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On the existence of solutions to the Navier-Stokes equations of a two-dimensional compressible flow

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Summary: We consider the Navier-Stokes equations for compressible, barotropic flow in two space dimensions. We introduce useful tools from the theory of Orlicz spaces. Then we prove the existence of globally defined finite energy weak solutions for the pressure satisfying $p(\varrho) = a\varrho \log^d(1+\varrho)$ for large ϱ. Here $d > 1$ and $a > 0$. This result fills the gap between the isentropic flow (where the existence of solutions was already established) and the isothermic flow (where the existence is still an open problem).

1 Introduction

The Navier-Stokes equations for compressible, barotropic flow in two space dimensions can be written in the form:

$$\frac{\partial \varrho}{\partial t} + \text{div } (\varrho \vec{u}) = 0, \tag{1.1}$$

$$\frac{\partial \varrho \vec{u}}{\partial t} + \text{div } (\varrho \vec{u} \otimes \vec{u}) + \nabla p(\varrho) = \mu \Delta \vec{u} + (\lambda + \mu) \nabla (\text{div } \vec{u}) + \varrho \vec{f}, \tag{1.2}$$

where the density $\varrho = \varrho(t,x)$ and the velocity $\vec{u} = [u^1(t,x), u^2(t,x)]$ are functions of the time $t \in (0,T)$ and the spatial coordinate $x \in \Omega$ where $\Omega \subset \mathbb{R}^2$ is a bounded regular domain, $\vec{f} = [f^1(t,x), f^2(t,x)]$ is a given external force and $p(\varrho)$ is the pressure. The viscosity coefficients μ and λ satisfy

$$\mu > 0, \quad \lambda + \mu \geq 0.$$

We prescribe the initial conditions for the density and the momentum:

$$\varrho(0) = \varrho_0, \quad (\varrho u^i)(0) = q^i, \quad i = 1, 2; \tag{1.3}$$

together with the no-slip boundary conditions for the velocity:

$$u^i|_{\partial\Omega} = 0, \quad i = 1, 2. \tag{1.4}$$

The problem (1.1) – (1.4) was studied by many authors (e.g. [6], [2] in 3D, [9] in 2D where λ is supposed to be a power of ρ etc.). One can find in [6] the global existence result for the pressure function $p \in \mathrm{C}^1[0,\infty)$ satisfying the condition (in 2D)

$$\liminf_{z \to \infty} \frac{p(z)}{z^\gamma} > 0 \quad \text{for} \quad \gamma \geq \frac{3}{2}.$$

This result includes the case of isentropic fluid where $p(z) = az^\gamma$, $a > 0$, for $\gamma \geq 3/2$. If we use the technique introduced in [2] in our two-dimensional case, we will obtain the global existence result for the pressure of the form $p(z) = az^\gamma$, $a > 0$, where $\gamma > 1$. Thus, we can cover all physical interesting cases of isentropic fluid in two space dimensions. However, the case of isothermal flow where $p(z) = az$, $a > 0$, remains still open.

The motivation of this article is to fill the gap in global existence results between isothermal and isentropic fluid, in particular, we shall assume that

$$\left\{ \begin{array}{l} p \in \mathrm{C}^1[0,\infty),\ p \text{ is convex, non-decreasing and there exist numbers} \\ d > 1, a > 0, z_0 > 1 \text{ such that} \quad p(z) = az\log^d(z), \quad \text{for all } z \geq z_0, \end{array} \right\} \tag{1.5}$$

If we formally multiply (1.6) by $\vec{u}$ and integrate by parts, we obtain the energy inequality (using also (1.1))

$$E(t) + \int_0^t \int_\Omega \mu \left|\nabla \vec{u}\right|^2 + (\lambda+\mu) \left|\mathrm{div}\ \vec{u}\right|^2 \ \mathrm{d}x \leq E(0) \tag{1.7}$$

where

$$E(t) = E[\varrho, \vec{u}](t) = \int_\Omega \frac{1}{2} \varrho \left|\vec{u}\right|^2 \ \mathrm{d}x + \int_\Omega \int_{z_0}^{\varrho} \int_{z_0}^{\sigma} \frac{p'(s)}{s} \,\mathrm{d}s\,\mathrm{d}\sigma\,\mathrm{d}x \tag{1.8}$$

and

$$E(0) = \int\limits_{\{x\in\Omega:\, \varrho_0(x)>0\}} \frac{1}{2} \frac{\left|\vec{q}\right|^2}{\varrho_0} \,\mathrm{d}x + \int_\Omega \int_{z_0}^{\varrho_0} \int_{z_0}^{\sigma} \frac{p'(s)}{s} \,\mathrm{d}s\,\mathrm{d}\sigma\,\mathrm{d}x.$$

If $\varrho \geq z_0$ then we can compute

$$\int_{z_0}^{\varrho} \int_{z_0}^{\sigma} \frac{p'(s)}{s} \,\mathrm{d}s\,\mathrm{d}\sigma = \int_{z_0}^{\varrho} (\varrho - \sigma) \frac{p'(\sigma)}{\sigma} \,\mathrm{d}\sigma = \varrho \int_{z_0}^{\varrho} \frac{p'(\sigma)}{\sigma} \,\mathrm{d}\sigma - p(\varrho) + p(z_0) =$$

$$= \varrho \int_{z_0}^{\varrho} \frac{ad \log^{d-1}(\sigma) + a \log^d(\sigma)}{\sigma} \,\mathrm{d}\sigma - a\varrho \log^d(\varrho) + a z_0 \log^d(z_0) =$$

$$= \frac{a}{d+1} \varrho \log^{d+1}(\varrho) - \varrho \left(\frac{a}{d+1} \log^{d+1}(z_0) + a \log^d(z_0) \right) + a z_0 \log^d(z_0).$$

Motivated by this relation, we introduce the concept of finite energy weak solutions (see [2] and [6]) of the problem (1.1), (1.6), (1.4) (let us note that the definition of the function space $\mathrm{L}_{y \log^{d+1}(y)}(\Omega)$ can be found in the next section).

Definition: We shall say that ϱ, $\vec{u}$ is finite energy weak solution of the problem (1.1), (1.6) and (1.4) if the following four conditions are satisfied:

- $\varrho \geq 0$, $\varrho \in \mathrm{L}^\infty(0,T;\mathrm{L}_{y \log^{d+1}(y)}(\Omega))$, $u^i \in \mathrm{L}^2(0,T;\mathrm{W}_0^{1,2}(\Omega))$, $i = 1, 2$;
- the energy (1.8) satisfies the energy inequality (1.7) for a.a. $t \in (0,T)$;
- the equations (1.1), (1.6) are satisfied in $D'((0,T)\times\Omega)$; moreover, (1.1) holds in $D'((0,T)\times\mathbb{R}^2)$ provided ϱ, $\vec{u}$ were prolonged to be zero on $\mathbb{R}^2 - \Omega$;
- the equation (1.1) is satisfied in the sense of renormalized solutions, it means that

$$b(\varrho)_t + \mathrm{div}\ (b(\varrho)\vec{u}) + (b'(\varrho)\varrho - b(\varrho))\mathrm{div}\ \vec{u} = 0 \tag{1.9}$$

holds in $D'((0,T)\times\Omega)$ for any $b \in \mathrm{C}^1(\mathbb{R})$ such that

In the following, we shall introduce that the initial data ϱ_0, q^i, $i = 1, 2$, satisfy compatibility conditions of the form:

$$\varrho_0 \in \mathrm{L}_{y \log^{d+1}(y)}(\Omega),\ \varrho_0 \geq 0,\ q^i(x) = 0 \text{ whenever } \varrho_0(x) = 0,\ \frac{|q^i|^2}{\varrho_0} \in \mathrm{L}^1(\Omega),\ i = 1, 2. \tag{1.10}$$

Our main result reads as follows:

Theorem 1: *Assume* $\Omega \subset \mathbb{R}^2$ *is a bounded domain of the class* $\mathrm{C}^{2+\upsilon}$, $\upsilon > 0$. *Let the data* ϱ_0, q^i *satisfy the compatibility conditions* (1.10) *and let the pressure* p *satisfies* (1.5).

Then given $T > 0$ *arbitrary, there exists a finite energy weak solution* ϱ, $\vec{u}$ *of the problem* (1.1), (1.6), (1.4) *satisfying the initial conditions* (1.3).

Remark: As we shall see, the definition of the finite energy weak solutions implies

$$\varrho \in C([0, T]; \mathrm{L}^{*-weak}_{y \log^{d+1}(y)}(\Omega)) \text{ and } \varrho\vec{u} \in C([0, T]; \mathrm{L}^{*-weak}_{y \log^{(d+1)/2}(y)}(\Omega))$$

(let us note that the precise definitions of this functions spaces are given in the next sections), and, consequently, the initial conditions (1.3) make sense. Moreover, by virtue of (1.10), we see that $E(0) < \infty$.

At first, in the next two sections, we shall introduce useful tools for the proof of Theorem 1.

2 Excursion to the theory of Orlicz spaces

In this section, we recall several definitions and well known results concerning Orlicz spaces (see [4] for details). We shall assume that Ω is a bounded domain.

Let ϕ be a Young function. We will denote by $\mathrm{L}_\phi(\Omega)$ the Orlicz space corresponding to the Young function ϕ, it means: a measurable function u is in $\mathrm{L}_\phi(\Omega)$ if and only if the Luxembourg norm of u is finite, i. e.

$$\|u\|_\phi = \inf \left\{ k > 0\,;\ \int_\Omega \phi\left(\frac{|u(x)|}{k}\right)\, \mathrm{d}x \leq 1 \right\} < \infty.$$

Let us denote by $\mathrm{E}_\phi(\Omega)$ the closure of the set of all bounded measurable functions on Ω with respect to the Luxembourg norm $\|\cdot\|_\phi$. Then the space $\mathrm{E}_\phi(\Omega)$ is separable and $\mathrm{L}_\psi(\Omega) = (\mathrm{E}_\phi(\Omega))^*$, where ψ is the complementary function to ϕ.

Definition: Let $d > 0$. Let us denote by $\mathrm{L}_{y \log^d(y)}(\Omega)$ the Orlicz space generated by the

Remark: The space $\mathrm{E}_\phi(\Omega)$ corresponding to the space $\mathrm{L}_{y\log^d(y)}(\Omega)$, resp. $\mathrm{L}_{e(\beta)}(\Omega)$, will be denoted by $\mathrm{E}_{y\log^d(y)}(\Omega)$, resp $\mathrm{E}_{e(\beta)}(\Omega)$. It holds $\mathrm{L}_{y\log^d(y)}(\Omega) = \mathrm{E}_{y\log^d(y)}(\Omega)$,

$$\left(\mathrm{E}_{e(\beta)}(\Omega)\right)^* = \mathrm{L}_{y\log^{1/\beta}(y)}(\Omega) \quad \text{and} \quad \left(\mathrm{L}_{y\log^d(y)}(\Omega)\right)^* = \mathrm{L}_{e(1/d)}(\Omega).$$

Definition: Let $d \in (-\infty, \infty)$. Let us denote by $\mathrm{L}_{y^2\log^d(y)}(\Omega)$ the Orlicz space generated by the Young function ϕ satisfying for sufficiently large y, say, for $y \geq c_d$, the formula $\phi(y) = y^2\log^d(y)$.

Definition: Let ϕ be a Young function. Let us denote by $\mathrm{W\,L}_\phi(\Omega)$ and $\mathrm{W\,E}_\phi(\Omega)$ the set of all functions $u \in \mathrm{L}_\phi(\Omega)$, and $u \in \mathrm{E}_\phi(\Omega)$, respectively, such that all distributional derivatives $\partial_{x_i}u$, $i = 1, 2$, are elements of the spaces $\mathrm{L}_\phi(\Omega)$, and $\mathrm{E}_\phi(\Omega)$, respectively.

Remark: $\mathrm{W\,L}_\phi(\Omega)$ and $\mathrm{W\,E}_\phi(\Omega)$ are Banach spaces with norm $\|u\|_{WL_\phi} = \|u\|_\phi + \|\partial_{x_1}u\|_\phi + \|\partial_{x_2}u\|_\phi$. The space $\mathrm{W\,E}_\phi(\Omega)$ is separable. If Ω be a domain with Lipschitz boundary then

$$\overline{\mathrm{C}^\infty(\overline{\Omega})} = \mathrm{W\,E}_\phi(\Omega), \tag{2.1}$$

where the closure is taken with respect to the norm of the space $\mathrm{W\,E}_\phi(\Omega)$. The spaces $\mathrm{W\,L}_\phi(\Omega)$ and $\mathrm{W\,E}_\phi(\Omega)$ are called Sobolev-Orlicz spaces (see [4], chapter 7).

3 Auxiliary results

In this section, we shall prove several useful lemmas.

Lemma 3.1: *Let Ω be a bounded domain, let b be a mapping from $\mathbb{R}^2$ to $\mathbb{R}$ and*

$$|\xi|^{|\alpha|}|D^\alpha b(\xi)| \leq C < \infty \quad (\textit{for } |\alpha| \leq 2,\ \alpha \textit{ is multiindex}).$$

Then for $d > 0$ there exists a constant D_d such that for all $g \in \mathrm{L}_{y\log^{(d+1)}(y)}(\Omega)$

$$\|\left(\mathcal{F}^{-1}b\right) * g\|_{y\log^d(y)} \leq D_d\|g\|_{y\log^{(d+1)}(y)}, \tag{3.1}$$

where g is prolonged to be 0 on $\mathbb{R}^2 - \Omega$, $\mathcal{F}$ denotes the Fourier transform and $*$ denotes the convolution.

Proof: Lemma 3.1 has the same assumptions as the standard Mikhlin multiplier theorem, see [1], Theorem 6.1.6. As in [1], we can show that $\mathcal{F}^{-1}b\,*$ is bounded mapping as

$$\mathcal{F}^{-1}b* : \mathrm{L}^2(\mathbb{R}^2) \to \mathrm{L}^2(\mathbb{R}^2), \qquad \mathcal{F}^{-1}b* : \mathrm{L}^1(\mathbb{R}^2) \to \mathrm{L}^{1\infty}(\mathbb{R}^2), \tag{3.2}$$

Q.E.D.

Lemma 3.2: *Let Ω be a bounded domain, let b be a mapping from $\mathbb{R}^2$ to $\mathbb{R}$ and*

$$|\xi|^{|\alpha|}|D^\alpha b(\xi)| \le C < \infty \quad (\textit{for } |\alpha| \le 2,\ \alpha \textit{ is multiindex}).$$

Then for $\beta > 0$ there exists a constant D_β such that for all $g \in \mathrm{L}_{e(\beta)}(\Omega)$

$$\| \left(\mathcal{F}^{-1}b\right) * g\|_{e\left(\frac{\beta}{\beta+1}\right)} \le D_\beta \|g\|_{e(\beta)}, \tag{3.3}$$

where g is prolonged to be 0 on $\mathbb{R}^2 - \Omega$.

Proof: Let $g \in \mathrm{L}_{e(\beta)}(\Omega)$, $h \in \mathrm{L}_{y \log^{\frac{\beta+1}{\beta}}(y)}(\Omega)$ and let us extend g, h to be 0 on $\mathbb{R}^2 - \Omega$. Then, by virtue of the Lemma 3.1,

$$\left|\int_\Omega \left(\mathcal{F}^{-1}b * g\right) h \,\mathrm{d}x\right| = \left|\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \mathcal{F}^{-1}b(x-y)h(x)\,\mathrm{d}x\, g(y)\,\mathrm{d}y\right| \le D_\beta \|h\|_{y \log^{\frac{\beta+1}{\beta}}(y)} \|g\|_{e(\beta)},$$

which, combined with the definition of the Orlicz norm of the Orlicz space (see [4], Definition 3.6.1), gives (3.3).

Q.E.D.

Lemma 3.3: *Let Ω be a bounded domain, let ϕ, ϕ_1 and ϕ_2 be Young functions and let there exist numbers $c > 0$, $k \ge 0$, $y_0 \ge 0$, $z_0 \ge 0$ such that for all $y \ge y_0$, $z \ge z_0$*

$$\phi\left(\frac{y\,z}{c}\right) \le \phi_1(y) + \phi_2(z) + k,$$

then there is a number m such that for all measurable f_1 and f_2

$$\|f_1 f_2\|_\phi \le m\|f_1\|_{\phi_1}\|f_2\|_{\phi_2}.$$

Proof: See [7], Theorem 10.4.

Let Ω be a bounded domain. By virtue of Lemma 3.3, the following inequalities hold:

- *Let $d > 0$, $\beta > 0$ and $\gamma = d - \frac{1}{\beta} > 0$. Then there exists a constant a such that*

$$\|f_1 f_2\|_{y \log^\gamma(y)} \le a\|f_1\|_{y \log^d(y)}\|f_2\|_{e(\beta)}. \tag{3.4}$$

- *Let $d > 0$. Then there exists a constant a such that*

$$\|f_1 f_2\|_{y\log^d(y)} \leq a\|f_1\|_{y^2\log^{2d}(y)}\|f_2\|_{L^2}. \tag{3.6}$$

- *Let $d > 0$ and let $\beta > 2/d$. Then there exists a constant a such that*

$$\|f_1 f_2\|_{L^2} \leq a\|f_1\|_{y^2\log^d(y)}\|f_2\|_{e(\beta)}. \tag{3.7}$$

- *Let $\alpha > 0$, $\beta > 0$, $\gamma > 0$ and $(\alpha-\gamma)(\beta-\gamma) \geq \gamma^2$. Then there exists a constant a such that*

$$\|f_1 f_2\|_{e(\gamma)} \leq a\|f_1\|_{e(\alpha)}\|f_2\|_{e(\beta)}. \tag{3.8}$$

- *Let $d > 0$ and let $\beta > 2/d$. Then there exists a constant a such that*

$$\|f_1 f_2\|_{y^2\log^{-d}(y)} \leq a\|f_1\|_{e(\beta)}\|f_2\|_{L^2}. \tag{3.9}$$

Lemma 3.4: *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with the Lipschitz boundary. Then*

$$\mathrm{W}^{1,2}(\Omega) \hookrightarrow \mathrm{L}_{e(2)}(\Omega),$$

and

$$\mathrm{W}^{1,2}(\Omega) \hookrightarrow\hookrightarrow \mathrm{L}_{e(\beta)}(\Omega), \qquad \text{for } \beta < 2.$$

Proof: See [8] and [4] (Lemma 7.4.1).

Lemma 3.5: *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with the Lipschitz boundary. Let h_n be a sequence of functions such that*

$$h_n \to h \text{ weakly star in } \mathrm{L}^\infty(0,T;\mathrm{L}_{y\log^d(y)}(\Omega)) \quad \text{with } d > \frac{1}{2}. \tag{3.10}$$

Let $\mathcal{Z} \subset \mathrm{E}_{e(1/d)}(\Omega)$ be dense in $\mathrm{E}_{e(1/d)}(\Omega)$ and assume that the functions *and*

$$h_n \to h \text{ in } C([0,T]; \mathrm{W}^{-1,2}(\Omega)). \tag{3.13}$$

Remark: Here, the convergence with respect to the weak star topology in (3.12) means that

$$t \to \int_\Omega h_n(t) g \,\mathrm{d}x \text{ converges uniformly to } t \to \int_\Omega h(t) g \,\mathrm{d}x$$

for any $g \in \mathrm{E}_{e(1/d)}(\Omega)$.

Proof: The set $\mathcal{Z}$ is dense in $\mathrm{E}_{e(1/d)}(\Omega)$ which is a separable space. Thus, there exists a sequence $\psi_j \in \mathcal{Z}$, $j = 1, 2, \ldots$, $\|\psi_j\|_{e(1/d)} = 1$,

$$\mathrm{E}_{e(1/d)}(\Omega) = \overline{\text{span }\{\psi_j\}_{j=1}^{\infty}}^{E_{e(1/d)}(\Omega)}. \tag{3.14}$$

Because of (3.10), there exists a constant m such that

$$\operatorname{ess}\sup_{t\in[0,T]} \|h_n(t)\|_{y \log^d(y)} \le m, \quad n = 1, 2, \ldots. \tag{3.15}$$

Let $t \in [0,T]$, $n \in \mathbb{N}$ and let $\varepsilon > 0$. Then by virtue of the definition of the Orlicz norm of the Orlicz space $\mathrm{L}_{y \log^d(y)}(\Omega)$ (see [4], Remark 3.12.8), there exists a $g \in \mathrm{E}_{e(1/d)}(\Omega)$, $\|g\|_{e(1/d)} = 1$ such that

$$\|h_n(t)\|_{y \log^d(y)} \le \int_\Omega h_n(t) g \,\mathrm{d}x + \varepsilon \tag{3.16}$$

Because of (3.11), (3.14) and (3.15), we can find the sequence $t_k \to t$ such that

$$\lim_{k\to\infty} \int_\Omega h_n(t_k) g \,\mathrm{d}x = \int_\Omega h_n(t) g \,\mathrm{d}x, \quad \text{and} \quad \int_\Omega h_n(t_k) g \,\mathrm{d}x \le \|h_n(t_k)\|_{y \log^d(y)} \le m. \tag{3.17}$$

Substituting (3.17) in (3.16), we have $\|h_n(t)\|_{y \log^d(y)} \le m + \varepsilon$. As $\varepsilon > 0$ is arbitrary, we obtain (using also (3.10))

$$\sup_{t\in[0,T]} \|h_n(t)\|_{y \log^d(y)} \le m \quad \text{and} \quad \sup_{t\in[0,T]} \|h(t)\|_{y \log^d(y)} \le m.$$

Let $\psi \in A = \text{span }\{\psi_j\}_{j=1}^{\infty}$. Then, we can use Arzela-Ascoli theorem to deduce that

$$\int_\Omega h_n(t) \psi \,\mathrm{d}x \rightrightarrows \int_\Omega h(t) \psi \,\mathrm{d}x. \tag{3.18}$$

Thus

$$\sup_{t\in[0,T]}\left|\int_\Omega (h_n(t,x)-h(t,x))g\,\mathrm{d}x\right| \leq \sup_{t\in[0,T]}\left|\int_\Omega (h_n(t,x)-h(t,x))(g-\psi_\varepsilon)\,\mathrm{d}x\right| +$$

$$+\sup_{t\in[0,T]}\left|\int_\Omega (h_n(t,x)-h(t,x))\psi_\varepsilon\,\mathrm{d}x\right| \leq \varepsilon + \sup_{t\in[0,T]}\left|\int_\Omega (h_n(t,x)-h(t,x))\psi_\varepsilon\,\mathrm{d}x\right|,$$

which, combined with (3.18), gives (3.12).

Finally, (3.13) is equivalent to the statement

$$\sup_{t\in[0,T]}\left|\sup_{\|\psi\|_{W_0^{1,2}}=1}\int_\Omega (h_n(t)-h(t))\psi\,\mathrm{d}x\right| \to 0.$$

Because of Lemma 3.4, we have the compact imbedding

$$\mathrm{W}^{1,2}(\Omega) \hookrightarrow\hookrightarrow \mathrm{L}_{e(1/d)}(\Omega),$$

thus, there exists a function $\zeta(t) \in \mathrm{L}_{e(1/d)}(\Omega)$,

$$\|\zeta(t)\|_{e(1/d)} \leq c, \tag{3.19}$$

such that

$$\sup_{t\in[0,T]}\left|\sup_{\|\psi\|_{W_0^{1,2}}=1}\int_\Omega (h_n(t)-h(t))\psi\,\mathrm{d}x\right| = \sup_{t\in[0,T]}\left|\int_\Omega (h_n(t)-h(t))\zeta(t)\,\mathrm{d}x\right|.$$

Thus

$$\sup_{t\in[0,T]}\left|\sup_{\|\psi\|_{W_0^{1,2}}=1}\int_\Omega (h_n(t)-h(t))\psi\,\mathrm{d}x\right| \leq \sup_{t\in[0,T]}\left|\sup_{s\in[0,T]}\int_\Omega (h_n(t)-h(t))\zeta(s)\,\mathrm{d}x\right|. \tag{3.20}$$

Using the estimate (3.19) and $(\mathrm{L}_{y\log^d(y)})^* = \mathrm{L}_{e(1/d)}$, we deduce that there exists a function $\xi \in \mathrm{L}_{e(1/d)}(\Omega)$ such that

$$\sup_{s\in[0,T]}\int_\Omega (h_n(t)-h(t))\zeta(s)\,\mathrm{d}x = \int_\Omega (h_n(t)-h(t))\xi\,\mathrm{d}x$$

which, combined with (3.20) and (3.12), gives (3.13).

Q.E.D.

Let the functions

$$t \to \int_{\Omega} h_n(t)\psi \, dx \ \ \textit{are uniformly bounded and uniformly continuous on } [0,T]$$

for any $\psi \in D(\Omega)$. *Then*

$$h_n \to h \ \textit{in} \ C([0,T]; \mathrm{L}^{*-weak}_{e(\beta)}(\Omega)) \qquad (3.22).$$

Proof: Similar to the proof of Lemma 3.5.

Lemma 3.7: *Let* v_n, w_n *be two sequences,*

$$v_n \to v \quad \textit{weakly in } \mathrm{L}^p(\mathbb{R}^2), \quad w_n \to w \quad \textit{weakly in } \mathrm{L}^q(\mathbb{R}^2)$$

where $1/p + 1/q = 1/r < 1$.

Then

$$v_n \mathcal{R}_{i,j}[w_n] - w_n \mathcal{R}_{i,j}[v_n] \to v\mathcal{R}_{i,j}[w] - w\mathcal{R}_{i,j}[v] \quad \textit{weakly in } L^r(\mathbb{R}^2)$$

where $\mathcal{R}_{i,j}$ *are defined by their Fourier symbol*

$$\mathcal{R}_{i,j}[u] = \mathcal{F}^{-1}\left(\frac{\xi_i \xi_j}{|\xi|^2}\right) * u, \quad i,j = 1,2. \qquad (3.23)$$

Proof: See [2], Lemma 3.4.

Lemma 3.8: *Let* v_n, w_n *be two sequences,*

$$v_n(x) = w_n(x) = 0 \quad \textit{in } \mathbb{R}^2 - \Omega,$$

$$v_n \to v \quad \textit{weakly star in } \mathrm{L}_{e(\beta)}(\Omega), \quad w_n \to w \quad \textit{weakly star in } \mathrm{L}_{y \log^d(y)}(\Omega).$$

Let

$$\gamma = d - \frac{1}{\beta} - 1 > 0.$$

Then

$$v_n \mathcal{R}_{i,j}[w_n] - w_n \mathcal{R}_{i,j}[v_n] \to v\mathcal{R}_{i,j}[w] - w\mathcal{R}_{i,j}[v] \quad \textit{weakly star in } \mathrm{L}_{y \log^{\gamma}(y)}(\Omega)$$

$$\|\mathcal{R}_{i,j}[v]\|_{e\left(\frac{p}{p+1}\right)} \leq c(p)\|v\|_{e(p)}, \qquad \text{for } p > 0. \tag{3.25}$$

Let us introduce the cut-off operator

$$Q_k(z) = \begin{cases} -k & \text{for } z \in (-\infty, -k], \\ z & \text{for } z \in (-k, k), \\ k & \text{for } z \in [k, \infty). \end{cases}$$

thus, using (3.24), we see

$$\|\mathcal{R}_{i,j}[Q_k(w_n) - w_n]\|_{y \log^{\frac{1}{\beta}}(y)} \leq c\|Q_k(w_n) - w_n\|_{y \log^{\frac{\beta+1}{\beta}}(y)} =$$

$$= c\|\chi_{[w_n \geq k]}\left(Q_k(w_n) - w_n\right)\|_{y \log^{\frac{\beta+1}{\beta}}(y)}. \tag{3.26}$$

As w_n are bounded in $\mathrm{L}_{y \log^d(y)}(\Omega)$ uniformly in n, we see that there exists a constant c independent of n such that

$$\int_\Omega w_n \, \mathrm{d}x \leq c, \quad \text{for all } n \in \mathbb{N},$$

thus, there exists a function $r : (0, \infty) \to (0, \infty)$ such that

$$\lim_{k \to \infty} r(k) = 0, \quad \mu\{x \in \Omega : w_n(x) \geq k\} \leq r(k), \quad \text{for all } n \in \mathbb{N}. \tag{3.27}$$

Now, we can use (3.4) to deduce

$$\|\chi_{[w_n \geq k]}\left(Q_k(w_n) - w_n\right)\|_{y \log^{\frac{\beta+1}{\beta}}(y)} \leq c\|Q_k(w_n) - w_n\|_{y \log^d(y)} \|\chi_{[w_n \geq k]}\|_{e\left(\frac{1}{\gamma}\right)} \leq$$

$$\leq c \sup_n \|w_n\|_{y \log^d(y)} \|\chi_{[w_n \geq k]}\|_{e\left(\frac{1}{\gamma}\right)} \leq c\|\chi_{[w_n \geq k]}\|_{e\left(\frac{1}{\gamma}\right)} \tag{3.28}$$

By virtue of the definition of the Luxembourg norm and using (3.27), we easily observe

$$\|\chi_{[w_n \geq k]}\|_{e\left(\frac{1}{\gamma}\right)} \leq \left(\phi^{-1}\left(\frac{1}{r(k)}\right)\right)^{-1} \tag{3.29}$$

where ϕ is the Young function corresponding to the space $\mathrm{L}_{e\left(\frac{1}{\gamma}\right)}(\Omega)$. Thus, the right hand side in (3.29) tends to 0 for $k \to \infty$ uniformly in n. Summing up the results (3.26) – (3.29), there exists a function $q : (0, \infty) \to (0, \infty)$ such that

$$\lim_{k \to \infty} q(k) = 0$$

Consequently,

$$\|\mathcal{R}_{i,j}[\overline{Q_k(w)} - w]\|_{y \log^{\frac{1}{\beta}}(y)} \leq c \|\overline{Q_k(w)} - w\|_{y \log^{\frac{\beta+1}{\beta}}(y)} \leq q(k) \tag{3.31}$$

where $\overline{Q_k(w)}$ stands for a weak limit of $Q_k(w_n)$.

Finally, we write

$$\Big[v_n \mathcal{R}_{i,j}[w_n] - w_n \mathcal{R}_{i,j}[v_n]\Big] - \Big[v \mathcal{R}_{i,j}[w] - w \mathcal{R}_{i,j}[v]\Big] =$$

$$= \Big(\Big[v_n \mathcal{R}_{i,j}[Q_k(w_n)] - Q_k(w_n)\mathcal{R}_{i,j}[v_n]\Big] - \Big[v \mathcal{R}_{i,j}[\overline{Q_k(w)}] - \overline{Q_k(w)}\mathcal{R}_{i,j}[v]\Big]\Big) +$$

$$+ \Big[v_n \mathcal{R}_{i,j}[w_n - Q_k(w_n)] - (w_n - Q_k(w_n))\mathcal{R}_{i,j}[v_n]\Big] +$$

$$+ \Big[v \mathcal{R}_{i,j}[\overline{Q_k(w)} - w] - (\overline{Q_k(w)} - w)\mathcal{R}_{i,j}[v]\Big]$$

where the firts term on the right-hand side converges to zero in $D'(\Omega)$ because of Lemma 3.7 and the rest is uniformly small for large k in view of the estimates (3.30) and (3.31) together with (3.24) and (3.25). Thus Lemma 3.8 have been proved.

Q.E.D.

Lemma 3.9: *Let $z > y \geq 0$. Let $d \geq 1$. Then*

$$z \log^d(1+z) - y \log^d(1+y) \geq (z-y) \log^d(1+z-y).$$

Proof: Let us define $g(t) = t \log^d(1+t)$ for $t \in [0, \infty)$. Then g is convex, $g(0) = 0$. Thus

$$z \log^d(1+z) - y \log^d(1+y) = \int_y^z g'(t)\,\mathrm{d}t \geq \int_y^z g'(t-y)\,\mathrm{d}t = (z-y)\log^d(1+z-y).$$

Q.E.D.

Lemma 3.10: *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and let $d > 0$. Then there exists a constant D_d such that for all $g \in \mathrm{L}_{y^2 \log^{-d}(y)}(\Omega)$*

$$\|\mathcal{A}_i[g]\|_{e\left(\frac{2}{d+2}\right)} \leq D_d \|g\|_{y^2 \log^{-d}(y)}, \tag{3.32}$$

where $\mathcal{A}_i$, $i = 1, 2$, are defined by their Fourier symbol

Proof: We shall prove Lemma 3.10 in two steps.
(1) Let Φ_d be a Young function satisfying $\Phi_d(y) = y^2 \log^{-d}(y)$ for $y \geq y_0$ and

$$\int_0^1 \frac{\Phi_d^{-1}(s)}{s^{3/2}}\, \mathrm{d}s < \infty. \tag{3.34}$$

At first, we shall prove the inequality

$$\|\mathcal{A}_i[g]\|_{e\left(\frac{2}{d+2}\right)} \leq c(d)\, \|\mathcal{A}_i[g]\|_{WL_{\Phi_d}}. \tag{3.35}$$

To this end, we will use imbedding theorems for Sobolev-Orlicz spaces (see [4], Section 7.2). By virtue of (3.34), we can define the Sobolev conjugate Φ_d^* by the formula

$$\left(\Phi_d^*\right)^{-1}(y) = \int_0^y \frac{\left(\Phi_d\right)^{-1}(s)}{s^{3/2}}\, \mathrm{d}s.$$

Then, using [4], Theorem 7.2.3, we get

$$\|\mathcal{A}_i[g]\|_{\Phi_d^*} \leq c(d) \|\mathcal{A}_i[g]\|_{WL_{\Phi_d}}. \tag{3.36}$$

Let us define an auxiliary function $\xi : [0,\infty) \to [0,\infty)$ by the formula

$$\xi(z) = \begin{cases} \sqrt{z} \log^{\frac{d}{2}} z & \text{for } z \geq z_0, \\ \left(\Phi_d\right)^{-1}(z) & \text{for } z < z_0, \end{cases}$$

where z_0 is chosen so that

$$\Phi_d(\Phi_d^{-1}(z)) = z \leq \Phi_d(\xi(z)) \quad \text{for all } z \geq z_0.$$

Thus, $\Phi_d^{-1}(z) \leq \xi(z)$ and consequently

$$\left(\Phi_d^*\right)^{-1}(z) = \int_0^z \frac{\left(\Phi_d\right)^{-1}(s)}{s^{3/2}}\, \mathrm{d}s \leq \int_0^z \frac{\xi(s)}{s^{3/2}}\, \mathrm{d}s \leq c + \frac{2}{d+2} \log^{\frac{d+2}{2}} z \quad \text{for } z \geq z_0$$

which implies

$$\exp\left(y^{\frac{2}{d+2}}\right) \leq \Phi_d^*(y+c) \quad \text{for } y \geq \log^{\frac{d+2}{2}} z_0,$$

in particular

$$\|\mathcal{A}_i[g]\|_{e\left(\frac{2}{d+2}\right)} \leq c(d) \|\mathcal{A}_i[g]\|_{\Phi_d^*}. \tag{3.37}$$

To this end, let us introduce the operator $\mathcal{R}_{i,j} = \partial_{x_j}\mathcal{A}_i$, $i,j = 1,2$, defined by the formula (3.23). The operator $\mathcal{R}_{i,j}$ satisfies the assumptions of the standard Mikhlin multiplier theorem, see [1], Theorem 6.1.6. Thus, we can show (compare with [1] and the proof of our Lemma 3.1) that $\mathcal{R}_{i,j}$ is bounded mapping as

$$\mathcal{R}_{i,j} : \mathrm{L}^p(\mathbb{R}^2) \to \mathrm{L}^p(\mathbb{R}^2),\ p > 1, \qquad \mathcal{R}_{i,j} : \mathrm{L}^1(\mathbb{R}^2) \to \mathrm{L}^{1\infty}(\mathbb{R}^2),$$

where $\mathrm{L}^{1\infty}(\mathbb{R}^2)$ is one of the Lorentz spaces (we have $u \in \mathrm{L}^{1\infty}(\mathbb{R}^2)$ if and only if $\sup_\sigma \sigma m(\sigma, u) < \infty$ where $m(\sigma,u) = \mu\{x : |u(x)| > \sigma\}$).

Then, by virtue of [3], Theorem B.2, we have (3.38).

Finally, combining (3.35) and (3.38), we get (3.32). Thus Lemma 3.10 has been proved.

Q.E.D.

4 Proof of Theorem 1

A starting point of our proof of Theorem 1 will be the following lemma:

Lemma 4.1: *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with C^{2+v} boundary and let $\beta \geq 4$. Let $\delta > 0$ and let the pressure p satisfies (1.5). Then, given initial data ϱ_0, $\vec{q}$ satisfying the compatibility conditions (1.10), there exists $\varrho_\delta \in \mathrm{L}^\infty(0,T;\mathrm{L}^\beta(\Omega))$, $\varrho_\delta \geq 0$, $u^i_\delta \in \mathrm{L}^2(0,T;\mathrm{W}^{1,2}_0(\Omega))$, $i = 1,2$, such that*

$$(\varrho_\delta)_t + \mathrm{div}\ (\varrho_\delta \vec{u}_\delta) = 0 \quad \text{in } D'((0,T)\times\Omega), \tag{4.1}$$

$$(\varrho_\delta u^i_\delta)_t + \mathrm{div}\ (\varrho_\delta u^i_\delta \vec{u}_\delta) + \Big(p(\varrho_\delta) + \delta\varrho_\delta^\beta\Big)_{x_i} = \mu\Delta u^i_\delta + (\lambda+\mu)(\mathrm{div}\ \vec{u}_\delta)_{x_i},$$

$$i = 1,2, \quad \text{in } D'((0,T)\times\Omega), \tag{4.2}$$

$$\vec{u}_\delta|_{\partial\Omega} = 0, \tag{4.3}$$

$$\varrho_\delta(0) = \varrho_0, \quad (\varrho_\delta u^i_\delta)(0) = q_i,\ i = 1,2. \tag{4.4}$$

Moreover, $\varrho_\delta \in \mathrm{L}^{\beta+1}((0,T)\times\Omega)$ and the equation (1.9) holds in $D'((0,T)\times\mathbb{R}^2)$ provided ϱ_δ, $\vec{u}_\delta$ were prolonged to be zero on $\mathbb{R}^2 - \Omega$, for any continuously differentiable function b such that b' is uniformly bounded.
Moreover, ϱ_δ, $\vec{u}_\delta$ satisfy the estimates:

$$\sup_{t\in[0,T]} \|\varrho_\delta(t)\|_{y\log^{(d+1)}(y)} \leq c(\varrho_0, \vec{q}), \tag{4.5}$$

$$\|\vec{u}_\delta\|_{L^2(0,T;W_0^{1,2}(\Omega))} \le c(\varrho_0, \vec{q}) \tag{4.8}$$

where the constant c is independent of $\delta > 0$.
Finally, the energy $E_\delta(t)$ given by the formula

$$E_\delta(t) = E[\varrho_\delta, \vec{u}_\delta](t) = \int_\Omega \frac{1}{2}\varrho_\delta \left|\vec{u}_\delta\right|^2 \, dx + \int_\Omega \int_{z_0}^{\varrho_\delta} \int_{z_0}^{\sigma} \frac{p'(s)}{s} \, ds \, d\sigma \, dx + \int_\Omega \frac{\delta}{\beta - 1} \varrho_\delta^\beta \, dx$$

satisfies the energy inequality

$$E_\delta(t) + \int_0^t \int_\Omega \mu \left|\nabla \vec{u}_\delta\right|^2 + (\lambda + \mu) \left|\mathrm{div}\ \vec{u}_\delta\right|^2 \, dx \le E_\delta(0) \tag{4.9}$$

where

$$E_\delta(0) = \int_{\{x \in \Omega \,:\, \varrho_0(x) > 0\}} \frac{1}{2} \frac{|\vec{q}|^2}{\varrho_0} \, dx + \int_\Omega \int_{z_0}^{\varrho_0} \int_{z_0}^{\sigma} \frac{p'(s)}{s} \, ds \, d\sigma \, dx + \int_\Omega \frac{\delta}{\beta - 1} \varrho_0^\beta \, dx.$$

Remark: We can prove the global existence of solutions to the Navier-Stokes equations provided we have good apriori estimates. If we add the artificial pressure term $\delta\varrho_\delta^\beta$, we can derive the apriori estimate $\varrho_\delta \in \mathrm{L}^\infty(0, T, \mathrm{L}^\beta(\Omega))$. This estimate is better than the apriori estimate obtained from the pressure term $p(\varrho_\delta)$ and we can use results of [6] or [2] to prove the global existence result for the modified system (with artificial pressure $\delta\varrho_\delta^\beta$).

Proof of Lemma 4.1 : See either [6], chapter 7.5, or [2], chapters 2 and 3. Our assumptions are nearly the same as in [2]. Thus we can use the approximation scheme developed in [2] to prove Lemma 4.1 (compare with [2], Proposition 3.1).

Q.E.D.

The idea of the proof of Theorem 1 is now simple: To pass to the limits for $\delta \to 0$ in (4.1) – (4.4).

4.1 On the integrability of the density

We first derive an estimate of the density ϱ_δ to make possible passing to the limit in the term $\delta\varrho_\delta^\beta$ as $\delta \to 0$. We shall prove the following lemma:

Lemma 4.2: *Let $d > 1$ and $0 < \theta < \min(1, d-1)$. Let $\psi \in D(0,T)$ and $\phi \in D(\Omega)$.*

Proof: Let us consider the operators $\mathcal{A}_i$, $i = 1, 2$, defined by (3.33). Note, that $\partial_{x_i}\mathcal{A}_i[v] = v$ and, by virtue of the classical Mikhlin multiplier theorem (see [1], Theorem 6.1.6):

$$\|\mathcal{A}_i[v]\|_{W^{1,s}(\Omega)} \leq c(s)\|v\|_{L^s(\mathbb{R}^2)}, \quad 1 < s < \infty, \quad \text{in particular,} \tag{4.10}$$

$$\|\mathcal{A}_i[v]\|_{L^\infty(\Omega)} \leq c(s)\|v\|_{L^s(\mathbb{R}^2)} \text{ for } s > 2.$$

Similarly, according to Lemma 3.2, we have

$$\|\mathcal{A}_i[v]\|_{WL_{e\left(\frac{\gamma}{\gamma+1}\right)}(\Omega)} \leq c(\gamma)\|v\|_{L_{e(\gamma)}(\Omega)}, \quad \text{for all } v \in L_{e(\gamma)}(\Omega),\ \gamma > 0. \tag{4.11}$$

Next, let us define the operators $\mathcal{R}_{i,j}$, $i, j = 1, 2$, by (3.23), thus $\mathcal{R}_{i,j}[v] = \partial_{x_j}\mathcal{A}_i[v]$ and we have the estimates (3.24) and (3.25).

Using renormalized continuity equation (1.9) for $b(y) = \log^\theta(1 + y)$, we obtain

$$\Big(\log^\theta(1 + \varrho_\delta)\Big)_t + \text{div }(\log^\theta(1 + \varrho_\delta)\vec{u}_\delta) +$$

$$+ \left(\theta \log^{\theta-1}(1 + \varrho_\delta)\frac{\varrho_\delta}{1 + \varrho_\delta} - \log^\theta(1 + \varrho_\delta)\right) \text{div } \vec{u}_\delta = 0 \qquad \text{in } D'((0, T) \times \mathbb{R}^2). \tag{4.12}$$

Let us consider the functions

$$\varphi_i(t, x) = \psi(t)\phi(x)\mathcal{A}_i[\log^\theta(1 + \varrho_\delta)],\ i = 1, 2, \ \text{ where } \psi \in D(0, T),\ \phi \in D(\Omega),$$

where, as always, ϱ_δ is prolonged by zero outside Ω.

By virtue of (4.5), we get

$$\sup_{t\in[0,T]} \|\log^\theta(1 + \varrho_\delta(t))\|_{e\left(\frac{1}{\theta}\right)} \leq c(\varrho_0, \vec{q}), \tag{4.13}$$

where the constant $c(\varrho_0, \vec{q})$ does not depend on δ. Consequently, by virtue of (4.11)

$$\mathcal{A}_i[\log^\theta(1 + \varrho_\delta)] \in C([0, T]; \mathrm{W\,L}_{e\left(\frac{1}{\theta+1}\right)}(\Omega)),$$

in particular, using the definition of the space $\mathrm{E}_\phi(\Omega)$, we get

$$\mathcal{A}_i[\log^\theta(1 + \varrho_\delta)] \in C([0, T]; \mathrm{W\,E}_{e(\beta)}(\Omega)), \ \text{ for } 0 < \beta < \frac{1}{\theta + 1}.$$

Now, (2.1) justifies the choice of φ_i as test functions for the equation (4.2) to obtain

$$+ \mu \int_0^T \psi \int_\Omega \partial_{x_j}\phi\, \partial_{x_j} u_\delta^i \mathcal{A}_i[\log^\theta(1+\varrho_\delta)]\,\mathrm{d}x\,\mathrm{d}t +$$

$$+ \mu \int_0^T \psi \int_\Omega \phi\, \partial_{x_j} u_\delta^i \mathcal{R}_{i,j}[\log^\theta(1+\varrho_\delta)]\,\mathrm{d}x\,\mathrm{d}t +$$

$$+ (\lambda+\mu) \int_0^T \psi \int_\Omega \partial_{x_i}\phi\, \mathrm{div}\; \vec{u}_\delta \mathcal{A}_i[\log^\theta(1+\varrho_\delta)]\,\mathrm{d}x\,\mathrm{d}t +$$

$$+ (\lambda+\mu) \int_0^T \psi \int_\Omega \phi\, \mathrm{div}\; (\vec{u}_\delta) \log^\theta(1+\varrho_\delta)\,\mathrm{d}x\,\mathrm{d}t +$$

$$+ \int_0^T \psi \int_\Omega \phi \varrho_\delta u_\delta^i \mathcal{R}_{i,j}[\log^\theta(1+\varrho_\delta) u_\delta^j]\,\mathrm{d}x\,\mathrm{d}t - \int_0^T \partial_t\psi \int_\Omega \phi\varrho_\delta u_\delta^i \mathcal{A}_i[\log^\theta(1+\varrho_\delta)]\,\mathrm{d}x\,\mathrm{d}t +$$

$$+ \int_0^T \psi \int_\Omega \phi\varrho_\delta u_\delta^i \mathcal{A}_i\left[\left(\theta \log^{\theta-1}(1+\varrho_\delta)\frac{\varrho_\delta}{1+\varrho_\delta} - \log^\theta(1+\varrho_\delta)\right) \mathrm{div}\; \vec{u}_\delta\right]\,\mathrm{d}x\,\mathrm{d}t -$$

$$- \int_0^T \psi \int_\Omega \partial_{x_i}\phi \left(p(\varrho_\delta) + \delta\varrho_\delta^\beta\right) \mathcal{A}_i\left[\log^\theta(1+\varrho_\delta)\right]\,\mathrm{d}x\,\mathrm{d}t = \sum_{k=1}^{10} I_k$$

where the integrals $I_1 - I_{10}$ may be treated as follows:

(i) Using Lemma 3.4, (3.4) (4.10), (4.5), (4.8) and (4.13), we have

$$|I_1| = \left| \int_0^T \psi \int_\Omega \partial_{x_j}\phi\, \varrho_\delta u_\delta^i u_\delta^j \mathcal{A}_i[\log^\theta(1+\varrho_\delta)]\,\mathrm{d}x\,\mathrm{d}t \right| \le$$

$$\le c \int_0^T \|\varrho_\delta u_\delta^i u_\delta^j\|_{y\log^d(y)} \|\mathcal{A}_i[\log^\theta(1+\varrho_\delta)]\|_{L^\infty}\,\mathrm{d}t \le$$

$$\le c \int_0^T \|\varrho_\delta\|_{y\log^{d+1}(y)} \|\vec{u}_\delta\|_{e(2)} \|\vec{u}_\delta\|_{e(2)} \|\log^\theta(1+\varrho_\delta)\|_{e\left(\frac{1}{\theta}\right)}\,\mathrm{d}t \le c.$$

(ii) Similarly, by virtue of Lemma 3.4, (3.4), (3.25), (4.5), (4.8) and (4.13), we obtain

$$|I_2| = \left| \int_0^T \psi \int_\Omega \phi\varrho_\delta u_\delta^i u_\delta^j \mathcal{R}_{i,j}[\log^\theta(1+\varrho_\delta)]\,\mathrm{d}x\,\mathrm{d}t \right| \le$$

$$\le c \int_0^T \|\varrho_\delta u_\delta^i u_\delta^j\|_{y\log^d(y)} \|\mathcal{R}_{i,j}[\log^\theta(1+\varrho_\delta)]\|_{e\left(\frac{1}{d}\right)}\,\mathrm{d}t \le$$

(iii) In view of (4.10), (4.8) and (4.13), it holds

$$|I_3| = \left| \mu \int_0^T \psi \int_\Omega \partial_{x_j} \phi \, \partial_{x_j} u_\delta^i \mathcal{A}_i[\log^\theta (1 + \varrho_\delta)] \, \mathrm{d}x \, \mathrm{d}t \right| \le c.$$

(iv) Furthermore, (3.25), (4.8) and (4.13) imply

$$|I_4| = \left| \mu \int_0^T \psi \int_\Omega \phi \, \partial_{x_j} u_\delta^i \mathcal{R}_{i,j}[\log^\theta (1 + \varrho_\delta)] \, \mathrm{d}x \, \mathrm{d}t \right| \le c.$$

(v) Similarly as in (iii):

$$|I_5| = \left| (\lambda + \mu) \int_0^T \psi \int_\Omega \partial_{x_i} \phi \, \mathrm{div} \, \vec{u}_\delta \mathcal{A}_i[\log^\theta (1 + \varrho_\delta)] \, \mathrm{d}x \, \mathrm{d}t \right| \le c.$$

(vi) Using (4.8) and (4.13), we obtain

$$|I_6| = \left| (\lambda + \mu) \int_0^T \psi \int_\Omega \phi \, \mathrm{div} \, (\vec{u}_\delta) \log^\theta (1 + \varrho_\delta) \, \mathrm{d}x \, \mathrm{d}t \right| \le$$

$$\le c \|\vec{u}_\delta\|_{L^2(0,T;W^{1,2}(\Omega))} \sup_{t \in [0,T]} \| \log^\theta (1 + \varrho_\delta(t)) \|_{e\left(\frac{1}{\theta}\right)} \le c.$$

(vii) By virtue of (3.5), it holds

$$|I_7| \le \int_0^T \psi \int_\Omega \phi \varrho_\delta u_\delta^i \mathcal{R}_{i,j}[\log^\theta (1 + \varrho_\delta) u_\delta^j] \, \mathrm{d}x \, \mathrm{d}t \le$$

$$\le c \int_0^T \|\varrho_\delta u_\delta^i\|_{y \log^{d+1/2}(y)} \|\mathcal{R}_{i,j}[\log^\theta (1 + \varrho_\delta) u_\delta^j]\|_{e\left(\frac{1}{d+1/2}\right)} \, \mathrm{d}t. \tag{4.15}$$

In view of (3.25) and (3.8), we can estimate

$$\|\mathcal{R}_{i,j}[\log^\theta (1 + \varrho_\delta) u_\delta^j]\|_{e\left(\frac{1}{d+1/2}\right)} \le c \| \log^\theta (1 + \varrho_\delta) u_\delta^j \|_{e\left(\frac{1}{d-1/2}\right)} \le$$

$$\le c \| \log^\theta (1 + \varrho_\delta) \|_{e\left(\frac{1}{d-1}\right)} \|u_\delta^j\|_{e(2)},$$

consequently (4.15) implies

provided $\theta < d-1$.

(viii) Similarly as in (i), we have

$$|I_8| = \left| \int_0^T \partial_t \psi \int_\Omega \phi \varrho_\delta u^i_\delta \mathcal{A}_i[\log^\theta(1+\varrho_\delta)] \,\mathrm{d}x \,\mathrm{d}t \right| \leq$$

$$\leq c \int_0^T \|\varrho_\delta u^i_\delta\|_{y \log^{d+1/2}(y)} \|\mathcal{A}_i[\log^\theta(1+\varrho_\delta)]\|_{L^\infty} \,\mathrm{d}t \leq$$

$$\leq c \int_0^T \|\varrho_\delta\|_{y \log^{d+1}(y)} \|\vec{u}_\delta\|_{e(2)} \|\log^\theta(1+\varrho_\delta)\|_{e\left(\frac{1}{\theta}\right)} \,\mathrm{d}t \leq c.$$

(ix) By virtue of (3.5), we get

$$|I_9| = \left| \int_0^T \psi \int_\Omega \phi \varrho_\delta u^i_\delta \mathcal{A}_i \Big[\Big(\theta \log^{\theta-1}(1+\varrho_\delta) \frac{\varrho_\delta}{1+\varrho_\delta} - \log^\theta(1+\varrho_\delta) \Big) \mathrm{div}\, \vec{u}_\delta \Big] \,\mathrm{d}x \,\mathrm{d}t \right| \leq$$
$$\leq c \int_0^T \|\varrho_\delta u^i_\delta\|_{y \log^{d+1/2}(y)} \Big\| \mathcal{A}_i \Big[\Big(\log^{\theta-1}(1+\varrho_\delta) \frac{\theta \varrho_\delta}{1+\varrho_\delta} - \log^\theta(1+\varrho_\delta) \Big) \mathrm{div}\, \vec{u}_\delta \Big] \Big\|_{e\left(\frac{2}{2d+1}\right)} \,\mathrm{d}t. \tag{4.16}$$

Using (3.9) and Lemma 3.10, we obtain

$$\Big\| \mathcal{A}_i \Big[\Big(\log^{\theta-1}(1+\varrho_\delta) \frac{\theta \varrho_\delta}{1+\varrho_\delta} - \log^\theta(1+\varrho_\delta) \Big) \mathrm{div}\, \vec{u}_\delta \Big] \Big\|_{e\left(\frac{2}{2d+1}\right)} \leq$$

$$\leq \Big\| \Big(\log^{\theta-1}(1+\varrho_\delta) \frac{\theta \varrho_\delta}{1+\varrho_\delta} - \log^\theta(1+\varrho_\delta) \Big) \mathrm{div}\, \vec{u}_\delta \Big\|_{y^2 \log^{-2d+1}(y)} \leq$$

$$\leq c \|\log^{\theta-1}(1+\varrho_\delta) \frac{\theta \varrho_\delta}{1+\varrho_\delta} - \log^\theta(1+\varrho_\delta)\|_{e\left(\frac{1}{\theta}\right)} \|\mathrm{div}\, \vec{u}_\delta\|_{L^2} \leq$$

$$\leq c \|\log^\theta(1+\varrho_\delta)\|_{e\left(\frac{1}{\theta}\right)} \|\vec{u}_\delta\|_{W^{1,2}}$$

which, combined with (4.16), yields $|I_9| \leq c$.

(x) By virtue of (1.5), (4.5), (4.6), (4.10) and (4.13), we have

$$|I_{10}| = \left| \int_0^T \psi \int_\Omega \partial_{x_i} \phi \left(p(\varrho_\delta) + \delta \varrho_\delta^\beta \right) \mathcal{A}_i \left[\log^\theta(1+\varrho_\delta) \right] \,\mathrm{d}x \,\mathrm{d}t \right| \leq$$

$$\leq c \int_0^T \|p(\varrho_\delta) + \delta \varrho_\delta^\beta\|_{L^1} \|\mathcal{A}_i[\log^\theta(1+\varrho_\delta)]\|_{L^\infty} \leq c.$$

Thus Lemma 4.2 has been proved.

Q.E.D.

Let $\theta \in \big(0, \min(1, d-1)\big)$. Let $\psi \in D(0,T)$ and $\phi \in D(\Omega)$. Put

$$M = \overline{\{x \in \Omega \,:\, \phi(x) \neq 0\}} \times \overline{\{t \in (0,T) \,:\, \psi(t) \neq 0\}}.$$

Then, by virtue of Lemma 4.2, it holds

$$\iint_M \delta \varrho_\delta^\beta \log^\theta (1+\varrho_\delta)\,\mathrm{d}x\,\mathrm{d}t \le c \tag{4.17}$$

where c does not depend on δ.

Let us define the set

$$J_k^\delta = \{(x,t) \in M \,:\, \varrho_\delta(x,t) \le k\} \text{ for } k>0 \text{ and } \delta \in (0,1).$$

In view of (4.5), there exists a constant $s \in (0,\infty)$ such that for all $\delta \in (0,1)$ and $k>0$

$$\mu\left(M - J_k^\delta\right) \le \frac{s}{k}. \tag{4.18}$$

We can estimate

$$\left|\int_0^T \int_\Omega \psi\phi\delta\varrho_\delta^\beta \,\mathrm{d}x\,\mathrm{d}t\right| \le c \iint_M \delta\varrho_\delta^\beta \,\mathrm{d}x\,\mathrm{d}t \le c\iint_{J_k^\delta} \delta\varrho_\delta^\beta \,\mathrm{d}x\,\mathrm{d}t + c\iint_{M-J_k^\delta} \delta\varrho_\delta^\beta \,\mathrm{d}x\,\mathrm{d}t \le$$

$$\le c\delta k^\beta \mu(\Omega) + c\delta \iint_M \chi_{(M-J_k^\delta)} \varrho_\delta^\beta \,\mathrm{d}x\,\mathrm{d}t. \tag{4.19}$$

Let us denote by Φ the Young function corresponding to the Orlicz space $\mathrm{L}_{e\left(\frac{1}{\theta}\right)}(\Omega \times (0,T))$. Then, using (3.5), (4.17) and (4.18), we get

$$\delta \iint_M \chi_{(M-J_k^\delta)} \varrho_\delta^\beta \,\mathrm{d}x\,\mathrm{d}t \le c\|\chi_{M-J_k^\delta}\|_{e\left(\frac{1}{\theta}\right)} \delta \|\varrho_\delta^\beta\|_{y\log^\theta(y)} \le$$

$$\le c\left(\Phi^{-1}\left(\frac{k}{s}\right)\right)^{-1} \iint_M \delta\varrho_\delta^\beta \log^\theta(1+\varrho_\delta)\,\mathrm{d}x\,\mathrm{d}t \le c\left(\Phi^{-1}\left(\frac{k}{s}\right)\right)^{-1}. \tag{4.20}$$

Combining (4.20) with (4.19), we derive the estimate

$$\left|\int_0^T \int_\Omega \psi\phi\delta\varrho_\delta^\beta \,\mathrm{d}x\,\mathrm{d}t\right| \le c\delta k^\beta \mu(\Omega) + c\left(\Phi^{-1}\left(\frac{k}{s}\right)\right)^{-1}$$

The right hand side of (4.21) tends to zero as $k \to \infty$, therefore, passing to the limit for $k \to \infty$ in (4.21), we infer

$$\lim_{\delta \to 0} \int_0^T \int_\Omega \psi \phi \delta \varrho_\delta^\beta \, \mathrm{d}x \, \mathrm{d}t = 0$$

which means that

$$\delta \varrho_\delta^\beta \to 0 \quad \text{in } D'((0,T) \times \Omega). \tag{4.22}$$

4.2 The limit passage

As ϱ_δ satisfies equation (4.1), we infer, by virtue of the estimate (4.5), Lemma 3.5,

$$\varrho_\delta \to \varrho \text{ in } C([0,T]; \mathrm{L}^{*-weak}_{y \log^{d+1}(y)}(\Omega)) \text{ and } \varrho_\delta \to \varrho \text{ in } C([0,T]; \mathrm{W}^{-1,2}(\Omega)), \tag{4.23}$$

next, the estimate (4.8) yields

$$\vec{u}_\delta \to \vec{u} \text{ weakly in } \mathrm{L}^2(0,T; \mathrm{W}^{1,2}_0(\Omega)) \tag{4.24}$$

and, combining (4.23), (4.24) with (4.7), (4.2), (3.6) and Lemma 3.5, we infer

$$\varrho_\delta \vec{u}_\delta \to \varrho \vec{u} \text{ in } C([0,T]; \mathrm{L}^{*-weak}_{y \log^{(d+1)/2}(y)}(\Omega)) \text{ and } \varrho_\delta \vec{u}_\delta \to \varrho \vec{u} \text{ in } C([0,T]; \mathrm{W}^{-1,2}(\Omega)), \tag{4.25}$$

finally, (4.24) and (4.25) yield

$$\varrho_\delta u^i_\delta u^j_\delta \to \varrho u^i u^j \text{ weakly in } \mathrm{L}^1((0,T) \times \Omega), \ i,j = 1,2, \tag{4.26}$$

passing to subsequences as the case may be.

By virtue of (4.5) and (1.5), there exists a constant c such that for all $\delta > 0$

$$\|p(\varrho_\delta)\|_{L^\infty(0,T; L_{y \log(y)}(\Omega))} \leq c,$$

thus

$$p(\varrho_\delta) \to \overline{p(\varrho)} \text{ weakly star in } \mathrm{L}^\infty(0,T; \mathrm{L}_{y \log(y)}(\Omega)). \tag{4.27}$$

In view of (4.22), we get

$$\delta \varrho_\delta^\beta \to 0 \text{ in } \mathrm{L}^1_{loc}((0,T) \times \Omega). \tag{4.28}$$

Finally, using (4.23) and (4.25), we see that the limits ϱ, $\varrho \vec{u}$ satisfy the initial conditions in $D'((0,T)\times\mathbb{R}^2)$,

$$(\varrho u^i)_t + \text{div } (\varrho u^i \vec{u}) + \left(\overline{p(\varrho)}\right)_{x_i} = \mu\Delta u^i + (\lambda+\mu)(\text{div } \vec{u})_{x_i}, \quad i = 1,2, \tag{4.30}$$

in $D'((0,T)\times\Omega)$. Thus the only thing to complete the proof of Theorem 1 is to show

$$\overline{p(\varrho)} = p(\varrho). \tag{4.31}$$

Let us introduce a family of functions (see [2])

$$T_k(z) = kT\left(\frac{z}{k}\right) \text{ for } z\in\mathbb{R},\ k = 1,2,3,\dots$$

where $T\in \mathrm{C}^\infty(\mathbb{R})$ is chosen so that

$$T(z) = z \text{ for } z\le 1, \quad T(z) = 2 \text{ for } z\ge 3, \quad T \text{ concave.}$$

Then, since ϱ_δ, $\vec{u}_\delta$ is a renormalized solution of the continuity equation (4.1) in $D'((0,T)\times\mathbb{R}^2)$, we have

$$T_k(\varrho_\delta)_t + \text{div } (T_k(\varrho_\delta)\vec{u}_\delta) + (T_k'(\varrho_\delta)\varrho_\delta - T_k(\varrho_\delta)) \text{ div } \vec{u}_\delta = 0 \text{ in } D'((0,T)\times\mathbb{R}^2). \tag{4.32}$$

Passing to the limit for $\delta\to 0+$ we obtain (using Lemma 3.6)

$$\partial_t\overline{T_k(\varrho)} + \text{div } (\overline{T_k(\varrho)}\vec{u}) + \overline{(T_k'(\varrho)\varrho - T_k(\varrho)) \text{ div } \vec{u}} = 0 \text{ in } D'((0,T)\times\mathbb{R}^2) \tag{4.33}$$

where

$$(T_k'(\varrho_\delta)\varrho_\delta - T_k(\varrho_\delta)) \text{ div } \vec{u}_\delta \to \overline{(T_k'(\varrho)\varrho - T_k(\varrho)) \text{ div } \vec{u}} \text{ weakly in } \mathrm{L}^2((0,T)\times\Omega) \tag{4.34}$$

and

$$T_k(\varrho_\delta)\to\overline{T_k(\varrho)} \text{ in } C([0,T];\mathrm{L}^{*-weak}_{e(\beta)}(\Omega)) \text{ for all } \beta > 0. \tag{4.35}$$

4.3 The effective viscous flux

We introduce the quantity $p(\varrho) - (\lambda+2\mu)\text{div } \vec{u}$ called usually the effective viscous flux. Here we shall prove the result analogous to the result in [2] (see Lemma 4.2).

Lemma 4.3: *Let ϱ_δ, $\vec{u}_\delta$ be the sequence of approximate solutions, the existence of*

Then

$$\lim_{\delta\to 0_+}\int_0^T \psi\int_\Omega \phi\,(p(\varrho_\delta)-(\lambda+2\mu)\mathrm{div}\;\vec{u}_\delta)\,T_k(\varrho_\delta)\,dx\,dt =$$

$$=\int_0^T \psi\int_\Omega \phi\Big(\overline{p(\varrho)}-(\lambda+2\mu)\mathrm{div}\;\vec{u}\Big)\overline{T_k(\varrho)}\,dx\,dt$$

for any $\psi\in D(0,T)$, $\phi\in D(\Omega)$.

Proof: Let us consider the functions of the form

$$\varphi_i(t,x)=\psi(t)\phi(x)\mathcal{A}_i[T_k(\varrho_\delta)],\ i=1,2,\ \text{ where } \psi\in D(0,T),\ \phi\in D(\Omega),$$

where, as always, ϱ_δ is prolonged by zero outside Ω. By virtue of (4.11)

$$\mathcal{A}_i[T_k(\varrho_\delta)]\in C([0,T];\mathrm{W\,L}_{e(\beta)}(\Omega)),\ \text{for } \beta\in(0,1),$$

in particular, using the definition of the space $\mathrm{E}_{e(\beta)}(\Omega)$, we get

$$\mathcal{A}_i[T_k(\varrho_\delta)]\in C([0,T];\mathrm{W\,E}_{e(\beta)}(\Omega)),\ \text{for } \beta\in(0,1),\ t\in[0,T].$$

Now, (2.1) justifies the choice of φ_i as test functions for the equation (4.2) to obtain (using also (4.32), (3.24), (3.25) and the assumption $d>1$)

$$\int_0^T\psi\int_\Omega\phi\left(p(\varrho_\delta)+\delta\varrho_\delta^\beta-(\lambda+2\mu)\mathrm{div}\;\vec{u}_\delta\right)T_k(\varrho_\delta)\,\mathrm{d}x\,\mathrm{d}t= \tag{4.36}$$

$$=\int_0^T\psi\int_\Omega\partial_{x_i}\phi\left((\lambda+\mu)\mathrm{div}\;(\vec{u}_\delta)-p(\varrho_\delta)-\delta\varrho_\delta^\beta\right)\mathcal{A}_i[T_k(\varrho_\delta)]\,\mathrm{d}x\,\mathrm{d}t+$$

$$+\mu\int_0^T\psi\int_\Omega\left(\partial_{x_j}\phi\right)u_\delta^jT_k(\varrho_\delta)\,\mathrm{d}x\,\mathrm{d}t-\mu\int_0^T\psi\int_\Omega\left(\partial_{x_j}\phi\right)u_\delta^i\mathcal{R}_{i,j}[T_k(\varrho_\delta)]\,\mathrm{d}x\,\mathrm{d}t+$$

$$+\mu\int_0^T\psi\int_\Omega\left(\partial_{x_j}\phi\right)\partial_{x_j}u_\delta^i\mathcal{A}_i[T_k(\varrho_\delta)]\,\mathrm{d}x\,\mathrm{d}t-\int_0^T\psi_t\int_\Omega\phi\varrho_\delta u_\delta^i\mathcal{A}_i[T_k(\varrho_\delta)]\,\mathrm{d}x\,\mathrm{d}t+$$

$$+\int_0^T\psi\int_\Omega\phi\varrho_\delta u_\delta^i\mathcal{A}_i[(T_k'(\varrho_\delta)\varrho_\delta-T_k(\varrho_\delta))\mathrm{div}\;\vec{u}_\delta]\,\mathrm{d}x\,\mathrm{d}t-$$

$$-\int_0^T\psi\int_\Omega\left(\partial_{x_j}\phi\right)\varrho_\delta u_\delta^iu_\delta^j\mathcal{A}_i[T_k(\varrho_\delta)]\,\mathrm{d}x\,\mathrm{d}t+$$

$$+\int_0^T\psi\int_\Omega u_\delta^i\left(T_k(\varrho_\delta)\mathcal{R}_{i,j}[\phi\varrho_\delta u_\delta^j]-\phi\varrho_\delta u_\delta^j\mathcal{R}_{i,j}[T_k(\varrho_\delta)]\right)\,\mathrm{d}x\,\mathrm{d}t,$$

ϱ being set zero outside Ω.

Similarly as in (4.36), we can deduce (using also (4.33)):

$$\int_0^T \psi \int_\Omega \phi\Big(\overline{p(\varrho)} - (\lambda+2\mu)\text{div } \vec{u}\Big)\overline{T_k(\varrho)}\,\mathrm{d}x\,\mathrm{d}t = \tag{4.37}$$

$$= \int_0^T \psi \int_\Omega \partial_{x_i}\phi\Big((\lambda+\mu)\text{div }(\vec{u}) - \overline{p(\varrho)}\Big)\mathcal{A}_i[\overline{T_k(\varrho)}]\,\mathrm{d}x\,\mathrm{d}t +$$

$$+\mu \int_0^T \psi \int_\Omega \left(\partial_{x_j}\phi\right) u^j \overline{T_k(\varrho)}\,\mathrm{d}x\,\mathrm{d}t - \mu \int_0^T \psi \int_\Omega \left(\partial_{x_j}\phi\right) u^i \mathcal{R}_{i,j}[\overline{T_k(\varrho)}]\,\mathrm{d}x\,\mathrm{d}t +$$

$$+\mu \int_0^T \psi \int_\Omega \left(\partial_{x_j}\phi\right) \partial_{x_j} u^i \mathcal{A}_i[\overline{T_k(\varrho)}]\,\mathrm{d}x\,\mathrm{d}t - \int_0^T \psi_t \int_\Omega \phi\varrho u^i \mathcal{A}_i[\overline{T_k(\varrho)}]\,\mathrm{d}x\,\mathrm{d}t +$$

$$+ \int_0^T \psi \int_\Omega \phi\varrho u^i \mathcal{A}_i[\overline{(T_k'(\varrho)\varrho - T_k(\varrho))\text{div } \vec{u}}]\,\mathrm{d}x\,\mathrm{d}t -$$

$$- \int_0^T \psi \int_\Omega \left(\partial_{x_j}\phi\right) \varrho u^i u^j \mathcal{A}_i[\overline{T_k(\varrho)}]\,\mathrm{d}x\,\mathrm{d}t +$$

$$+ \int_0^T \psi \int_\Omega u^i \left(\overline{T_k(\varrho)}\mathcal{R}_{i,j}[\phi\varrho u^j] - \phi\varrho u^j \mathcal{R}_{i,j}[\overline{T_k(\varrho)}]\right)\,\mathrm{d}x\,\mathrm{d}t.$$

Now, we shall prove that the integrals on the right-hand side of (4.36) converge for $\delta \to 0$ to their counterparts in (4.37).

Now, (4.11), (3.25), (4.34) and (4.35) imply

$$\mathcal{A}_i[T_k(\varrho_\delta)] \to \mathcal{A}_i[\overline{T_k(\varrho)}] \text{ in } C(\overline{(0,T)\times\Omega}), \tag{4.38}$$

$$\mathcal{R}_{i,j}[T_k(\varrho_\delta)] \to \mathcal{R}_{i,j}[\overline{T_k(\varrho)}] \text{ in } C([0,T]; \mathrm{L}^{*-weak}_{e(\gamma)}(\Omega)) \text{ for } \gamma \in (0,1) \tag{4.39}$$

and

$$\mathcal{A}_i[(T_k'(\varrho_\delta)\varrho_\delta - T_k(\varrho_\delta))\text{div } \vec{u}_\delta] \to \mathcal{A}_i[\overline{(T_k'(\varrho)\varrho - T_k(\varrho))\text{div } \vec{u}}]$$
$$\text{weakly in } \mathrm{L}^2(0,T;\mathrm{W}^{1,2}(\Omega)). \tag{4.40}$$

With the relations (4.23) – (4.28), (4.38) – (4.40) in mind, it is easy to see that it is enough to show

$$\int_0^T \psi \int_\Omega u^i_\delta \left(T_k(\varrho_\delta)\mathcal{R}_{i,j}[\phi\varrho_\delta u^j_\delta] - \phi\varrho_\delta u^j_\delta \mathcal{R}_{i,j}[T_k(\varrho_\delta)]\right)\,\mathrm{d}x\,\mathrm{d}t \ \to \tag{4.41}$$

$$\to \int_0^T \psi \int_\Omega u^i \left(\overline{T_k(\varrho)}\mathcal{R}_{i,j}[\phi\varrho u^j] - \phi\varrho u^j \mathcal{R}_{i,j}[\overline{T_k(\varrho)}]\right)\,\mathrm{d}x\,\mathrm{d}t$$

Lemma 3.8, together with (4.42) and (4.35), gives

$$T_k(\varrho_\delta)\mathcal{R}_{i,j}[\phi\varrho_\delta u_\delta^j] - \phi\varrho_\delta u_\delta^j\mathcal{R}_{i,j}[T_k(\varrho_\delta)] \;\to\; \overline{T_k(\varrho)}\mathcal{R}_{i,j}[\phi\varrho u^j] - \phi\varrho u^j\mathcal{R}_{i,j}[\overline{T_k(\varrho)}]$$

in $\mathrm{L}^2(0,T;\mathrm{W}^{-1,2}(\Omega))$ which, combined with (4.24), implies (4.41) and the proof of Lemma 4.3 is finished.

Q.E.D.

4.4 The amplitude of oscillations

The main result of this part is following (compare with [2], Lemma 4.3):

Lemma 4.4: *There exists a constant c independent of k such that*

$$\limsup_{\delta\to 0+} \|T_k(\varrho_\delta) - T_k(\varrho)\|_{y^2\log^d(y)} \le c$$

where $\|\cdot\|_{y^2\log^d(y)}$ is the Luxembourg norm in the Orlicz space $\mathrm{L}_\phi((0,T)\times\Omega)$ generated by the Young function $\phi(y) = y^2\log^d(1+y)$.

Proof: By the convexity of functions $z \to p(z)$ and $z \to -T_k(z)$, we obtain

$$\limsup_{\delta\to 0+} \int_0^T \int_\Omega p(\varrho_\delta)T_k(\varrho_\delta) - \overline{p(\varrho)}\,\overline{T_k(\varrho)}\,\mathrm{d}x\,\mathrm{d}t = \tag{4.43}$$

$$= \limsup_{\delta\to 0+} \int_0^T \int_\Omega (p(\varrho_\delta) - p(\varrho))\,(T_k(\varrho_\delta) - T_k(\varrho))\,\mathrm{d}x\,\mathrm{d}t +$$

$$+ \int_0^T \int_\Omega \left(\overline{p(\varrho)} - p(\varrho)\right)(T_k(\varrho) - \overline{T_k(\varrho)})\,\mathrm{d}x\,\mathrm{d}t \ge$$

$$\ge \limsup_{\delta\to 0+} \int_0^T \int_\Omega (p(\varrho_\delta) - p(\varrho))\,(T_k(\varrho_\delta) - T_k(\varrho))\,\mathrm{d}x\,\mathrm{d}t.$$

By virtue of Lemma 3.9, we have

$$(z\log^d(1+z) - y\log^d(1+y))(T_k(z) - T_k(y)) \ge |T_k(z) - T_k(y)|^2 \log^d(1 + |T_k(z) - T_k(y)|)$$

for all $z, y \ge 0$, consequently, (1.5) implies

$$(p(z) - p(y))(T_k(z) - T_k(y)) \ge a\,|T_k(z) - T_k(y)|^2 \log^d(1 + |T_k(z) - T_k(y)|)$$

$$\geq \limsup_{\delta\to 0+} \|T_k(\varrho_\delta) - T_k(\varrho)\|^2_{y^2\log^d(y)}. \tag{4.44}$$

On the other hand,

$$\limsup_{\delta\to 0+} \int_0^T \int_\Omega \text{div } \vec{u}_\delta T_k(\varrho_\delta) - \text{div } \vec{u}\,\overline{T_k(\varrho)}\,\mathrm{d}x\,\mathrm{d}t =$$

$$\limsup_{\delta\to 0+} \int_0^T \int_\Omega \left(T_k(\varrho_\delta) - T_k(\varrho) + T_k(\varrho) - \overline{T_k(\varrho)}\right) \text{div } \vec{u}_\delta\,\mathrm{d}x\,\mathrm{d}t \leq$$

$$\leq 2\sup_\delta \|\text{div } \vec{u}_\delta\|_{L^2((0,T)\times\Omega)} \limsup_{\delta\to 0+} \|T_k(\varrho_\delta) - T_k(\varrho)\|_{L^2((0,T)\times\Omega)} \leq$$

$$\leq c \limsup_{\delta\to 0+} \|T_k(\varrho_\delta) - T_k(\varrho)\|_{y^2\log^d(y)} \leq c + \frac{1}{2}\limsup_{\delta\to 0+} \|T_k(\varrho_\delta) - T_k(\varrho)\|^2_{y^2\log^d(y)} \tag{4.45}$$

where we used also (3.7) and the Young inequality in the final step.

The relations (4.44), (4.45) combined with Lemma 4.3 yield the desired conclusion.

Q.E.D.

4.5 The renormalized solutions

We are going to use the conclusion of Lemma 4.4 to show the following result (compare [2], Lemma 4.4):

Lemma 4.5: *The limit functions ϱ, $\vec{u}$ solve (4.29) in the sense of renormalized solution, i.e.,*

$$\partial_t b(\varrho) + \text{div } (b(\varrho)\vec{u}) + (b'(\varrho)\varrho - b(\varrho))\text{div } \vec{u} = 0 \tag{4.46}$$

holds in $D'((0,T)\times\mathbb{R}^2)$ provided ϱ, $\vec{u}$ are set zero outside Ω for any $b \in \mathrm{C}^1(\mathbb{R})$ satisfying

$$b'(z) \equiv 0 \text{ for all } z \in \mathbb{R} \text{ large enough, say, } z \geq M \tag{4.47}$$

where the constant M may vary for different functions b.

Proof: As in [5], Lemma 2.3, we can regularize equation (4.33) to obtain

$$\partial_t S_m[\overline{T_k(\varrho)}] + \text{div } (S_m[\overline{T_k(\varrho)}]\vec{u}) + S_m[\overline{(T_k'(\varrho)\varrho - T_k(\varrho))\,\text{div } \vec{u}}] = r_m \tag{4.48}$$

where $S_m[v] = \vartheta_m * v$ are the standard smoothing operators and $r_m \to 0$ in $\mathrm{L}^2((0,T)\times\mathbb{R}^2)$ as $m \to \infty$ for any fixed k.

Now, we are allowed to multiply (4.48) by $b'(S_m[\overline{T_k(\varrho)}])$. Letting $m \to \infty$ we deduce

Our next goal will be passing $k \to \infty$ in (4.49).

By virtue of (3.4), we have

$$\|\overline{T_k(\varrho)} - \varrho\|_{y\log^\gamma(y)} \leq \liminf_{\delta\to 0+} \|T_k(\varrho_\delta) - \varrho_\delta\|_{y\log^\gamma(y)} \leq$$

$$\leq \liminf_{\delta\to 0+} \|\chi_{[\varrho_\delta \geq k]} \left(T_k(\varrho_\delta) - \varrho_\delta\right)\|_{y\log^\gamma(y)} \leq$$

$$\leq \liminf_{\delta\to 0+} \|\chi_{[\varrho_\delta\geq k]}\|_{e\left(\frac{1}{d+1-\gamma}\right)} \|T_k(\varrho_\delta) - \varrho_\delta\|_{y\log^{d+1}(y)} \text{ for } \gamma \in (0, d+1). \tag{4.50}$$

As ϱ_δ are bounded in $\mathrm{L}_{y\log^{d+1}(y)}(\Omega)$ uniformly in δ, we see that there exists a function $r : (0,\infty) \to (0,\infty)$ such that

$$\lim_{k\to\infty} r(k) = 0, \quad \text{and} \quad \mu\big\{(x,t) \in (0,T)\times\Omega \,|\, \varrho_\delta(x,t) \geq k\big\} \leq r(k), \text{ for } \delta \in (0,1). \tag{4.51}$$

By virtue of the definition of the Luxembourg norm and using (4.51), we easily observe that for any $\omega > 0$ it holds

$$\|\chi_{[\varrho_\delta\geq k]}\|_{e\left(\frac{1}{\omega}\right)} \leq \left(\phi^{-1}\left(\frac{1}{r(k)}\right)\right)^{-1} = q_\omega(k), \qquad \lim_{k\to\infty} q_\omega(k) = 0, \tag{4.52}$$

where ϕ is the Young function corresponding to the space $\mathrm{L}_{e\left(\frac{1}{\omega}\right)}((0,T)\times\Omega)$.

The relations (4.50) and (4.52) (for $\omega = d+1-\gamma > 0$) imply that

$$\overline{T_k(\varrho)} \to \varrho \text{ as } k \to \infty \text{ in } \mathrm{L}_{y\log^\gamma(y)}((0,T)\times\Omega) \text{ for any } \gamma \in (0, d+1).$$

Thus (4.49) will imply (4.46) provided we show

$$b'(\overline{T_k(\varrho)}) \left[\overline{(T_k(\varrho) - T_k'(\varrho)\varrho)\,\mathrm{div}\,\vec{u}}\right] \to 0 \text{ in } \mathrm{L}^1((0,T)\times\Omega) \text{ as } k \to \infty. \tag{4.53}$$

To this end, let us denote

$$J_{k,M} = \big\{(t,x) \in (0,T)\times\Omega \,|\, \overline{T_k(\varrho)} \leq M\big\},$$

where M is the constant related to b by (4.47). One has

$$\int_0^T \int_\Omega \left| b'(\overline{T_k(\varrho)}) \left[\overline{(T_k(\varrho) - T_k'(\varrho)\varrho)\,\mathrm{div}\,\vec{u}}\right]\right| \,\mathrm{d}x\,\mathrm{d}t \leq$$

$$\leq \sup_{0\leq z\leq M} |b'(z)| \iint_{J_{k,M}} \left|\overline{(T_k(\varrho) - T_k'(\varrho)\varrho)\,\mathrm{div}\,\vec{u}}\right| \,\mathrm{d}x\,\mathrm{d}t \leq$$

$$\leq c \liminf_{\delta\to 0+} \|T_k(\varrho_\delta) - T_k'(\varrho_\delta)\varrho_\delta\|_{L^2(J_{k,M})}. \tag{4.54}$$

Now, using (3.7) and (4.52) we see

$$\|T_k(\varrho_\delta) - T_k'(\varrho_\delta)\varrho_\delta\|_{L^2(J_{k,M})} = \|\chi_{[\varrho_\delta \geq k]}\left(T_k(\varrho_\delta) - T_k'(\varrho_\delta)\varrho_\delta\right)\|_{L^2(J_{k,M})} \leq$$

$$\leq \|\chi_{[\varrho_\delta\geq k]}\|_{e\left(\frac{2}{d}+1\right)} \|T_k(\varrho_\delta) - T_k'(\varrho_\delta)\varrho_\delta\|_{L_{y^2\log^d(y)}(J_{k,M})} \leq$$

$$\leq q_{\frac{d}{2+d}}(k)\|T_k(\varrho_\delta) - T_k'(\varrho_\delta)\varrho_\delta\|_{L_{y^2\log^d(y)}(J_{k,M})}, \tag{4.55}$$

and, since $T_k'(z)z \leq T_k(z)$,

$$\|T_k(\varrho_\delta) - T_k'(\varrho_\delta)\varrho_\delta\|_{L_{y^2\log^d(y)}(J_{k,M})} \leq 2\|T_k(\varrho_\delta)\|_{L_{y^2\log^d(y)}(J_{k,M})} \leq$$

$$\leq 2\|T_k(\varrho_\delta) - T_k(\varrho)\|_{y^2\log^d(y)} + 2\|T_k(\varrho) - \overline{T_k(\varrho)}\|_{y^2\log^d(y)} + 2\|\overline{T_k(\varrho)}\|_{L_{y^2\log^d(y)}(J_{k,M})} \leq$$

$$\leq 2\|T_k(\varrho_\delta) - T_k(\varrho)\|_{y^2\log^d y} + 2\|T_k(\varrho) - \overline{T_k(\varrho)}\|_{y^2\log^d(y)} + 2M\mu(\Omega), \tag{4.56}$$

and, by virtue of Lemma 4.4 and (4.56), one gets

$$\limsup_{\delta\to 0+} \|T_k(\varrho_\delta) - T_k'(\varrho_\delta)\varrho_\delta\|_{L_{y^2\log^d(y)}(J_{k,M})} \leq 4c + 2M\mu(\Omega)$$

which, together with (4.54), (4.55) and (4.52), completes the proof of (4.53). Thus, Lemma 4.5 has been proved.

Q.E.D.

4.6 Strong convergence of the density

We are going to complete the proof of Theorem 1. To this end, we introduce a family of functions $L_k \in \mathrm{C}^1(\mathbb{R})$:

$$L_k(z) = \begin{cases} z\log(z) & \text{for } 0 \leq z < k, \\ z\log(k) + z\int_k^z \frac{T_k(s)}{s^2}\,\mathrm{d}s & \text{for } z \geq k. \end{cases}$$

Seeing that L_k can be written as

$$L_k(z) = \beta_k z + b_k(z) \tag{4.57}$$

Similarly, by virtue of Lemma 4.5 and (4.29),

$$\partial_t L_k(\varrho) + \mathrm{div}\ (L_k(\varrho)\vec{u}) + T_k(\varrho)\mathrm{div}\ \vec{u} = 0 \quad \text{in } D'((0,T)\times\mathbb{R}^2). \tag{4.59}$$

In view of (4.58), (4.5) and Lemma 3.5, we can assume

$$L_k(\varrho_\delta) \to \overline{L_k(\varrho)} \text{ in } C([0,T]; \mathrm{L}^{*-weak}_{y\log^{d+1}(y)}(\Omega)) \quad \text{and} \quad \text{in } C([0,T]; \mathrm{W}^{-1,2}(\Omega)). \tag{4.60}$$

Taking the difference of (4.58) and (4.59) and integrating with respect to t, we get

$$\int_\Omega (L_k(\varrho_\delta) - L_k(\varrho))(t)\phi \,\mathrm{d}x =$$

$$+\int_0^t \int_\Omega (L_k(\varrho_\delta)\vec{u}_\delta - L_k(\varrho)\vec{u})\cdot\nabla\phi + (T_k(\varrho)\mathrm{div}\ \vec{u} - T_k(\varrho_\delta)\mathrm{div}\ \vec{u}_\delta)\phi \,\mathrm{d}x\,\mathrm{d}t$$

for any $\phi \in D(\Omega)$. Passing to the limit for $\delta \to 0$ and making use of (4.24) and (4.60), one obtains

$$\int_\Omega (\overline{L_k(\varrho)} - L_k(\varrho))(t)\phi \,\mathrm{d}x = \tag{4.61}$$

$$= \int_0^t \int_\Omega \left(\overline{L_k(\varrho)} - L_k(\varrho)\right)\vec{u}\cdot\nabla\phi \,\mathrm{d}x\,\mathrm{d}t + \lim_{\delta\to 0+}\int_0^t\int_\Omega (T_k(\varrho)\mathrm{div}\ \vec{u} - T_k(\varrho_\delta)\mathrm{div}\ \vec{u}_\delta)\phi\,\mathrm{d}x\,\mathrm{d}t$$

for any $\phi \in D(\Omega)$. As the velocity components u^i, $i = 1,2$ belong to $\mathrm{L}^2(0,T;\mathrm{W}^{1,2}_0(\Omega))$, it holds

$$\frac{|\vec{u}|}{\mathrm{dist}[x,\partial\Omega]} \in \mathrm{L}^2(0,T;\mathrm{L}^2(\Omega)). \tag{4.62}$$

Let us consider a sequence of functions $\phi_m \in D(\Omega)$ such that

$$0 \le \phi_m \le 1,\ \phi_m(x) = 1 \ \text{ for all } x \text{ such that } \ \mathrm{dist}[x,\partial\Omega] \ge \frac{1}{m} \quad \text{and}$$

$$|\nabla\phi_m(x)| \le 2m \quad \text{for all } x \in \Omega.$$

Taking functions ϕ_m as test functions in (4.61), passing to the limit for $m \to \infty$ and making use of (4.62), one derives

$$\int_\Omega (\overline{L_k(\varrho)} - L_k(\varrho))(t)\,\mathrm{d}x = \int_0^t\int_\Omega T_k(\varrho)\mathrm{div}\ \vec{u}\,\mathrm{d}x\,\mathrm{d}t - \lim_{\delta\to 0+}\int_0^t\int_\Omega T_k(\varrho_\delta)\mathrm{div}\ \vec{u}_\delta\,\mathrm{d}x\,\mathrm{d}t. \tag{4.63}$$

Observe that the term $\overline{L_k(\varrho)} - L_k(\varrho)$ is bounded in view of (4.57).

At this stage, the main idea is to let $k \to \infty$ in (4.63). By virtue of (4.5), we can assume since, making of use (3.4), (4.52) and (4.5),

$$\|\overline{L_k(\varrho)} - \overline{\varrho\log(\varrho)}\|_{L^\infty(0,T;L_{y\log^\gamma(y)}(\Omega))} \le \liminf_{\delta\to0+} \sup_{t\in[0,T]} \|L_k(\varrho_\delta) - \varrho_\delta\log(\varrho_\delta)\|_{y\log^\gamma(y)} \le$$

$$\le c\, q_{d-\gamma}(k) \sup_\delta \sup_{t\in[0,T]} \|\varrho_\delta(t)\|_{y\log^{d+1}(y)} \to 0 \text{ as } k\to\infty.$$

Similarly, we obtain

$$\overline{L_k(\varrho)} \to \overline{\varrho\log(\varrho)} \text{ in } \mathrm{L}^\infty(0,T;\mathrm{L}_{y\log^\gamma(y)}(\Omega)) \quad \text{for any } \gamma\in(1,d). \tag{4.65}$$

Finally, by virtue of Lemma 4.3 and the monotonicity of the pressure (cf. (4.43) and (4.44)), we can estimate the right hand side of (4.63):

$$\int_0^t\int_\Omega T_k(\varrho)\mathrm{div}\,\vec{u}\,\mathrm{d}x\,\mathrm{d}t - \lim_{\delta\to0+}\int_0^t\int_\Omega T_k(\varrho_\delta)\mathrm{div}\,\vec{u}_\delta\,\mathrm{d}x\,\mathrm{d}t \le$$

$$\le \int_0^t\int_\Omega (T_k(\varrho) - \overline{T_k(\varrho)})\mathrm{div}\,\vec{u}\,\mathrm{d}x\,\mathrm{d}t. \tag{4.66}$$

By virtue of Lemma 4.4 and (4.52), the right-hand side of (4.66) tends to zero as $k\to\infty$. Now, we can pass to the limit for $k\to\infty$ in (4.63) to conclude

$$\int_\Omega \overline{\varrho\log(\varrho)} - \varrho\log(\varrho)\,\mathrm{d}x\,(t) = 0 \;\text{ for a.e. } t\in[0,T]. \tag{4.67}$$

Because of the convexity of the function $z\to z\log z$, we have

$$\overline{\varrho\log(\varrho)} \ge \varrho\log(\varrho) \;\text{ a.e. in } (0,T)\times\Omega$$

which, combined with (4.67), gives

$$\overline{\varrho\log(\varrho)} = \varrho\log(\varrho) \;\text{ a.e. in } (0,T)\times\Omega. \tag{4.68}$$

By virtue of (4.5), we can assume

$$\left(\frac{\varrho+\varrho_\delta}{2}\right)\log\left(\frac{\varrho+\varrho_\delta}{2}\right) \to w \;\text{ weakly star in } \mathrm{L}^\infty(0,T;\mathrm{L}_{y\log^d(y)}(\Omega))$$

where, in view of convexity, $w\ge\varrho\log\varrho$. Thus, using convexity and (4.68),

$$0\le h_\delta = \frac{1}{2}\varrho_\delta\log(\varrho_\delta) + \frac{1}{2}\varrho\log(\varrho) - \left(\frac{\varrho+\varrho_\delta}{2}\right)\log\left(\frac{\varrho+\varrho_\delta}{2}\right) \to \varrho\log(\varrho) - w$$

weakly star in $\mathrm{L}^\infty(0,T;\mathrm{L}_{y\log^d(y)}(\Omega))$. As $\varrho\log(\varrho)-w\le0$, we have weak star convergence

Now, we are able to prove that even the sequence ϱ_δ converge in measure to ϱ. To this end, let us fix $\sigma > 0$. Then

$$\mu\{(t,x) \in (0,T) \times \Omega : |\varrho_\delta(x,t) - \varrho(x,t)| \geq \sigma\} \leq$$

$$\leq \tilde{r}(k) + \mu\Big\{(t,x) \in (0,T) \times \Omega : \varrho_\delta \leq k \ \& \ \varrho \leq k \ \& \ |\varrho_\delta(x,t) - \varrho(x,t)| \geq \sigma\Big\} \tag{4.69}$$

where $\tilde{r}(k)$ tends to zero for $k \to \infty$ independently of δ (cf. (4.51)). The second term on the right-hand side of (4.69) tends to zero for $\delta \to 0$ and fixed k since $h_\delta \to 0$ in measure and

$$h_\delta \geq \frac{1}{8k}|\varrho - \varrho_\delta|^2 \ \text{ for } \varrho \leq k \text{ and } \varrho_\delta \leq k,$$

consequently, the sequence ϱ_δ tends to ϱ in measure.

The convergence in measure and the convergence of L^1-norms, in particular

$$\|\varrho_\delta\|_{L^1((0,T)\times\Omega)} = \int_\Omega \varrho_\delta \,\mathrm{d}t \to \int_\Omega \varrho \,\mathrm{d}t = \|\varrho\|_{L^1((0,T)\times\Omega)}, \tag{4.70}$$

imply strong convergence of the sequence ϱ_δ in $L^1((0,T) \times \Omega)$. In particular, it implies (4.31).

By virtue of (4.24), (4.25) and (4.70), we can pass in (4.9) to the limit for $\delta \to 0$ and we can derive the energy inequality (1.7) (using also convexity of the functions behind the integral in (1.8)).

Thus Theorem 1 has been proved.

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