d}t = \sum_{k=1}^9 I_k$$

where the integrals $I_1 - I_9$ may be treated as follows:

(i) Using Lemma 3.3, Lemma 3.7, (5.15), (5.36), (5.39) and (6.6), we have

$$|I_1| = \left|\int_0^T \psi \int_\Omega \partial_{x_j}\phi\, \varrho_\delta u_\delta^i u_\delta^j \mathcal{A}_i[\log^\theta(1+\varrho_\delta)]\,\mathrm{d}x\,\mathrm{d}t\right| \le$$

$$\le c\int_0^T \|\varrho_\delta u_\delta^i u_\delta^j\|_{y\log^d(y)} \|\mathcal{A}_i[\log^\theta(1+\varrho_\delta)]\|_{L^\infty}\,\mathrm{d}t \le$$

$$\le c\int_0^T \|\varrho_\delta\|_{y\log^{d+1}(y)} \|\vec{u}_\delta\|_{e(2)} \|\vec{u}_\delta\|_{e(2)} \|\log^\theta(1+\varrho_\delta)\|_{e\left(\frac{1}{\theta}\right)}\,\mathrm{d}t \le c.$$

(ii) Similarly, by virtue of Lemma 3.3, Lemma 3.7, (5.19), (5.36), (5.39) and (6.6), we obtain

$$|I_2| = \left| \int_0^T \psi \int_\Omega \phi \varrho_\delta u_\delta^i u_\delta^j \mathcal{R}_{i,j}[\log^\theta(1+\varrho_\delta)] \,\mathrm{d}x \,\mathrm{d}t \right| \le$$

$$\le c \int_0^T \|\varrho_\delta u_\delta^i u_\delta^j\|_{y \log^d(y)} \|\mathcal{R}_{i,j}[\log^\theta(1+\varrho_\delta)]\|_{e\left(\frac{1}{d}\right)} \,\mathrm{d}t \le$$

$$\le c \int_0^T \|\varrho_\delta\|_{y \log^{d+1}(y)} \|\vec{u}_\delta\|_{e(2)} \|\vec{u}_\delta\|_{e(2)} \|\mathcal{R}_{i,j}[\log^\theta(1+\varrho_\delta)]\|_{e\left(\frac{1}{1+\theta}\right)} \,\mathrm{d}t \le$$

$$\le c \int_0^T \|\vec{u}_\delta\|_{e(2)}^2 \,\mathrm{d}t \sup_{t\in[0,T]} \|\varrho_\delta\|_{y \log^{d+1}(y)} \sup_{t\in[0,T]} \|\log^\theta(1+\varrho_\delta)\|_{e\left(\frac{1}{\theta}\right)} \le c.$$

(iii) In view of (5.15), (5.39), it holds

$$|I_3| = \left| \mu \int_0^T \psi \int_\Omega \partial_{x_j}\phi \, \partial_{x_j} u_\delta^i \mathcal{A}_i[\log^\theta(1+\varrho_\delta)] \,\mathrm{d}x \,\mathrm{d}t \right| \le c.$$

(iv) Furthermore, (5.19), (5.39) and (6.6) imply

$$|I_4| = \left| \mu \int_0^T \psi \int_\Omega \phi \, \partial_{x_j} u_\delta^i \mathcal{R}_{i,j}[\log^\theta(1+\varrho_\delta)] \,\mathrm{d}x \,\mathrm{d}t \right| \le c.$$

(v) Similarly as in (iii):

$$|I_5| = \left| (\lambda+\mu) \int_0^T \psi \int_\Omega \partial_{x_i}\phi \,\mathrm{div}\; \vec{u}_\delta \mathcal{A}_i[\log^\theta(1+\varrho_\delta)] \,\mathrm{d}x \,\mathrm{d}t \right| \le c.$$

(vi) Using (5.39) and (6.6), we obtain

$$|I_6| = \left| (\lambda+\mu) \int_0^T \psi \int_\Omega \phi \,\mathrm{div}\; (\vec{u}_\delta) \log^\theta(1+\varrho_\delta) \,\mathrm{d}x \,\mathrm{d}t \right| \le$$

$$\le c \|\vec{u}_\delta\|_{L^2(0,T;W^{1,2}(\Omega))} \sup_{t\in[0,T]} \|\log^\theta(1+\varrho_\delta(t))\|_{e\left(\frac{1}{\theta}\right)} \le c.$$

(vii) By virtue of Lemma 3.3, it holds

$$|I_7| \le \int_0^T \psi \int_\Omega \phi \varrho_\delta u_\delta^i \mathcal{R}_{i,j}[\log^\theta(1+\varrho_\delta) u_\delta^j] \,\mathrm{d}x \,\mathrm{d}t \le$$

$$\le c \int_0^T \|\varrho_\delta u_\delta^i\|_{y \log^{d+1/2}(y)} \|\mathcal{R}_{i,j}[\log^\theta(1+\varrho_\delta) u_\delta^j]\|_{e\left(\frac{1}{d+1/2}\right)} \,\mathrm{d}t. \tag{6.8}$$

In view of (5.19) and Lemma 3.6, we can estimate

$$\|\mathcal{R}_{i,j}[\log^\theta(1+\varrho_\delta)u^j_\delta]\|_{e\left(\frac{1}{d+1/2}\right)} \le c\|\log^\theta(1+\varrho_\delta)u^j_\delta\|_{e\left(\frac{1}{d-1/2}\right)} \le$$

$$\le c\|\log^\theta(1+\varrho_\delta)\|_{e\left(\frac{1}{d-1}\right)}\|u^j_\delta\|_{e(2)},$$

consequently (6.8) implies

$$|I_7| \le c\int_0^T \|\varrho_\delta u^i_\delta\|_{y\log^{d+1/2}(y)}\|\log^\theta(1+\varrho_\delta)\|_{e\left(\frac{1}{d-1}\right)}\|u^j_\delta\|_{e(2)}\,\mathrm{d}t \le$$

$$\le c\int_0^T \|\varrho_\delta\|_{y\log^{d+1}(y)}\|u^i_\delta\|_{e(2)}\|\log^\theta(1+\varrho_\delta)\|_{e\left(\frac{1}{\theta}\right)}\|u^j_\delta\|_{e(2)}\,\mathrm{d}t \le c$$

provided $\theta < d-1$.

(viii) Similarly as in (i), we have

$$|I_8| = \left|\int_0^T \partial_t\psi\int_\Omega \phi\varrho_\delta u^i_\delta\mathcal{A}_i[\log^\theta(1+\varrho_\delta)]\,\mathrm{d}x\,\mathrm{d}t\right| \le$$

$$\le c\int_0^T \|\varrho_\delta u^i_\delta\|_{y\log^{d+1/2}(y)}\|\mathcal{A}_i[\log^\theta(1+\varrho_\delta)]\|_{L^\infty}\,\mathrm{d}t \le$$

$$\le c\int_0^T \|\varrho_\delta\|_{y\log^{d+1}(y)}\|\vec{u}_\delta\|_{e(2)}\|\log^\theta(1+\varrho_\delta)\|_{e\left(\frac{1}{\theta}\right)}\,\mathrm{d}t \le c.$$

(ix) By virtue of Lemma 3.3, we get

$$|I_9| = \left|\int_0^T \psi\int_\Omega \phi\varrho_\delta u^i_\delta\mathcal{A}_i\Big[\Big(\theta\log^{\theta-1}(1+\varrho_\delta)\frac{\varrho_\delta}{1+\varrho_\delta} - \log^\theta(1+\varrho_\delta)\Big)\mathrm{div}\ \vec{u}_\delta\Big]\,\mathrm{d}x\,\mathrm{d}t\right| \le$$

$$\le c\int_0^T\|\varrho_\delta u^i_\delta\|_{y\log^{d+1/2}(y)}\left\|\mathcal{A}_i\Big[\Big(\log^{\theta-1}(1+\varrho_\delta)\frac{\theta\varrho_\delta}{1+\varrho_\delta} - \log^\theta(1+\varrho_\delta)\Big)\mathrm{div}\ \vec{u}_\delta\Big]\right\|_{e\left(\frac{2}{2d+1}\right)}\mathrm{d}t. \tag{6.9}$$

Using Lemma 3.14 and Lemma 3.15, we obtain

$$\left\|\mathcal{A}_i\Big[\Big(\log^{\theta-1}(1+\varrho_\delta)\frac{\theta\varrho_\delta}{1+\varrho_\delta} - \log^\theta(1+\varrho_\delta)\Big)\mathrm{div}\ \vec{u}_\delta\Big]\right\|_{e\left(\frac{2}{2d+1}\right)} \le$$

$$\le \left\|\Big(\log^{\theta-1}(1+\varrho_\delta)\frac{\theta\varrho_\delta}{1+\varrho_\delta} - \log^\theta(1+\varrho_\delta)\Big)\mathrm{div}\ \vec{u}_\delta\right\|_{y^2\log^{-2d+2}(y)} \le$$

$$\le c\|\log^{\theta-1}(1+\varrho_\delta)\frac{\theta\varrho_\delta}{1+\varrho_\delta} - \log^\theta(1+\varrho_\delta)\|_{e\left(\frac{1}{\theta}\right)}\|\mathrm{div}\ \vec{u}_\delta\|_{L^2} \le$$

$$\le c\|\log^\theta(1+\varrho_\delta)\|_{e\left(\frac{1}{\theta}\right)}\|\vec{u}_\delta\|_{W^{1,2}}$$

which, combined with (6.9), yields $|I_9| \le c$.

Thus, we see that all integrals on the right hand side of (6.7) are bounded, consequently, in view of (5.36) and $\theta < 1$,

$$\int_0^T \int_\Omega \psi\phi\delta\varrho_\delta^\beta \log^\theta(1+\varrho_\delta)\,\mathrm{d}x\,\mathrm{d}t \le c.$$

Thus Lemma 6.1 has been proved.

Q.E.D.

Let $\theta \in (0,1)$ satisfies $\theta < d-1$. Let $\psi \in D(0,T)$ and $\phi \in D(\Omega)$. Put

$$M = \overline{\{x \in \Omega \,:\, \phi(x) \neq 0\}} \times \overline{\{t \in (0,T) \,:\, \psi(t) \neq 0\}}.$$

Then, by virtue of Lemma 6.1, it holds

$$\iint_M \delta\varrho_\delta^\beta \log^\theta(1+\varrho_\delta)\,\mathrm{d}x\,\mathrm{d}t \le c \tag{6.10}$$

where c does not depend on δ.

Let us define the set

$$J_k^\delta = \{(x,t) \in M \,:\, \varrho_\delta(x,t) \le k\} \ \text{ for } k > 0 \text{ and } \delta \in (0,1).$$

In view of (5.36), there exists a constant $s \in (0,\infty)$ such that for all $\delta \in (0,1)$ and $k > 0$

$$\mu\left(M - J_k^\delta\right) \le \frac{s}{k}. \tag{6.11}$$

We can estimate

$$\left|\int_0^T \int_\Omega \psi\phi\delta\varrho_\delta^\beta\,\mathrm{d}x\,\mathrm{d}t\right| \le c\iint_M \delta\varrho_\delta^\beta\,\mathrm{d}x\,\mathrm{d}t \le c\iint_{J_k^\delta} \delta\varrho_\delta^\beta\,\mathrm{d}x\,\mathrm{d}t + c\iint_{M-J_k^\delta} \delta\varrho_\delta^\beta\,\mathrm{d}x\,\mathrm{d}t \le$$

$$\le c\delta k^\beta \mu(\Omega) + c\delta \iint_M \chi_{(M-J_k^\delta)}\varrho_\delta^\beta\,\mathrm{d}x\,\mathrm{d}t. \tag{6.12}$$

Let us denote by Φ the Young function corresponding to the Orlicz space $\mathrm{L}_{e\left(\frac{1}{\theta}\right)}(\Omega \times (0,T))$. Then, using Lemma 3.3, (6.10) and (6.11), we get

$$\delta\iint_M \chi_{(M-J_k^\delta)}\varrho_\delta^\beta\,\mathrm{d}x\,\mathrm{d}t \le c\|\chi_{M-J_k^\delta}\|_{e\left(\frac{1}{\theta}\right)}\delta\|\varrho_\delta^\beta\|_{y\log^\theta(y)} \le$$

$$\le c\left(\Phi^{-1}\left(\frac{k}{s}\right)\right)^{-1}\iint_M \delta\varrho_\delta^\beta \log^\theta(1+\varrho_\delta)\,\mathrm{d}x\,\mathrm{d}t \le c\left(\Phi^{-1}\left(\frac{k}{s}\right)\right)^{-1}. \tag{6.13}$$

Combining (6.13) with (6.12), we derive the estimate

$$\left|\int_0^T \int_\Omega \psi\phi\delta\varrho_\delta^\beta \,\mathrm{d}x\,\mathrm{d}t\right| \leq c\delta k^\beta \mu(\Omega) + c\left(\Phi^{-1}\left(\frac{k}{s}\right)\right)^{-1}$$

where c does not depend on δ and k, consequently

$$\limsup_{\delta\to 0}\left|\int_0^T \int_\Omega \psi\phi\delta\varrho_\delta^\beta \,\mathrm{d}x\,\mathrm{d}t\right| \leq c\left(\Phi^{-1}\left(\frac{k}{s}\right)\right)^{-1}. \tag{6.14}$$

The right hand side of (6.14) tends to zero as $k\to\infty$, therefore, passing to the limit for $k\to\infty$ in (6.14), we infer

$$\lim_{\delta\to 0}\int_0^T \int_\Omega \psi\phi\delta\varrho_\delta^\beta \,\mathrm{d}x\,\mathrm{d}t = 0$$

which means that

$$\delta\varrho_\delta^\beta \to 0 \quad \text{in } D'((0,T)\times\Omega). \tag{6.15}$$

6.2 The limit passage

As ϱ_δ satisfies equation (5.32), we infer, by virtue of the estimate (5.36), Lemma 3.8,

$$\varrho_\delta \to \varrho \text{ in } C([0,T];\mathrm{L}^{*-weak}_{y\log^{d+1}(y)}(\Omega)) \text{ and } \varrho_\delta \to \varrho \text{ in } C([0,T];\mathrm{W}^{-1,2}(\Omega)), \tag{6.16}$$

next, the estimate (5.39) yields

$$\vec{u}_\varepsilon \to \vec{u} \text{ weakly in } \mathrm{L}^2(0,T;\mathrm{W}^{1,2}_0(\Omega)) \tag{6.17}$$

and, combining (6.16), (6.17) with (5.38), (5.33), Lemma 3.4 and Lemma 3.8, we infer

$$\varrho_\delta\vec{u}_\delta \to \varrho\vec{u} \text{ in } C([0,T];\mathrm{L}^{*-weak}_{y\log^{d+1}(y)}(\Omega)) \text{ and } \varrho_\delta\vec{u}_\delta \to \varrho\vec{u} \text{ in } C([0,T];\mathrm{W}^{-1,2}(\Omega)), \tag{6.18}$$

finally, (6.17) and (6.18) yield

$$\varrho_\delta u^i_\delta u^j_\delta \to \varrho u^i u^j \text{ in } \mathrm{L}^1((0,T)\times\Omega),\ i,j=1,2, \tag{6.19}$$

passing to subsequences as the case may be.

By virtue of (5.36), there exists a constant c such that for all $\delta > 0$

$$\|\varrho_\delta \log^d(1+\varrho_\delta)\|_{L^\infty(0,T;L_{y\log(y)}(\Omega))} \leq c,$$

thus

$$\varrho_\delta \log^d(1+\varrho_\delta) \to \overline{\varrho \log^d(1+\varrho)} \text{ weakly star in } \mathrm{L}^\infty(0,T;\mathrm{L}_{y\log(y)}(\Omega)). \tag{6.20}$$

In view of (6.15), we get

$$\delta \varrho_\delta^\beta \to 0 \text{ in } \mathrm{L}^1_{loc}((0,T)\times\Omega). \tag{6.21}$$

Finally, using (6.2), (6.4), (6.16) and (6.18), we see that the limits ϱ, $\varrho\vec{u}$ satisfy the initial conditions

$$\varrho(0) = \varrho_0, \quad \left(\varrho u^i\right)(0) = q^i, \ i = 1,2.$$

Now, using (6.16) – (6.21), we can pass to the limit in (5.32), (5.33) to obtain that ϱ, $\vec{u}$ satisfy

$$\varrho_t + \text{div } (\varrho\vec{u}) = 0, \tag{6.22}$$

in $D'((0,T)\times\mathbb{R}^2)$,

$$(\varrho u^i)_t + \text{div } (\varrho u^i\vec{u}) + a\left(\overline{\varrho\log^d(1+\varrho)}\right)_{x_i} = \mu\Delta u^i + (\lambda+\mu)(\text{div } \vec{u})_{x_i}, \quad i=1,2, \tag{6.23}$$

in $D'((0,T)\times\Omega)$. Thus the only thing to complete the proof of Theorem 1 is to show

$$\overline{\varrho\log^d(1+\varrho)} = \varrho\log^d(1+\varrho). \tag{6.24}$$

Let us introduce a family of functions (see [2])

$$T_k(z) = kT\left(\frac{z}{k}\right) \text{ for } z\in\mathbb{R}, \ k = 1,2,3,\ldots$$

where $T\in C^\infty(\mathbb{R})$ is chosen so that

$$T(z) = z \text{ for } z\le 1, \quad T(z) = 2 \text{ for } z\ge 3, \quad T \text{ concave.}$$

Then, since ϱ_δ, $\vec{u}_\delta$ is a renormalized solution of the continuity equation (5.32) in $D'((0,T)\times\mathbb{R}^2)$, we have

$$T_k(\varrho_\delta)_t + \text{div } (T_k(\varrho_\delta)\vec{u}_\delta) + (T'_k(\varrho_\delta)\varrho_\delta - T_k(\varrho_\delta)) \text{ div } \vec{u}_\delta = 0 \text{ in } D'((0,T)\times\mathbb{R}^2). \tag{6.25}$$

Passing to the limit for $\delta\to 0+$ we obtain (using Lemma 3.9)

$$\partial_t\overline{T_k(\varrho)} + \text{div } (\overline{T_k(\varrho)}\vec{u}) + \overline{(T'_k(\varrho)\varrho - T_k(\varrho)) \text{ div } \vec{u}} = 0 \text{ in } D'((0,T)\times\mathbb{R}^2) \tag{6.26}$$

where

$$(T'_k(\varrho_\delta)\varrho_\delta - T_k(\varrho_\delta)) \text{ div } \vec{u}_\delta \to \overline{(T'_k(\varrho)\varrho - T_k(\varrho)) \text{ div } \vec{u}} \text{ weakly in } \mathrm{L}^2((0,T)\times\Omega) \tag{6.27}$$

and

$$T_k(\varrho_\delta) \to \overline{T_k(\varrho)} \text{ in } C([0,T]; \mathrm{L}^{*-weak}_{e(\beta)}(\Omega)) \text{ for all } \beta > 0. \tag{6.28}$$

6.3 The effective viscous flux

We shall prove the following auxiliary result similar to Lemma 5.3.

Lemma 6.2: *Let ϱ_δ, $\vec{u}_\delta$ be the sequence of approximate solutions, the existence of which is guaranteed by Theorem 3, and let*

$$\varrho,\ \vec{u},\ \overline{\varrho \log^d(1+\varrho)},\ \text{and } \overline{T_k(\varrho)}$$

be the limits appearing in (6.16), (6.17), (6.20), and (6.28) respectively.
Then

$$\lim_{\delta\to 0_+} \int_0^T \psi \int_\Omega \phi \left(a\varrho_\delta \log^d(1+\varrho_\delta) - (\lambda+2\mu)\mathrm{div}\ \vec{u}_\delta \right) T_k(\varrho_\delta)\, dx\, dt =$$

$$= \int_0^T \psi \int_\Omega \phi \Big(\overline{a\varrho \log^d(1+\varrho)} - (\lambda+2\mu)\mathrm{div}\ \vec{u} \Big) \overline{T_k(\varrho)}\, dx\, dt$$

for any $\psi \in D(0,T)$, $\phi \in D(\Omega)$.

Proof: Persuing step by step the proof of Lemma 5.3, let us consider the functions of the form

$$\varphi_i(t,x) = \psi(t)\phi(x)\mathcal{A}_i[T_k(\varrho_\delta)],\ i=1,2, \quad \text{where } \psi \in D(0,T),\ \phi \in D(\Omega),$$

where, as always, ϱ_δ is prolonged by zero outside Ω. By virtue of (5.16)

$$\mathcal{A}_i[T_k(\varrho_\delta)] \in C([0,T]; \mathrm{W\,L}_{e(\beta)}(\Omega)),\ \text{for } \beta \in (0,1),$$

in particular, using the definition of the space $\mathrm{E}_\phi(\Omega)$, we get

$$\mathcal{A}_i[T_k(\varrho_\delta)] \in C([0,T]; \mathrm{W\,E}_{e(\beta)}(\Omega)),\ \text{for } \beta \in (0,1),\ t \in [0,T].$$

Now, (2.2) justifies the choice of φ_i as test functions for the equation (5.33) to obtain (using also (6.25), (5.18), (5.19) and the assumption $d > 1$)

$$\int_0^T \psi \int_\Omega \phi \left(a\varrho_\delta \log^d(1+\varrho_\delta) + \delta\varrho_\delta^\beta - (\lambda+2\mu)\mathrm{div}\ \vec{u}_\delta \right) T_k(\varrho_\delta)\, \mathrm{d}x\, \mathrm{d}t = \tag{6.29}$$

$$= \int_0^T \psi \int_\Omega \partial_{x_i}\phi \left((\lambda+\mu)\mathrm{div}\ (\vec{u}_\delta) - a\varrho_\delta \log^d(1+\varrho_\delta) - \delta\varrho_\delta^\beta \right) \mathcal{A}_i[T_k(\varrho_\delta)]\, \mathrm{d}x\, \mathrm{d}t +$$

$$+\mu\int_0^T\psi\int_\Omega\left(\partial_{x_j}\phi\right)u^j_\delta T_k(\varrho_\delta)\,\mathrm{d}x\,\mathrm{d}t-\mu\int_0^T\psi\int_\Omega\left(\partial_{x_j}\phi\right)u^i_\delta\mathcal{R}_{i,j}[T_k(\varrho_\delta)]\,\mathrm{d}x\,\mathrm{d}t+$$

$$+\mu\int_0^T\psi\int_\Omega\left(\partial_{x_j}\phi\right)\partial_{x_j}u^i_\delta\mathcal{A}_i[T_k(\varrho_\delta)]\,\mathrm{d}x\,\mathrm{d}t-\int_0^T\psi_t\int_\Omega\phi\varrho_\delta u^i_\delta\mathcal{A}_i[T_k(\varrho_\delta)]\,\mathrm{d}x\,\mathrm{d}t+$$

$$+\int_0^T\psi\int_\Omega\phi\varrho_\delta u^i_\delta\mathcal{A}_i[(T'_k(\varrho_\delta)\varrho_\delta-T_k(\varrho_\delta))\mathrm{div}\;\vec{u}_\delta]\,\mathrm{d}x\,\mathrm{d}t-$$

$$-\int_0^T\psi\int_\Omega\left(\partial_{x_j}\phi\right)\varrho_\delta u^i_\delta u^j_\delta\mathcal{A}_i[T_k(\varrho_\delta)]\,\mathrm{d}x\,\mathrm{d}t+$$

$$+\int_0^T\psi\int_\Omega u^i_\delta\Big(T_k(\varrho_\delta)\mathcal{R}_{i,j}[\phi\varrho_\delta u^j_\delta]-\phi\varrho_\delta u^j_\delta\mathcal{R}_{i,j}[T_k(\varrho_\delta)]\Big)\,\mathrm{d}x\,\mathrm{d}t.$$

Analogously, we can repeat the same procedure as above with the limit equation (6.23) making use of the test functions

$$\varphi_i(t,x)=\psi(t)\phi(x)\mathcal{A}_i[\overline{T_k(\varrho)}],\quad i=1,2,\ \text{ where }\psi\in D(0,T),\ \phi\in D(\Omega),$$

ϱ being set zero outside Ω.

Similarly as in (6.29), we can deduce (using also (6.28)):

$$\int_0^T\psi\int_\Omega\phi\Big(\overline{a\varrho\log^d(1+\varrho)}-(\lambda+2\mu)\mathrm{div}\;\vec{u}\Big)\overline{T_k(\varrho)}\,\mathrm{d}x\,\mathrm{d}t= \tag{6.30}$$

$$=\int_0^T\psi\int_\Omega\partial_{x_i}\phi\Big((\lambda+\mu)\mathrm{div}\;(\vec{u})-\overline{a\varrho\log^d(1+\varrho)}\Big)\mathcal{A}_i[\overline{T_k(\varrho)}]\,\mathrm{d}x\,\mathrm{d}t+$$

$$+\mu\int_0^T\psi\int_\Omega\left(\partial_{x_j}\phi\right)u^j\overline{T_k(\varrho)}\,\mathrm{d}x\,\mathrm{d}t-\mu\int_0^T\psi\int_\Omega\left(\partial_{x_j}\phi\right)u^i\mathcal{R}_{i,j}[\overline{T_k(\varrho)}]\,\mathrm{d}x\,\mathrm{d}t+$$

$$+\mu\int_0^T\psi\int_\Omega\left(\partial_{x_j}\phi\right)\partial_{x_j}u^i\mathcal{A}_i[\overline{T_k(\varrho)}]\,\mathrm{d}x\,\mathrm{d}t-\int_0^T\psi_t\int_\Omega\phi\varrho u^i\mathcal{A}_i[\overline{T_k(\varrho)}]\,\mathrm{d}x\,\mathrm{d}t+$$

$$+\int_0^T\psi\int_\Omega\phi\varrho u^i\mathcal{A}_i[\overline{(T'_k(\varrho)\varrho-T_k(\varrho))\mathrm{div}\;\vec{u}}]\,\mathrm{d}x\,\mathrm{d}t-$$

$$-\int_0^T\psi\int_\Omega\left(\partial_{x_j}\phi\right)\varrho u^iu^j\mathcal{A}_i[\overline{T_k(\varrho)}]\,\mathrm{d}x\,\mathrm{d}t+$$

$$+\int_0^T\psi\int_\Omega u^i\Big(\overline{T_k(\varrho)}\mathcal{R}_{i,j}[\phi\varrho u^j]-\phi\varrho u^j\mathcal{R}_{i,j}[\overline{T_k(\varrho)}]\Big)\,\mathrm{d}x\,\mathrm{d}t.$$

Similarly as in the proof of Lemma 5.3, it is sufficient for the proof of Lemma 6.2 to show that all the terms on the right-hand side of (6.29) converge to their counterparts in (6.30).

Now, (5.16), (5.19), (6.27) and (6.28) imply

$$\mathcal{A}_i[T_k(\varrho_\delta)] \to \mathcal{A}_i[\overline{T_k(\varrho)}] \text{ in } C(\overline{(0,T)\times\Omega}), \tag{6.31}$$

$$\mathcal{R}_{i,j}[T_k(\varrho_\delta)] \to \mathcal{R}_{i,j}[\overline{T_k(\varrho)}] \text{ in } C([0,T];\mathrm{L}^{*-weak}_{e(\gamma)}(\Omega)) \ \text{ for } \gamma \in (0,1) \tag{6.32}$$

and

$$\mathcal{A}_i[(T_k'(\varrho_\delta)\varrho_\delta - T_k(\varrho_\delta))\mathrm{div}\,\vec{u}_\delta] \to \mathcal{A}_i[\overline{(T_k'(\varrho)\varrho - T_k(\varrho))\mathrm{div}\,\vec{u}}] \\ \text{weakly in } \mathrm{L}^2(0,T;\mathrm{W}^{1,2}(\Omega)). \tag{6.33}$$

With the relations (6.16) – (6.21), (6.31) – (6.33) in mind, it is easy to see that it is enough to show

$$\int_0^T \psi \int_\Omega u^i_\delta \left(T_k(\varrho_\delta)\mathcal{R}_{i,j}[\phi\varrho_\delta u^j_\delta] - \phi\varrho_\delta u^j_\delta \mathcal{R}_{i,j}[T_k(\varrho_\delta)]\right) \mathrm{d}x\,\mathrm{d}t \ \to \tag{6.34}$$

$$\to \int_0^T \psi \int_\Omega u^i \left(\overline{T_k(\varrho)}\mathcal{R}_{i,j}[\phi\varrho u^j] - \phi\varrho u^j \mathcal{R}_{i,j}[\overline{T_k(\varrho)}]\right) \mathrm{d}x\,\mathrm{d}t.$$

Lemma 3.12, together with (6.18) and (6.28), gives

$$T_k(\varrho_\delta)\mathcal{R}_{i,j}[\phi\varrho_\delta u^j_\delta] - \phi\varrho_\delta u^j_\delta\mathcal{R}_{i,j}[T_k(\varrho_\delta)] \ \to \ \overline{T_k(\varrho)}\mathcal{R}_{i,j}[\phi\varrho u^j] - \phi\varrho u^j\mathcal{R}_{i,j}[\overline{T_k(\varrho)}]$$

in $C([0,T];\mathrm{W}^{-1,2}(\Omega))$ which, combined with (6.17), implies (6.34) and the proof of Lemma 6.2 is finished.

Q.E.D.

6.4 The amplitude of oscillations

The main result of this part is following (compare with [2], Lemma 4.3):

Lemma 6.3: *There exists a constant c independent of k such that*

$$\limsup_{\delta\to 0+} \|T_k(\varrho_\delta) - T_k(\varrho)\|_{y^2\log^d(y)} \le c$$

where $\|\cdot\|_{y^2\log^d(y)}$ is the Luxembourg norm in the Orlicz space $\mathrm{L}_\phi((0,T)\times\Omega)$ generated by the Young function $\phi(y) = y^2\log^d(1+y)$.

Proof: By the convexity of functions $z \to z\log^d(1+z)$ and $z \to -T_k(z)$, we obtain

$$\limsup_{\delta\to 0+} \int_0^T \int_\Omega \varrho_\delta \log^d(1+\varrho_\delta) T_k(\varrho_\delta) - \overline{\varrho\log^d(1+\varrho)}\,\overline{T_k(\varrho)}\,\mathrm{d}x\,\mathrm{d}t = \tag{6.35}$$

$$= \limsup_{\delta\to 0+} \int_0^T \int_\Omega \left(\varrho_\delta\log^d(1+\varrho_\delta) - \varrho\log^d(1+\varrho)\right)(T_k(\varrho_\delta) - T_k(\varrho))\,\mathrm{d}x\,\mathrm{d}t \ +$$

$$+\int_0^T\int_\Omega \left(\overline{\varrho\log^d(1+\varrho)} - \varrho\log^d(1+\varrho)\right)(T_k(\varrho)-\overline{T_k(\varrho)})\,\mathrm{d}x\,\mathrm{d}t \geq$$

$$\geq \limsup_{\delta\to 0+}\int_0^T\int_\Omega \left(\varrho_\delta\log^d(1+\varrho_\delta) - \varrho\log^d(1+\varrho)\right)(T_k(\varrho_\delta)-T_k(\varrho))\,\mathrm{d}x\,\mathrm{d}t.$$

By virtue of Lemma 3.13, we have

$$(z\log^d(1+z)-y\log^d(1+y))(T_k(z)-T_k(y)) \geq \frac{1}{2}|T_k(z)-T_k(y)|^2\log^d(1+|T_k(z)-T_k(y)|)$$

for all $z, y \geq 0$, consequently, (6.35) implies

$$1+2\limsup_{\delta\to 0+}\int_0^T\int_\Omega \varrho_\delta\log^d(1+\varrho_\delta)T_k(\varrho_\delta) - \overline{\varrho\log^d(1+\varrho)}\,\overline{T_k(\varrho)}\,\mathrm{d}x\,\mathrm{d}t \geq$$

$$\geq \limsup_{\delta\to 0+}\|T_k(\varrho_\delta)-T_k(\varrho)\|^2_{y^2\log^d(y)}. \tag{6.36}$$

On the other hand,

$$\limsup_{\delta\to 0+}\int_0^T\int_\Omega \mathrm{div}\ \vec{u}_\delta T_k(\varrho_\delta) - \mathrm{div}\ \vec{u}\,\overline{T_k(\varrho)}\,\mathrm{d}x\,\mathrm{d}t =$$

$$\limsup_{\delta\to 0+}\int_0^T\int_\Omega \left(T_k(\varrho_\delta)-T_k(\varrho)+T_k(\varrho)-\overline{T_k(\varrho)}\right)\mathrm{div}\ \vec{u}_\delta\,\mathrm{d}x\,\mathrm{d}t \leq$$

$$\leq 2\sup_\delta\|\mathrm{div}\ \vec{u}_\delta\|_{L^2((0,T)\times\Omega)}\limsup_{\delta\to 0+}\|T_k(\varrho_\delta)-T_k(\varrho)\|_{L^2((0,T)\times\Omega)} \leq$$

$$\leq c\limsup_{\delta\to 0+}\|T_k(\varrho_\delta)-T_k(\varrho)\|_{y^2\log^d(y)} \leq c+\frac{1}{2}\limsup_{\delta\to 0+}\|T_k(\varrho_\delta)-T_k(\varrho)\|^2_{y^2\log^d(y)} \tag{6.37}$$

where we used also Lemma 3.5 and the Young inequality in the final step.

The relations (6.36), (6.37) combined with Lemma 6.2 yield the desired conclusion.

Q.E.D.

6.5 The renormalized solutions

We are going to use the conclusion of Lemma 6.3 to show the following result (compare [2], Lemma 4.4):

Lemma 6.4: *The limit functions ϱ, $\vec{u}$ solve (6.22) in the sense of renormalized solution, i.e.,*

$$\partial_t b(\varrho) + \mathrm{div}\ (b(\varrho)\vec{u}) + (b'(\varrho)\varrho - b(\varrho))\mathrm{div}\ \vec{u} = 0 \tag{6.38}$$

holds in $D'((0,T)\times\mathbb{R}^2)$ provided ϱ, $\vec{u}$ are set zero outside Ω for any $b\in C^1(\mathbb{R})$ satisfying

$$b'(z)\equiv 0 \ \text{ for all } z\in\mathbb{R} \text{ large enough, say, } z\geq M \tag{6.39}$$

where the constant M may vary for different functions b.

Proof: As in (5.26), we can regularize equation (6.26) to obtain

$$\partial_t S_m[\overline{T_k(\varrho)}] + \operatorname{div}\,(S_m[\overline{T_k(\varrho)}]\vec{u}) + S_m[\overline{(T_k'(\varrho)\varrho - T_k(\varrho))\operatorname{div}\vec{u}}] = r_m \tag{6.40}$$

where $S_m[v]=\vartheta_m * v$ are the standard smoothing operators and $r_m\to 0$ in $\mathrm{L}^2((0,T)\times\mathbb{R}^2)$ as $m\to\infty$ for any fixed k.

Now, we are allowed to multiply (6.40) by $b'(S_m[\overline{T_k(\varrho)}])$. Letting $m\to\infty$ we deduce

$$\partial_t b(\overline{T_k(\varrho)}) + \operatorname{div}\,\Big(b(\overline{T_k(\varrho)})\vec{u}\Big) + \Big(b'(\overline{T_k(\varrho)})\overline{T_k(\varrho)} - b(\overline{T_k(\varrho)})\Big)\operatorname{div}\vec{u} = \tag{6.41}$$

$$= b'(\overline{T_k(\varrho)})\left[\overline{(T_k(\varrho)-T_k'(\varrho)\varrho)\operatorname{div}\vec{u}}\right] \text{ in } D'((0,T)\times\mathbb{R}^2).$$

Our next goal will be passing $k\to\infty$ in (6.41).

By virtue of Lemma 3.3, we have

$$\|\overline{T_k(\varrho)}-\varrho\|_{y\log^\gamma(y)} \leq \liminf_{\delta\to 0+}\|T_k(\varrho_\delta)-\varrho_\delta\|_{y\log^\gamma(y)} \leq$$

$$\leq \liminf_{\delta\to 0+}\|\chi_{[\varrho_\delta\geq k]}\left(T_k(\varrho_\delta)-\varrho_\delta\right)\|_{y\log^\gamma(y)} \leq$$

$$\leq \liminf_{\delta\to 0+}\|\chi_{[\varrho_\delta\geq k]}\|_{e\left(\frac{1}{d+1-\gamma}\right)}\|T_k(\varrho_\delta)-\varrho_\delta\|_{y\log^{d+1}(y)} \text{ for } \gamma\in(0,d+1). \tag{6.42}$$

As ϱ_δ are bounded in $\mathrm{L}_{y\log^{d+1}(y)}(\Omega)$ uniformly in δ, we see that there exists a function $r:(0,\infty)\to(0,\infty)$ such that

$$\lim_{k\to\infty} r(k)=0, \quad \text{and} \quad \mu\big\{(x,t)\in(0,T)\times\Omega \,|\, \varrho_\delta(x,t)\geq k\big\}\leq r(k), \ \text{ for } \delta\in(0,1). \tag{6.43}$$

By virtue of the definition of the Luxembourg norm and using (6.43), we easily observe that for any $\omega>0$ it holds

$$\|\chi_{[\varrho_\delta\geq k]}\|_{e\left(\frac{1}{\omega}\right)} \leq \left(\phi^{-1}\left(\frac{1}{r(k)}\right)\right)^{-1} = q_\omega(k), \qquad \lim_{k\to\infty} q_\omega(k)=0, \tag{6.44}$$

where ϕ is the Young function corresponding to the space $\mathrm{L}_{e\left(\frac{1}{\omega}\right)}((0,T)\times\Omega)$.

The relations (6.42) and (6.44) (for $\omega=d+1-\gamma>0$) imply that

$$\overline{T_k(\varrho)}\to\varrho \text{ as } k\to\infty \text{ in } \mathrm{L}_{y\log^\gamma(y)}((0,T)\times\Omega) \text{ for any } \gamma\in(0,d+1).$$

Thus (6.41) will imply (6.38) provided we show

$$b'(\overline{T_k(\varrho)})\left[\overline{(T_k(\varrho)-T_k'(\varrho)\varrho)\operatorname{div}\vec{u}}\right]\to 0 \text{ in } \mathrm{L}^1((0,T)\times\Omega) \text{ as } k\to\infty. \tag{6.45}$$

To this end, let us denote

$$J_{k,M} = \left\{(t,x) \in (0,T) \times \Omega \,|\, \overline{T_k(\varrho)} \leq M\right\},$$

where M is the constant related to b by (6.39). One has

$$\int_0^T \int_\Omega \left| b'(\overline{T_k(\varrho)}) \left[\overline{(T_k(\varrho) - T_k'(\varrho)\varrho) \operatorname{div} \vec{u}}\right] \right| \, \mathrm{d}x \, \mathrm{d}t \leq$$

$$\leq \sup_{0 \leq z \leq M} |b'(z)| \iint_{J_{k,M}} \left| \overline{(T_k(\varrho) - T_k'(\varrho)\varrho) \operatorname{div} \vec{u}} \right| \, \mathrm{d}x \, \mathrm{d}t \leq$$

$$\leq \sup_{0 \leq z \leq M} |b'(z)| \liminf_{\delta \to 0+} \| (T_k(\varrho_\delta) - T_k'(\varrho_\delta)\varrho_\delta) \operatorname{div} \vec{u}_\delta \|_{L^1(J_{k,M})} \leq$$

$$\leq \sup_{0 \leq z \leq M} |b'(z)| \sup_\delta \|\vec{u}_\delta\|_{L^2(0,T;W^{1,2}(\Omega))} \liminf_{\delta \to 0+} \|T_k(\varrho_\delta) - T_k'(\varrho_\delta)\varrho_\delta\|_{L^2(J_{k,M})} \leq$$

$$\leq c \liminf_{\delta \to 0+} \|T_k(\varrho_\delta) - T_k'(\varrho_\delta)\varrho_\delta\|_{L^2(J_{k,M})}. \tag{6.46}$$

Now, using Lemma 3.5 and (6.44) we see

$$\|T_k(\varrho_\delta) - T_k'(\varrho_\delta)\varrho_\delta\|_{L^2(J_{k,M})} = \|\chi_{[\varrho_\delta \geq k]} \left(T_k(\varrho_\delta) - T_k'(\varrho_\delta)\varrho_\delta\right)\|_{L^2(J_{k,M})} \leq$$

$$\leq \|\chi_{[\varrho_\delta \geq k]}\|_{e\left(\frac{2}{d}+1\right)} \|T_k(\varrho_\delta) - T_k'(\varrho_\delta)\varrho_\delta\|_{L_{y^2 \log^d(y)}(J_{k,M})} \leq$$

$$\leq q_{\frac{d}{2+d}}(k) \|T_k(\varrho_\delta) - T_k'(\varrho_\delta)\varrho_\delta\|_{L_{y^2 \log^d(y)}(J_{k,M})}, \tag{6.47}$$

and, since $T_k'(z)z \leq T_k(z)$,

$$\|T_k(\varrho_\delta) - T_k'(\varrho_\delta)\varrho_\delta\|_{L_{y^2 \log^d(y)}(J_{k,M})} \leq 2\|T_k(\varrho_\delta)\|_{L_{y^2 \log^d(y)}(J_{k,M})} \leq$$

$$\leq 2\|T_k(\varrho_\delta) - T_k(\varrho)\|_{y^2 \log^d(y)} + 2\|T_k(\varrho) - \overline{T_k(\varrho)}\|_{y^2 \log^d(y)} + 2\|\overline{T_k(\varrho)}\|_{L_{y^2 \log^d(y)}(J_{k,M})} \leq$$

$$\leq 2\|T_k(\varrho_\delta) - T_k(\varrho)\|_{y^2 \log^d y} + 2\|T_k(\varrho) - \overline{T_k(\varrho)}\|_{y^2 \log^d(y)} + 2M\mu(\Omega), \tag{6.48}$$

and, by virtue of Lemma 6.3 and (6.48), one gets

$$\limsup_{\delta \to 0+} \|T_k(\varrho_\delta) - T_k'(\varrho_\delta)\varrho_\delta\|_{L_{y^2 \log^d(y)}(J_{k,M})} \leq 4c + 2M\mu(\Omega)$$

which, together with (6.46), (6.47) and (6.44), completes the proof of (6.45). Thus, Lemma 6.4 has been proved.

Q.E.D.

6.6 Strong convergence of the density

We are going to complete the proof of Theorem 1. To this end, we introduce a family of functions $L_k \in C^1(\mathbb{R})$:

$$L_k(z) = \begin{cases} z\log(z) & \text{for } 0 \le z < k, \\ z\log(k) + z\int_k^z \frac{T_k(s)}{s^2}\,\mathrm{d}s & \text{for } z \ge k. \end{cases}$$

Seeing that L_k can be written as

$$L_k(z) = \beta_k z + b_k(z) \tag{6.49}$$

where b_k satisfy (6.39), we can use the fact that ϱ_δ, $\vec{u}_\delta$ are renormalized solutions of (5.32) to deduce

$$\partial_t L_k(\varrho_\delta) + \mathrm{div}\,(L_k(\varrho_\delta)\vec{u}_\delta) + T_k(\varrho_\delta)\mathrm{div}\,\vec{u}_\delta = 0 \quad \text{in } D'((0,T)\times\mathbb{R}^2). \tag{6.50}$$

Similarly, by virtue of Lemma 6.4 and (6.22),

$$\partial_t L_k(\varrho) + \mathrm{div}\,(L_k(\varrho)\vec{u}) + T_k(\varrho)\mathrm{div}\,\vec{u} = 0 \quad \text{in } D'((0,T)\times\mathbb{R}^2). \tag{6.51}$$

In view of (6.50), (5.36) and Lemma 3.8, we can assume

$$L_k(\varrho_\delta) \to \overline{L_k(\varrho)} \text{ in } C([0,T];\mathrm{L}^{*-weak}_{y\log^{d+1}(y)}(\Omega)) \quad \text{and} \quad \text{in } C([0,T];\mathrm{W}^{-1,2}(\Omega)). \tag{6.52}$$

Taking the difference of (6.50) and (6.51) and integrating with respect to t, we get

$$\int_\Omega (L_k(\varrho_\delta) - L_k(\varrho))(t)\phi\,\mathrm{d}x = \int_\Omega (L_k(\varrho_{0,\delta}) - L_k(\varrho_0))\phi\,\mathrm{d}x +$$

$$+ \int_0^t \int_\Omega (L_k(\varrho_\delta)\vec{u}_\delta - L_k(\varrho)\vec{u})\cdot\nabla\phi + (T_k(\varrho)\mathrm{div}\,\vec{u} - T_k(\varrho_\delta)\mathrm{div}\,\vec{u}_\delta)\phi\,\mathrm{d}x\,\mathrm{d}t$$

for any $\phi \in D(\Omega)$. Passing to the limit for $\delta \to 0$ and making use of (6.2), (6.17) and (6.52), one obtains

$$\int_\Omega (\overline{L_k(\varrho)} - L_k(\varrho))(t)\phi\,\mathrm{d}x = \tag{6.53}$$

$$= \int_0^t \int_\Omega \left(\overline{L_k(\varrho)} - L_k(\varrho)\right)\vec{u}\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}t + \lim_{\delta\to 0+}\int_0^t \int_\Omega (T_k(\varrho)\mathrm{div}\,\vec{u} - T_k(\varrho_\delta)\mathrm{div}\,\vec{u}_\delta)\phi\,\mathrm{d}x\,\mathrm{d}t$$

for any $\phi \in D(\Omega)$. As the velocity components u^i, $i = 1,2$ belong to $\mathrm{L}^2(0,T;\mathrm{W}^{1,2}_0(\Omega))$, it holds

$$\frac{|\vec{u}|}{\mathrm{dist}[x,\partial\Omega]} \in \mathrm{L}^2(0,T;\mathrm{L}^2(\Omega)). \tag{6.54}$$

Let us consider a sequence of functions $\phi_m \in D(\Omega)$ such that

$$0 \le \phi_m \le 1,\ \phi_m(x) = 1 \ \text{ for all } x \text{ such that } \mathrm{dist}[x,\partial\Omega] \ge \frac{1}{m} \quad \text{and}$$

$$|\nabla \phi_m(x)| \le 2m \quad \text{for all } x \in \Omega.$$

Taking functions ϕ_m as test functions in (6.53), passing to the limit for $m \to \infty$ and making use of (6.54), one derives

$$\int_\Omega (\overline{L_k(\varrho)} - L_k(\varrho))(t)\,\mathrm{d}x = \int_0^t \int_\Omega T_k(\varrho) \mathrm{div}\, \vec{u}\,\mathrm{d}x\,\mathrm{d}t - \lim_{\delta\to 0+} \int_0^t \int_\Omega T_k(\varrho_\delta) \mathrm{div}\, \vec{u}_\delta\,\mathrm{d}x\,\mathrm{d}t. \tag{6.55}$$

Observe that the term $\overline{L_k(\varrho)} - L_k(\varrho)$ is bounded in view of (6.49).

At this stage, the main idea is to let $k \to \infty$ in (6.55). By virtue of (5.36), we can assume

$$\varrho_\delta \log(\varrho_\delta) \to \overline{\varrho \log(\varrho)} \ \text{ weakly star in } \mathrm{L}^\infty(0,T;\mathrm{L}^{*-weak}_{y\log^d(y)}(\Omega)).$$

We also have

$$\overline{L_k(\varrho)} \to \overline{\varrho \log(\varrho)} \text{ in } \mathrm{L}^\infty(0,T;\mathrm{L}_{y\log^\gamma(y)}(\Omega)) \quad \text{for any } \gamma \in (1,d) \tag{6.56}$$

since, making of use Lemma 3.3, (6.44) and (5.36),

$$\|\overline{L_k(\varrho)} - \overline{\varrho\log(\varrho)}\|_{L^\infty(0,T;L_{y\log^\gamma(y)}(\Omega))} \le \liminf_{\delta\to 0+} \sup_{t\in[0,T]} \|L_k(\varrho_\delta) - \varrho_\delta \log(\varrho_\delta)\|_{y\log^\gamma(y)} \le$$

$$\le c\, q_{d-\gamma}(k) \sup_\delta \sup_{t\in[0,T]} \|\varrho_\delta(t)\|_{y\log^{d+1}(y)} \to 0 \text{ as } k \to \infty.$$

Similarly, we obtain

$$L_k(\varrho) \to \varrho \log(\varrho) \text{ in } \mathrm{L}^\infty(0,T;\mathrm{L}_{y\log^\gamma(y)}(\Omega)) \quad \text{for any } \gamma \in (1,d). \tag{6.57}$$

Finally, by virtue of Lemma 6.2 and the monotonicity of the pressure (cf. (6.35) and (6.36)), we can estimate the right hand side of (6.55):

$$\int_0^t \int_\Omega T_k(\varrho) \mathrm{div}\, \vec{u}\,\mathrm{d}x\,\mathrm{d}t - \lim_{\delta\to 0+} \int_0^t \int_\Omega T_k(\varrho_\delta) \mathrm{div}\, \vec{u}_\delta\,\mathrm{d}x\,\mathrm{d}t \le$$

$$\le \int_0^t \int_\Omega (T_k(\varrho) - \overline{T_k(\varrho)}) \mathrm{div}\, \vec{u}\,\mathrm{d}x\,\mathrm{d}t. \tag{6.58}$$

By virtue of Lemma 6.3, Lemma 3.3 and (6.44), the right-hand side of (6.58) tends to zero as $k \to \infty$. Now, we can pass to the limit for $k \to \infty$ in (6.55) to conclude

$$\int_\Omega \overline{\varrho\log(\varrho)} - \varrho\log(\varrho)\,\mathrm{d}x\,(t) = 0 \ \text{ for a.e. } t \in [0,T]. \tag{6.59}$$

Because of the convexity of the function $z \to z\log z$, we have

$$\overline{\varrho\log(\varrho)} \ge \varrho\log(\varrho) \ \text{ a.e. in } (0,T)\times\Omega$$

which, combined with (6.59), gives

$$\overline{\varrho\log(\varrho)} = \varrho\log(\varrho) \quad \text{a.e. in } (0,T)\times\Omega. \tag{6.60}$$

By virtue of (5.36), we can assume

$$\left(\frac{\varrho+\varrho_\delta}{2}\right)\log\left(\frac{\varrho+\varrho_\delta}{2}\right) \to w \quad \text{weakly star in } \mathrm{L}^\infty(0,T;\mathrm{L}_{y\log^d(y)}(\Omega))$$

where, in view of convexity, $w \geq \varrho\log\varrho$. Thus, using convexity and (6.60),

$$0 \leq h_\delta = \varrho_\delta\log(\varrho_\delta) + \varrho\log(\varrho) - \left(\frac{\varrho+\varrho_\delta}{2}\right)\log\left(\frac{\varrho+\varrho_\delta}{2}\right) \to \varrho\log(\varrho) - w$$

weakly star in $\mathrm{L}^\infty(0,T;\mathrm{L}_{y\log^d(y)}(\Omega))$. As $\varrho\log(\varrho) - w \leq 0$, we have weak star convergence $h_\delta \to 0$ which together with $h_\delta \geq 0$ yields even strong convergence $h_\delta \to 0$ in $\mathrm{L}^1((0,T)\times\Omega)$. Consequently

$$h_\delta = \varrho_\delta\log(\varrho_\delta) + \varrho\log(\varrho) - \left(\frac{\varrho+\varrho_\delta}{2}\right)\log\left(\frac{\varrho+\varrho_\delta}{2}\right) \to 0 \quad \text{in measure.}$$

Now, we are able to prove that even the sequence ϱ_δ converge in measure to ϱ. To this end, let us fix $\sigma > 0$. Then

$$\mu\{(t,x)\in(0,T)\times\Omega \,:\, |\varrho_\delta(x,t)-\varrho(x,t)| \geq \sigma\} \leq$$

$$\leq \tilde{r}(k) + \mu\Big\{(t,x)\in(0,T)\times\Omega \,:\, \varrho_\delta \leq k \;\&\; \varrho \leq k \;\&\; |\varrho_\delta(x,t)-\varrho(x,t)| \geq \sigma\Big\} \tag{6.61}$$

where $\tilde{r}(k)$ tends to zero for $k\to\infty$ independently of δ (cf. (6.43)). The second term on the right-hand side of (6.61) tends to zero for $\delta\to 0$ and fixed k since $h_\delta \to 0$ in measure and

$$h_\delta \geq k|\varrho-\varrho_\delta|^2 \quad \text{for } \varrho\leq k \text{ and } \varrho_\delta \leq k,$$

consequently, the sequence ϱ_δ tends to ϱ in measure.

The convergence in measure and the convergence of L^1-norms, in particular

$$\|\varrho_\delta\|_{L^1((0,T)\times\Omega)} = \int_\Omega \varrho_\delta\,\mathrm{d}t \to \int_\Omega \varrho\,\mathrm{d}t = \|\varrho\|_{L^1((0,T)\times\Omega)}, \tag{6.62}$$

imply strong convergence of the sequence ϱ_δ in $L^1((0,T)\times\Omega)$ and (6.24) easily follows.

Thus Theorem 1 has been proved.

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Radek Erban, Mathematical Institute, Academy of Sciences of the Czech Republic, Žitná 25, 115 67, Praha 1, Czech Republic

present address: Radek Erban, School of Mathematics, University of Minnesota, 127 Vincent Hall, 206 Church St. SE, Minneapolis, MN 55455, USA;

e-mail: erban@math.umn.edu