Brownian motion, random walks and Lévy processes as limits of shift-periodic maps

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Abstract

A shift-periodic map is a one-dimensional map from the real line to itself which is periodic up to a linear translation and allowed to have singularities. It is shown that iterative sequences generated by such maps display rich dynamical behaviour. They converge in certain limits to both discrete and continuous stochastic processes, including Brownian motion, more general Lévy processes and various types of random walks, depending on the properties of the generating shift-periodic map.

1 Introduction

Dynamical systems and their stochastic properties have been studied for more than a hundred years, starting with the pioneering works of Poincaré. He first connected probabilistic concepts with dynamics, conjecturing the Poincaré recurrence theorem [1]. Major advances in the field were made in the 1930s by Birkhoff [2] and von Neumann [3], via the proof of so called ergodic theorems, concerning time averages of functions along trajectories. Birkhoff also first used topological methods for the study of dynamical systems. In these early years differential equations were often the main focus of the study of dynamical systems. However, since the 1970s attention turned to simple dynamical systems, generated iteratively from a map $F : \Omega \to \Omega$ via an equation

$$x_{n+1} = F(x_n), \tag{1.1}$$

where Ω has been taken to be a low-dimensional set [4], such as interval $[0, 1]$. It has been observed that even very simple maps and systems can give rise to complicated, seemingly random behaviour of trajectories, a phenomenon Yorke and Li named "chaos" in their seminal paper [5]. An example of this concept was given by Robert May in [6] with the logistic map $F(x; r) = rx(1 - x)$. Depending on parameter r, it displays a wide array of behaviour of trajectories, highly sensitive to the initial value.

While sequence (x_n) generated from equation (1.1) is fully deterministic when x_0 is known, it can instead be viewed as a discrete-time stochastic process when initial value x_0 is chosen according to a probability distribution on Ω and a suitable scaling can also lead to continuous-time stochastic processes. A notable example is Brownian motion, obtained as a limit after an appropriate scaling in space and time for a specific class of maps F, as demonstrated by Beck and Roepstorff in [7, 8]. Their approach to convergence to the Ornstein-Uhlenbeck process was further investigated by various other authors, for example by Mackey and Tyran-Kamińska [9, 10], who derived various central limit theorems for chaotic, deterministic semi-dynamical systems and showed how Brownian motion can be obtained from a deterministic system using these results.

In some application areas, for example financial modelling [11] or network traffic [12], continuous-time stochastic processes with heavy-tailed increments, rather than normally distributed increments, have been observed to give a better model than Brownian motion. While such Lévy motions can arise as limits of a stochastic process [13], they have not yet been investigated as a limit of iterates (1.1) of one-dimensional maps, which we discuss in this paper. When the behaviour

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of (x_n) as a stochastic process is studied, it is common to choose the distribution of x_0 on Ω in such a way that F is measure preserving, ensuring that all x_n are identically distributed. This restricts the stochastic processes we might obtain in a limit. Although, for many maps on bounded intervals a corresponding invariant measure can be constructed [14], for maps on unbounded Ω this problem becomes harder. However, if we chose to restrict our attention to bounded intervals, the random variables generated by equation (1.1) would have finite variance, so that in a scaling limit, we would obtain normal distribution of increments.

The key issue in the above is that F shall map Ω to itself. However, some authors have also investigated the behaviour of trajectories generated instead by maps $F : \Omega \to F(\Omega)$ with $\Omega \subsetneqq F(\Omega)$ until the point of escape from Ω, especially when the holes, i.e. $\Omega \setminus F^{-1}(\Omega)$, are small compared to Ω. Early results are due to Pianigiani and Yorke [15], who motivated the discussion with the example of a billiard table with chaotic trajectories, and introduced the concept of conditionally invariant measures. This idea has been investigated further by other authors, for example, Demers and Young studied escape rates through the small holes [16].

In this paper, we combine these two approaches, studying sequences (x_n) generated via maps $F : [0, 1] \to \mathbb{R}$ with $x_{n+1} = F(\{x_n\}) + \lfloor x_n \rfloor$, where $\{x_n\}$ is the fractional part and $\lfloor x_n \rfloor$ is the integer part of x_n. Equivalently, we generate (x_n) via earlier equation (1.1) with a map satisfying $F(x) = F(\{x\}) + \lfloor x \rfloor$. This way, it is only necessary to find invariant distributions on $[0, 1]$, while simultaneously allowing the consideration of maps which generate random variables with infinite higher-order moments, so that we obtain a much larger variety of different behaviour of trajectories. Such shift-periodic maps are formally defined in the next section, when we illustrate the dynamics of (1.1) using simple examples of shift-periodic maps. In Section 3, we prove the convergence of dynamics of (1.1) to discrete time random walks, while the convergence to continuous-time random walks is established for a subclass of shift-periodic maps in Section 4.

Notation. We denote $\mathbb{R} \cup \{\infty\} \cup \{-\infty\}$ by $\mathbb{R}_\infty$. For any $x \in \mathbb{R}$ let $\{x\}$ denote the fractional part of x and $\lfloor x \rfloor$ denote the integer part of x. For any Lebesgue measurable set A we denote its Lebesgue measure by $\lambda(A)$. As it is common in the literature, the symbol $\sim$ will be used in two different contexts. First, for functions $f(x)$ and $g(x)$ we write $f \sim g$ if $f(x)/g(x) \to \infty$ as $x \to \infty$. Second, we also use symbol $\sim$ to specify the distribution of a random variable, for example, $X \sim N(0, 1)$ means that random variable X is normally distributed with zero mean and unit variance. In Section 4 we also make use of the sup-norm on the space of bounded functions from $[0, 1]$ to $\mathbb{R}$, defined by $||f||_\infty = \sup_{x\in[0,1]} |f(x)|$.

2 Shift-Periodic Maps

This paper studies the behaviour of iterative sequences given by (1.1) for functions F defined on the real line, which are periodic up to integer shifts. The key property of maps F is a shift-periodic formula given in the next definition as condition (i), together with technical restrictions (ii) and (iii) on properties of F. Note that, unlike in other works on this topic in the literature, we allow F to have singularities.

Definition 2.1. *A shift-periodic map is a map $F : \mathbb{R} \to \mathbb{R}_\infty$ with the following properties:*

(i) $F(x) = F(\{x\}) + \lfloor x \rfloor$ *for all* $x \in \mathbb{R}$*;*

(ii) *There exist* $0 = t_0 < t_1 < \cdots < t_k = 1$ *so that F is continuous, monotonic on* (t_{i-1}, t_i) *for* $i = 1, 2, \ldots, k$*;*

(iii) $|F(x) - F(y)| > |x - y|$ *holds for all distinct* $x, y \in (t_{i-1}, t_i)$ *when* $i = 1, 2, \ldots, k$*.*

This definition includes both continuous functions and functions with singularities. Points t_i, $i = 0, 1, 2, \ldots, k$, in Definition 2.1(ii) are either local extrema of map F or its singularities, i.e. points where $F(t_i) = \infty$ or $F(t_i) = -\infty$. Our first example of a shift-periodic map is a simple continuous piecewise linear map, which can demonstrate random walk properties of iterations (1.1). It has one local maximum and one local minimum in interval $[0, 1]$ and maps interval $[0, 1]$ to a larger interval, $[-\delta/4, 1 + \varepsilon/4]$, for parameters $\delta > 0$ and $\varepsilon > 0$. An example of this map is plotted in Figure 1(a) (as a red solid line) and it is formally defined below.

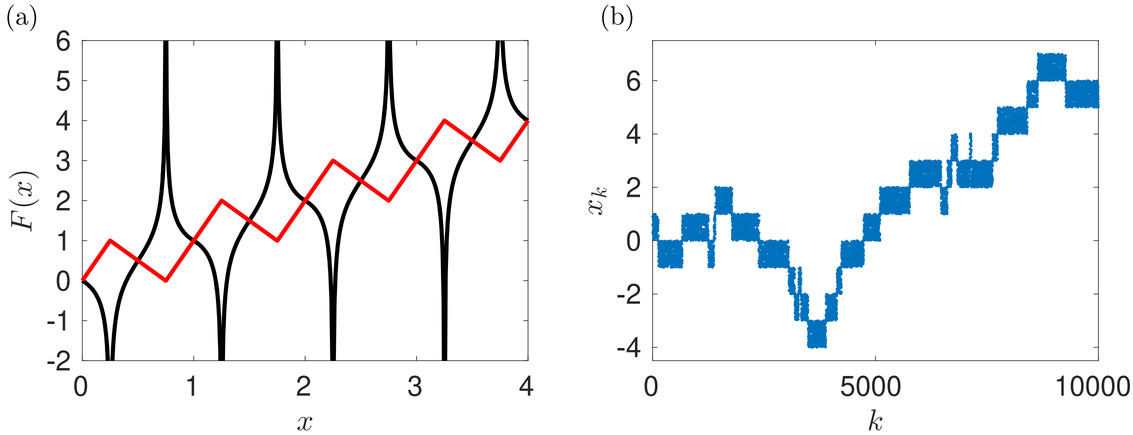

Figure 1: (a) *Two examples of shift-periodic maps. Continuous piecewise linear map* $F(x;\varepsilon,\delta)$ *given in Example 2.1 for* $\delta=\varepsilon=10^{-2}$ *(red solid line) and shift-periodic map with singularities* $F(x;\kappa)$ *given in Example 2.2 for* $\kappa=1$ *(black solid line).*
(b) *Illustrative dynamics of shift-periodic map* $F(x;\varepsilon,\delta)$ *from Example 2.1 for* $\delta=\varepsilon=10^{-2}$. *The first* 10^4 *iterations* $x_{k+1}=F(x_k;10^{-2},10^{-2})$ *are plotted for initial condition* $x_0=0.9$.

Example 2.1. We consider piecewise linear map $F:\mathbb{R}\to\mathbb{R}$ with parameters $\delta>0$ and $\varepsilon>0$, which is defined on $[0,1]$ by

$$F(x;\varepsilon,\delta):=\begin{cases}(4+\varepsilon)x, & \text{if} \quad x\in\left[0,\dfrac{1}{4}\right);\\ (-2-\varepsilon)x+\dfrac{(3+\varepsilon)}{2}, & \text{if} \quad x\in\left[\dfrac{1}{4},\dfrac{1}{2}\right);\\ (-2-\delta)x+\dfrac{(3+\delta)}{2}, & \text{if} \quad x\in\left[\dfrac{1}{2},\dfrac{3}{4}\right);\\ (4+\delta)x-(3+\delta), & \text{if} \quad x\in\left[\dfrac{3}{4},1\right],\end{cases}$$

and with $F(x;\varepsilon,\delta)=F(\{x\};\varepsilon,\delta)+\lfloor x\rfloor$ for $x\in\mathbb{R}$.

Choosing relatively small values $\delta=\varepsilon=10^{-2}$, first 10^4 iterations of map from Example 2.1 are shown in Figure 1(b). Identifying intervals $[i,i+1)$ with integer valued lattice points $\{i\}$ for $i\in\mathbb{Z}$, we observe that sequence x_k can be viewed as a random walk between these lattice points. More precisely, we can map sequence x_k to integer-valued sequence by $\lfloor x_k\rfloor$, which gives lattice positions of a random walker that is jumping from site $\{i\}$ to neighbouring sites $\{i-1\}$ and $\{i+1\}$ with certain probabilities. In Section 4, we show how properties of such random walks depend on properties of F and investigate limits in which the resulting random walk has independent waiting times. Note that the behaviour of iterations (1.1) depends on the initial condition x_0 and there are initial conditions for which the sequence is eventually constant. For instance, when considering the map in Example 2.1, if x_0 is an integer, then all values of the sequence are equal to x_0. However, such a behaviour is relatively rare since the set of these special initial conditions is of measure zero.

A more general example of a shift-periodic map is illustrated in Figure 1(a) as a black solid line and is formally defined as Example 2.2. It has two singularities in $[0,1]$, at one of them approaching ∞ and at the other one approaching $-\infty$ and maps interval $[0,1]$ to $\mathbb{R}_\infty$.

Example 2.2. We consider $F:\mathbb{R}\to\mathbb{R}_\infty$ with parameter $\kappa>0$, defined on $[0,1]$ by

$$F(x;\kappa)=\frac{4^{-1/\kappa}}{2(1-3^{-1/\kappa})}\left(\left|x-\frac{3}{4}\right|^{-1/\kappa}-\left|x-\frac{1}{4}\right|^{-1/\kappa}\right)+\frac{1}{2},$$

and with $F(x;\kappa)=F(\{x\};\kappa)+\lfloor x\rfloor$ for $x\in\mathbb{R}$. The prefactor is chosen so that $F(0;\kappa)=0$ and $F(1;\kappa)=1$.

In Figure 2, we plot illustrative trajectories for two different values of κ. For large κ (panel (a)), the behaviour of iterations $x_{k+1}=F(x_k;\kappa)$ resembles Brownian motion, while for small κ (panel

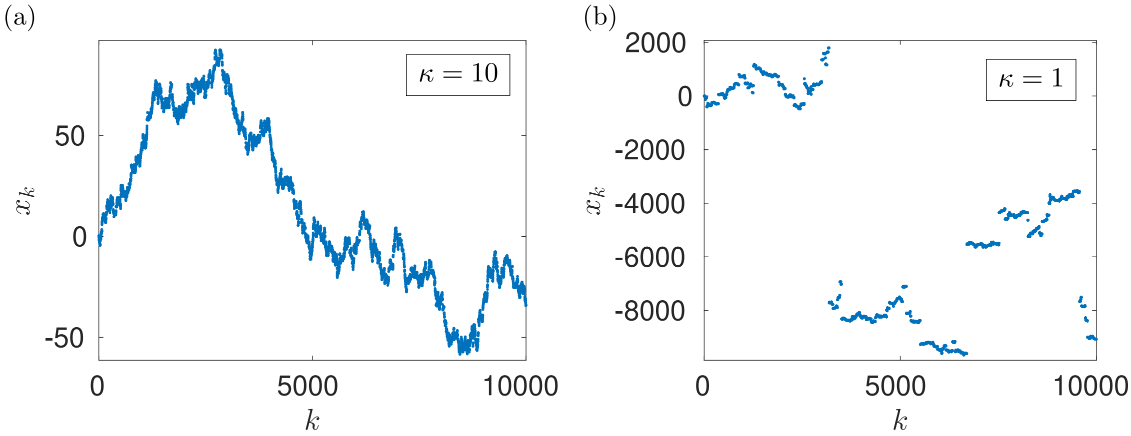

Figure 2: *Illustrative dynamics of shift-periodic map $F(x;\kappa)$ from Example 2.2. The first 10^4 iterations $x_{k+1} = F(x_k;\kappa)$ are plotted* (a) *for $\kappa = 10$ and initial condition $x_0 = 0.4$;* (b) *for $\kappa = 1$ and initial condition $x_0 = 0.3$.*

(b)) it resembles Lévy flights. We write "resembles" since we compare discrete dynamics with continuous time stochastic processes. In Section 3 we make these statements rigorous. To do this, we identify the index k in x_k with time and introduce suitable scaling of time to get convergence to a continuous time process. Before that, we study the random walk behaviour of a certain class of shift-periodic maps when time is left unscaled.

3 Discrete-time Random Walks

While the iterative formula (1.1) uniquely determines the next iterate x_{k+1} from the knowledge of x_k, Figure 1(b) suggests that the next value $\lfloor x_{k+1} \rfloor$ is determined from $\lfloor x_k \rfloor$ only with a certain probability. The goal of this section is to formalise this observation for certain shift-periodic maps by studying the connections between the dynamics of (1.1) and random walks.

Definition 3.1. *Consider a discrete-time stochastic process $(X_n)_{n\in\mathbb{N}}$ and define $Y_n = X_n - X_{n-1}$, for $n \in \mathbb{N}$, where we assume $X_0 = 0$. We say $(X_n)_{n\in\mathbb{N}}$ is a discrete-time random walk if Y_n, $n \in \mathbb{N}$, are independent and identically distributed.*

The next definition introduces a restriction on shift-periodic maps which for an appropriate distribution of the initial value guarantees that the behaviour of a sequence generated by such a shift-periodic map will be that of a random walk. The necessity of this condition will be discussed later on by considering Example 2.1.

Definition 3.2. *Let F be a shift-periodic map with $0 = t_0 < t_1 < \cdots < t_k = 1$ such that F is continuous and monotonic on (t_{i-1}, t_i). We then say that F has integer spikes if for $i \in \{0, 1, \ldots, k-1\}$ and for $j \in \{1, 2, \ldots, k\}$*

$$\lim_{x\to t_i^+} F(x) \in \mathbb{Z} \cup \{\infty\} \cup \{-\infty\} \qquad \text{and} \qquad \lim_{x\to t_j^-} F(x) \in \mathbb{Z} \cup \{\infty\} \cup \{-\infty\}.$$

Note that Example 2.2 satisfies the definition of being a shift-periodic map with integer spikes, where $t_0 = 0$, $t_1 = 1/4$, $t_2 = 3/4$ and $t_3 = 1$. On the other hand, Example 2.1 in general does not satisfy Definition 3.2, except when ε, δ are both integer multiples of 4. Utilising Definition 3.2, the following theorem holds.

Theorem 3.1. *Let $F : \mathbb{R} \to \mathbb{R}_\infty$ be a shift-periodic map with integer spikes and let U be uniformly distributed on $[0, 1]$. For any $m \in \mathbb{Z}$ let*

$$p_m = \lambda\{x \in [0, 1] : \lfloor F(x) \rfloor = m\}.$$

There exists a map $h : [0,1] \to [0,1]$ so that for $X_n = \lfloor F^n(h(U)) \rfloor$ and $X_0 = 0$, the stochastic process $(X_n)_{n\in\mathbb{N}}$ is an integer-valued discrete-time random walk and for any $m \in \mathbb{Z}$

$$\mathbb{P}(X_n - X_{n-1} = m) = p_m.$$

Before proving Theorem 3.1, we investigate a simple case, which will also be a key element in the proof of the full theorem in subsection 3.3.

3.1 Piecewise linear maps

Theorem 3.1 is easy to verify for maps which are linear in between integer function values, in this case map h can simply be taken to be the identity map. We will call such maps "linear between grid lines" and make the definition precise below.

Definition 3.3. *Let F be shift-periodic with integer spikes. F is linear between grid lines if whenever $x, y \in [0,1]$, $x < y$, with $F(x), F(y) \in \mathbb{Z}$, F continuous on (x,y), but $F((x,y)) \cap \mathbb{Z} = \emptyset$, then F is linear on (x,y).*

Lemma 3.1. *Let $F : \mathbb{R} \to \mathbb{R}_\infty$ be a shift-periodic map with integer spikes which is linear between grid lines and let U be uniformly distributed on $[0,1]$. Let*

$$p_m = \lambda\{x \in [0,1] : \lfloor F(x) \rfloor = m\}.$$

Then for $X_n = \lfloor F^n(U) \rfloor$ and $X_0 = 0$, the random process $(X_n)_{n\in\mathbb{N}}$ is a integer-valued discrete-time random walk with $\mathbb{P}(X_n - X_{n-1} = m) = p_m$.

In the proof of this Lemma and all further proofs we will associate with a shift-periodic map F a map on the unit interval, which we call the "restricted map", $F_r : [0,1] \to [0,1]$ given by

$$F_r(x) = \begin{cases} \{F(x)\} & \text{if } F(x) \notin \{\infty, -\infty\}; \\ 0 & \text{if } F(x) \in \{\infty, -\infty\}. \end{cases}$$

Note that the conditions placed on shift-periodic map F, in particular (i), ensure that whenever $\{F(x), F^2(x), \ldots, F^n(x)\}$ does not coincide with $\{\infty, -\infty\}$ then $F_r^n(x) = \{F^n(x)\}$ and

$$F^n(x) = F_r^n(x) + \lfloor F(x) \rfloor + \sum_{k=2}^{n} \lfloor F(F_r^{k-1}(x)) \rfloor \quad \text{for} \quad n \geq 2. \tag{3.1}$$

The crucial property of shift-periodic maps with integer spikes is that the corresponding restricted map consists of a countable number of linear pieces, each of which starts at 0 and ends at 1, or the other way around.

Proof of Lemma 3.1. Let $F : \mathbb{R} \to \mathbb{R}_\infty$ be a shift-periodic map with integer spikes which is linear between grid lines and let U be uniformly distributed on $[0,1]$. Let $X_n = \lfloor F^n(U) \rfloor$ for $n \geq 0$. There is a collection of open, pairwise disjoint intervals $\{(a_i, b_i) : i \in I\}$ with countable indexing set I, so that $a_i < b_i$,

$$[0,1] \setminus \bigcup_{i\in I}(a_i, b_i) \text{ is countable} \qquad \text{and} \qquad F_r \text{ is linear on } (a_i, b_i) \text{ with } F_r((a_i, b_i)) = (0,1).$$

$\lfloor F(x) \rfloor$ is constant on (a_i, b_i). For $m \in \mathbb{Z}$ let $I_m = \{i \in I : \lfloor F(x) \rfloor = m \text{ if } x \in (a_i, b_i)\}$ and then $p_m = \lambda(\cup_{i\in I_m}(a_i, b_i))$. Write $Y_n = X_n - X_{n-1}$. Now $Y_1 = \lfloor F(U) \rfloor$ and, using expression (3.1), we get that $Y_n = \lfloor F(F_r^{n-1}(U)) \rfloor$ for $n \geq 2$. For any subset $S \subseteq [0,1]$ we define

$$F_r^{-n+1}(S) = \{x \in [0,1] : F_r^{n-1}(x) \in S\}.$$

Then we see $\mathbb{P}(Y_n = m) = \lambda(\cup_{i\in I_m} F_r^{-n+1}(a_i, b_i))$. If we fix $k \in \mathbb{N}$ and choose $m_1, m_2, \ldots, m_k \in \mathbb{Z}$ then

$$\mathbb{P}(Y_1 = m_1, \ldots, Y_k = m_k) = \lambda\left(\bigcap_{n=1}^{k}\left(\bigcup_{i\in I_{m_n}} F_r^{-n+1}(a_i, b_i)\right)\right). \tag{3.2}$$

Now consider a measurable subset $S \subseteq (0,1)$ and interval (a_i, b_i) for some $i \in I_m$. F_r is linear on (a_i, b_i) with image $(0,1)$, so that $\lambda(F_r^{-1}(S) \cap (a_i, b_i)) = (b_i - a_i)\lambda(S)$. Summing over all $i \in I_m$, we find

$$\lambda\left(F_r^{-1}(S) \cap \left(\bigcup_{i \in I_m} (a_i, b_i)\right)\right) = p_m \lambda(S). \tag{3.3}$$

Now apply equation (3.3) with $S = S_k = \cup_{i \in I_{m_k}}(a_i, b_i)$ and $m = m_{k-1}$. Write $S_{k-1} = F^{-1}(S_k) \cap \cup_{i \in I_{m_{k-1}}}(a_i, b_i)$. Then equation (3.3) can be written as $\lambda(S_{k-1}) = p_{m_{k-1}}\lambda(S_k) = p_{m_{k-1}}p_{m_k}$. Similarly define $S_{j-1} = F^{-1}(S_j) \cap \cup_{i \in I_{m_{j-1}}}(a_i, b_i)$ for $j = k-1, \ldots, 2$. Apply equation (3.3) repeatedly to find $\lambda(S_j) = p_j \ldots p_k$ for $j = k-1, \ldots, 2$. But observing that

$$S_1 = \bigcap_{n=1}^{k} \left(\bigcup_{i \in I_{m_n}} F_r^{-n+1}(a_i, b_i)\right)$$

and using equation (3.2) we thus get $\mathbb{P}(Y_1 = m_1, \ldots, Y_k = m_k) = p_1 \ldots p_k$. By summing over all choices of $m_1, \ldots, m_{k-1}$, using $\sum_{m \in \mathbb{Z}} p_m = 1$, also $\mathbb{P}(Y_k = m_k) = p_k$. It follows immediately that $Y_1, Y_2, \ldots$ are independent, identically distributed and the Lemma holds. □

3.2 Topological conjugacy

To prove Theorem 3.1, we will identify the trajectories generated by a general shift-periodic map with integer spikes with those of a map which is piecewise linear between grid lines, like in Lemma 3.1.

Definition 3.4. *Maps $f : [0,1] \to [0,1]$ and $g : [0,1] \to [0,1]$ are topologically conjugate if there exists a homeomorphism $h : [0,1] \to [0,1]$ such that $g = h^{-1} \circ f \circ h$.*

In [17], Baldwin described all classes of topologically conjugate maps on $[0,1]$ which are continuous and piecewise monotonic. A slight adaptation of his proofs establishes the Lemma below, which is also valid for maps with infinitely, rather than finitely many, monotonic pieces, as was the case in the original proof.

Lemma 3.2. *Let $f : [0,1] \to [0,1]$ and suppose there exist pairwise disjoint intervals (a_i, b_i) with $i \in I$, where I is a countable indexing set disjoint from $[0,1]$, such that the following conditions are satisfied:*

(i) *For any $i \in I$, f is continuous and monotonic on (a_i, b_i) with $f((a_i, b_i)) = (0,1)$;*
(ii) *$|f(x) - f(y)| > |x - y|$ holds for all distinct $x, y \in (a_i, b_i)$, where $i \in I$.*
(iii) *$\Sigma = [0,1] \setminus \cup_{i \in I}(a_i, b_i)$ is countable and $f(\Sigma) = \{0\}$.*

Let $g : [0,1] \to [0,1]$ such that g is linear on (a_i, b_i) with $\lim_{x \to a_i^+} g(x) = \lim_{x \to a_i^+} f(x)$ and $\lim_{x \to b_i^-} g(x) = \lim_{x \to b_i^-} f(x)$ for any $i \in I$ and further $g(x) = 0$ for any $x \in \Sigma$. Then f and g are topologically conjugate.

Proof. With any $x \in [0,1]$ we associate a sequence $\mathbf{a}_f(x) = (a_0(x), a_1(x), \ldots)$ where $a_n(x) = i$ if $f^n(x) \in (a_i, b_i)$ and $a_n(x) = f^n(x)$ if $f^n(x) \in \Sigma$. Let $\mathbf{b}$ be a finite sequence of length n. For sequence $\mathbf{c}$, infinite or finite of length greater or equal n, we say $\mathbf{c}|_n = \mathbf{b}$ if the first n entries of $\mathbf{b}$ and $\mathbf{c}$ coincide. For a finite sequence $\mathbf{b}$ of length n with entries in $I \cup \Sigma$ then write

$$J_{\mathbf{b}}^f = \{x \in [0,1] : \mathbf{a}_f(x)|_n = \mathbf{b}\}.$$

We now make the following observations: Suppose $x \neq y$. Because of condition (ii) it is clear that for some $n \in \mathbb{N}$ the interval $(f^n(x), f^n(y))$ or $(f^n(y), f^n(x))$ intersects with Σ. We then easily deduce $\mathbf{a}_f(x) \neq \mathbf{a}_f(y)$.

Let again $\mathbf{b}$ be a finite sequence. From above we then note that $J_{\mathbf{b}}^f$ is empty or a singleton if at least one of the entries of $\mathbf{b}$ is in Σ. Otherwise $J_{\mathbf{b}}^f$ is an open interval. This can easily be shown by induction, and is a consequence of $f((a_i, b_i)) = (0,1)$ and of monotonicity of f on (a_i, b_i). Further, for another finite sequence $\mathbf{c}$, if $\mathbf{c}|_n = \mathbf{b}$ then $J_{\mathbf{c}}^f \subseteq J_{\mathbf{b}}^f$. The same observations also apply to g, where we define $J_{\mathbf{b}}^g = \{x \in [0,1] : \mathbf{a}_g(x)|_n = \mathbf{b}\}$. The construction of g, in particular equality to f

on Σ, ensures that $J_{\mathbf{b}}^g$ is empty, a singleton or open interval if and only if $J_{\mathbf{b}}^f$ is empty, a singleton or an open interval, respectively.

The sets $J_{\mathbf{b}}^f$, where $\mathbf{b}$ are sequences of length n in $I \cup \Sigma$, partition $[0, 1]$. Now we may define a continuous, monotonically increasing map $h_n : [0, 1] \to [0, 1]$ by sending each $J_{\mathbf{b}}^g$ to the corresponding set $J_{\mathbf{b}}^f$ via an increasing linear map. Note that the upper bound on the length of $J_{\mathbf{b}}^f$ for sequences $\mathbf{b}$ with n entries goes to 0 as n goes to infinity. h_n converges uniformly to a continuous, strictly monotonically increasing map h with $\mathbf{a}_g(x) = \mathbf{a}_f(h(x))$: Strict monotonicity can be seen by noting that for $x < y \in [0, 1]$ there must exist $z_1, z_2 \in [0, 1]$ with $x < z_1 < z_2 < y$, so that $\mathbf{a}_g(z_1)$ and $\mathbf{a}_g(z_2)$ have some entries contained in Σ. But then for large enough n the values of $h_n(z_1)$ and $h_n(z_2)$ will become constant and so in the limit we obtain $h(x) \le h(z_1) < h(z_2) \le h(y)$.

Now consider $\mathbf{b} = \mathbf{a}_g(x)|_n$ for some $n \in \mathbb{N}$. Since h maps x to $\overline{J_{\mathbf{b}}^f}$ for each such $\mathbf{b}$, $h(x)$ must either have $\mathbf{a}_g(x) = \mathbf{a}_f(h(x))$ or $\mathbf{a}_f(h(x))$ intersects with Σ. There will be a point y with $\mathbf{a}_g(y) = \mathbf{a}_f(h(x))$ and in the latter case h_n will be eventually constant and equal h at y. Then $\mathbf{a}_g(y)$ and $\mathbf{a}_f(h(y))$ agree up to arbitrary length and $\mathbf{a}_g(y) = \mathbf{a}_f(h(y))$. But $\mathbf{a}_f(h(x)) = \mathbf{a}_g(y) = \mathbf{a}_f(h(y))$ contradicts h being strictly increasing, unless $x = y$.

As h is continuous and strictly increasing, it is also a homeomorphism on $[0, 1]$ with $\mathbf{a}_g(x) = \mathbf{a}_f(h(x))$. Note also that $\mathbf{a}_g(x) = \mathbf{a}_f(x)$ gives $\mathbf{a}_g(g(x)) = \mathbf{a}_f(f(x))$. It is then easy to observe that $(h^{-1} \circ f \circ h)(x) = g(x)$ and thus f and g are topologically conjugate. □

3.3 Proof of Theorem 3.1

Let U be uniformly distributed on $[0, 1]$. Let $F : \mathbb{R} \to \mathbb{R}_\infty$ be a shift-periodic map with integer spikes. The conditions placed on F, in particular integer spikes, ensure that there exists a countable set I and $(a_i, b_i) \subseteq [0, 1]$ for $i \in I$, pairwise disjoint, so that F_r is continuous and monotonic on each (a_i, b_i) and $\Sigma = [0, 1] \setminus \cup_{i \in I}(a_i, b_i)$ is countable. Further, $F_r((a_i, b_i)) = (0, 1)$ for each such $i \in I$ and $|F_r(x) - F_r(y)| > |x - y|$ when $x, y \in (a_i, b_i)$ are distinct. Additionally, by the definition of F_r at the singularities of F we have that $F_r(x) = 0$ for $x \in \Sigma$. In short, all conditions in Lemma 3.2 are satisfied. F_r is then topologically conjugate to a map $G_r : [0, 1] \to [0, 1]$ linear on intervals (a_i, b_i) with $G_r((a_i, b_i)) = (0, 1)$. Let h be the homeomorphism with $G_r = h^{-1} \circ F_r \circ h$. Now consider $G : \mathbb{R} \to \mathbb{R}_\infty$ with $G(x) = G_r(x) + \lfloor F(x) \rfloor$ on $[0, 1]$ and $G(x) = G(\{x\}) + \lfloor x \rfloor$ on $\mathbb{R}$. By construction this is a shift-periodic map with integer spikes, linear between grid lines. So G satisfies Lemma 3.1 and therefore for $Z_n = \lfloor G^n(U) \rfloor$ the random process $(Z_n)_{n \in \mathbb{N}}$ is a discrete random walk with

$$\mathbb{P}(Z_n - Z_{n-1} = m) = \lambda\{x \in [0, 1] : \lfloor G(x) \rfloor = m\} = \lambda\{x \in [0, 1] : \lfloor F(x) \rfloor = m\} = p_m.$$

Let $i_1, i_2, \ldots, i_n$ be integers. Denote by $K^{G_r}(i_1, i_2, \ldots, i_n)$ the subset of $[0, 1]$ whose elements x satisfy $\lfloor G(x) \rfloor = i_1, \lfloor G(G_r(x)) \rfloor = i_2 - i_1, \ldots, \lfloor G(G_r^{n-1}(x)) \rfloor = i_n - i_{n-1}$. Similarly define $K^{F_r}(i_1, i_2, \ldots, i_n)$ to be the subset of $[0, 1]$ whose elements satisfy $\lfloor F(x) \rfloor = i_1, \lfloor F(F_r(x)) \rfloor = i_2 - i_1, \ldots, \lfloor F(F_r^{n-1}(x)) \rfloor = i_n - i_{n-1}$. Recall that for a sequence $\mathbf{b}$ of length n in $\Sigma \cup I$, h bijectively maps the set $J_{\mathbf{b}}^{G_r}$ of elements in $[0, 1]$ which have initial trajectory $\mathbf{b}$ with respect to G_r to the set $J_{\mathbf{b}}^{F_r}$ of such elements with respect to F_r. Recall also that $\lfloor F(x) \rfloor$ and $\lfloor G(x) \rfloor$ are equal to the same constant on set (a_i, b_i). Then there exists a collection B of sequences in $\Sigma \cup I$ such that $K^{G_r}(i_1, i_2, \ldots, i_n) = \cup_{\mathbf{b} \in B} J_{\mathbf{b}}^{G_r}$ and $K^{F_r}(i_1, i_2, \ldots, i_n) = \cup_{\mathbf{b} \in B} J_{\mathbf{b}}^{F_r}$. Using equation (3.1), we can then observe that

$$\begin{aligned} \{x \in [0, 1] : \lfloor F(h(x)) \rfloor = i_1, \ldots, \lfloor F^n(h(x)) \rfloor = i_n\} \\ = \{x : h(x) \in K^{F_r}(i_1, \ldots, i_n)\} = K^{G_r}(i_1, \ldots, i_n). \end{aligned}$$

Let $X_k = \lfloor F^k(h(U)) \rfloor$. Using the observation above and Lemma 3.1, we can then easily deduce

$$\begin{aligned} \mathbb{P}(X_1 = i_1, \ldots, X_n = i_n) &= \mathbb{P}(U \in K^{G_r}(i_1, \ldots, i_n)) = \mathbb{P}(Z_1 = i_1, \ldots, Z_n = i_n) \\ &= \mathbb{P}(Z_1 = i_1) \ldots \mathbb{P}(Z_n - Z_{n-1} = i_n - i_{n-1}) \\ &= \mathbb{P}(X_1 = i_1) \ldots \mathbb{P}(X_n - X_{n-1} = i_n - i_{n-1}) \end{aligned}$$

and $\mathbb{P}(X_k - X_{k-1}) = m) = \lambda\{x \in [0, 1] : \lfloor G(x) \rfloor = m\} = \lambda\{x \in [0, 1] : \lfloor F(x) \rfloor = m\}$.

□

Corollary 3.1. *Let U be distributed uniformly on $[0,1]$. Let h be the map from Theorem 3.1. Then $h(U)$ is an invariant distribution with respect to F_r, meaning that for any $x \in \mathbb{R}$*

$$\mathbb{P}(F_r(h(U)) \leq x) = \mathbb{P}(h(U)) \leq x).$$

Proof. This can be shown using the same methods as for the proof of Theorem 3.1: Let $\mathbf{b}$ be a finite sequence and let again $J_{\mathbf{b}}^{F_r} = \{x \in [0,1] : \mathbf{a}_{F_r}(x)|_n = \mathbf{b}\}$ and $J_{\mathbf{b}}^{G_r} = \{x \in [0,1] : \mathbf{a}_{G_r}(x)|_n = \mathbf{b}\}$. We denote by $a\mathbf{b}$ the sequence starting with a and followed by $\mathbf{b}$.

$$\begin{aligned}\mathbb{P}\left(F_r(h(U)) \in J_{\mathbf{b}}^{F_r}\right) &= \mathbb{P}\left(h(U) \in \bigcup_{a\in\Sigma\cup I} J_{a\mathbf{b}}^{F_r}\right) = \mathbb{P}\left(U \in \bigcup_{a\in\Sigma\cup I} J_{a\mathbf{b}}^{G_r}\right) \\ &= \mathbb{P}\left(G_r(U) \in J_{\mathbf{b}}^{G_r}\right) = \mathbb{P}\left(U \in J_{\mathbf{b}}^{G_r}\right) = \mathbb{P}\left(h(U) \in J_{\mathbf{b}}^{F_r}\right),\end{aligned}$$

where we used that U is an invariant distribution for G_R. The latter follows easily from linearity of G_r on each (a_i, b_i) and from $G_r((a_i, b_i)) = (0,1)$ for $i \in I$. Any interval $[0,x]$ with $x \in [0,1]$ can be approximated arbitrarily closely by a union of countably many $J_{\mathbf{b}}^{F_r}$ so that we also find $\mathbb{P}(F_r(h(U)) \leq x) = \mathbb{P}(h(U) \leq x)$, as desired. □

3.4 Alpha-stable processes

We now consider how sequences generated by shift-periodic maps F with integer spikes behave when the time between consecutive entries is scaled to go to zero. More precisely, we change discrete-time process $X_n = \lfloor F^n(h(U)) \rfloor$, used in Theorem 3.1, to a sequence of continuous-time process

$$V^{(n)}(t) = \frac{1}{b_n}\left(X_{\lfloor nt \rfloor} - a_n t\right), \qquad \text{where } n \in \mathbb{N}, \tag{3.4}$$

and a_n and b_n are appropriately chosen translation-scaling and space-scaling constants. Since the generated random variables $X_{\lfloor nt \rfloor}$ behave like a random walk, we can apply Functional Central Limit Theorems (FCLTs) [13]. The classical example is Donsker's theorem, which treats the convergence of processes of the form $\frac{1}{b_n}\left(\sum_{k=1}^{\lfloor nt \rfloor} Y_k - a_n t\right)$ to the Wiener process when the independent random variables Y_k are following a normal distribution. This convergence is with respect to the Skorohod metric on the space of right-continuous functions on with existing left limits [18, 13]. We later refer to such a space of functions on $[0,\infty)$ as $\mathcal{D}([0,\infty),\mathbb{R})$, when we study convergence to Lévy motions, which first require us to define a special class of α-strictly stable processes.

Definition 3.5. *For $\alpha \in (0,2]$ and $\beta \in [-1,1]$, with $\beta = 0$ when $\alpha = 1$, we define α-strictly stable distribution $S(\alpha,\beta)$ to be the distribution with characteristic function*

$$\phi(t;\alpha,\beta) = \exp\left(-|t|^\alpha \left(1 - i\beta \operatorname{sgn}(t) \tan\left(\frac{\pi\alpha}{2}\right)\right)\right).$$

With Definition 3.5 it can be shown that $\sum_{k=1}^{n} X_k \sim n^{1/\alpha} S(\alpha,\beta)$ where $X_k \sim S(\alpha,\beta)$ are independent [19]. In fact, this property is usually used to define α-strictly stable distributions, but it is not actually needed here. Note that $S(2,\beta)$ is a normal distribution with mean 0 for any value of β, while $S(1,0)$ is a Cauchy distribution.

A key component of the proof of Donsker's Theorem is the standard CLT, but a generalized version of the CLT due to Kolmogorov and Gnedenko [20] also applies to stable distributions with infinite variance and corresponds to a generalized FCLT for α-strictly stable processes, called Lévy motions [13].

Definition 3.6. *A Lévy motion is a stochastic process $V(t;\alpha,\beta), t \geq 0$ with $\alpha \in (0,2]$ and $\beta \in [-1,1]$, where $\beta = 0$ when $\alpha = 1$, satisfying the following properties:*

(a) *Every sample path of $V(t;\alpha,\beta)$ is contained in $D([0,\infty),\mathbb{R})$. Also, $V(0) = 0$.*

(b) *For $0 \leq s_1 < \cdots < s_n$, the increments $V(s_2;\alpha,\beta) - V(s_1;\alpha,\beta)$, ..., $V(s_n;\alpha,\beta) - V(s_{n-1};\alpha,\beta)$ are independent.*

(c) *For $s \geq 0, t > 0$ we have $V(s+t;\alpha,\beta) - V(s;\alpha,\beta)$ equal in distribution to $t^{1/\alpha} S(\alpha,\beta)$.*

A standard example of such a Lévy motion is the Wiener process for $\alpha = 2$. Lévy motions allow the following generalisation of Donsker's Theorem.

Theorem 3.2 (FCLT for α-stable Lévy motions)**.** *Let $Y_1, Y_2, \ldots$ be independent, identically distributed random variables with cumulative distribution $F_Y(x)$ satisfying*

$$1 - F_Y(M) \sim c_+ M^{-\kappa} \text{ as } x \to \infty \quad \text{and} \quad F_Y(M) \sim c_-|M|^{-\kappa} \text{ as } x \to -\infty$$

where $\kappa > 0$, $c_+ \geq 0$ and $c_- \geq 0$ are constants such that c_+ and c_- are not both zero and $c_+ = c_-$ if $\kappa = 1$. Let

$$\alpha = \min\{\kappa, 2\} \quad \text{and} \quad \beta = \frac{c_+ - c_-}{c_+ + c_-}$$

and depending on κ, choose a_n and b_n from the table below.

κ	a_n	b_n
0 ¡ κ ¡ 1	0	$(\pi(c_+ + c_-)(2\Gamma(\alpha)\sin(\alpha\pi/2))^{-1}n)^{1/\alpha}$
$\kappa = 1$	$\beta(c_+ + c_-)n\log(n)$	$\pi/2(c_+ + c_-)n$
1 ¡ κ ¡ 2	$n\mathbb{E}[Y_i]$	$(\pi(c_+ + c_-)(2\Gamma(\alpha)\sin(\alpha\pi/2))^{-1}n)^{1/\alpha}$
$\kappa = 2$	$n\mathbb{E}[Y_i]$	$(c_+ + c_-)^{1/2}(n\log(n))^{1/2}$
$\kappa > 2$	$n\mathbb{E}[Y_i]$	$(\mathrm{Var}(Y_i)/2)^{1/2}n^{1/2}$

Then the stochastic processes defined by

$$V^{(n)}(t) = \frac{1}{b_n}\left(\sum_{k=1}^{\lfloor nt \rfloor} Y_k - a_n t\right),$$

converge, with respect to the Skorohod metric on $D([0,\infty),\mathbb{R})$, to the Lévy motion $V(t;\alpha,\beta)$.

Proof. This is a simple consequence of combining Uchaikin's version of generalized CLT [19] with the discussion of FCLTs arising from CLTs in Whitt [13]. □

We can now rephrase this theorem in terms of our sequences of iterates of shift-periodic maps. The following statement is a direct corollary of Theorem 3.1 and Theorem 3.2.

Corollary 3.2. *Let $F : \mathbb{R} \to \mathbb{R}_\infty$ be a shift-periodic map with integer spikes. Suppose that*

$$\lambda\{y \in [0,1] : \lfloor F(y) \rfloor > M\} \sim c_+ M^{-\kappa} \text{ and } \lambda\{y \in [0,1] : \lfloor F(y) \rfloor < -M\} \sim c_- M^{-\kappa} \text{ as } M \to \infty,$$

where $\kappa > 0$, $c_+ \geq 0$ and $c_- \geq 0$ are constant with c_+ and c_- not both zero and $c_+ = c_-$ if $\kappa = 1$. Choose α, β, a_n and b_n as in Theorem 3.2. Let h be as described in Theorem 3.1, so that $h(U)$, with U uniformly distributed on $[0,1]$, is an invariant distribution of F. Define $V^{(n)}(t)$ by (3.4). Then these stochastic processes converge, with respect to the Skorohod metric on $D([0,\infty),\mathbb{R})$, to the Lévy motion $V(t;\alpha,\beta)$.

Note that while we required $c_+ > 0$ or $c_- > 0$ in Corollary 3.2, effectively excluding continuous shift-periodic maps, the random variables $Y_n = \lfloor F^n(h(U)) \rfloor - \lfloor F^{n-1}(h(U)) \rfloor$ generated by a continuous shift-periodic map F with integer spikes have finite second moments, so that Donsker's Theorem conveniently covers this case and we get Brownian motion upon an appropriate scaling in time and space.

Let us briefly return to map $F(x;\kappa)$ in Example 2.2. Simple calculations show us that that there is some $c > 0$ such that

$$\lambda\{y \in [0,1] : \lfloor F(y;\kappa) \rfloor > M\} \sim cM^{-\kappa} \text{ and } \lambda\{y \in [0,1] : \lfloor F(y;\kappa) \rfloor < -M\} \sim c|x|^{-\kappa} \text{ as } M \to \infty.$$

Corollary 3.2 then tells us that for appropriately chosen a_n and b_n the stochastic process $V^{(n)}(t)$, as defined in equation (3.4), behaves in a limit like a Lévy motion $Y(t;\alpha,\beta)$. Here $\alpha = \min\{\kappa, 2\}$. In particular, we get a Wiener process for $\kappa \geq 2$. This is also confirmed with a numerical example in Figure 2.

4 Continuous-time Random Walks

Simple examples show that the restriction on shift-periodic maps introduced in Section 3, namely that local extrema shall only take integer values, is necessary to obtain well-behaved stochastic processes.

Example 4.1. Consider the shift-periodic map $F : \mathbb{R} \to \mathbb{R}_\infty$ with $\kappa > 0$ and

$$F(x) = \begin{cases} 2x & \text{if } x \in [0, 1/4] \\ 1 - 2x & \text{if } x \in [1/4, 1/2] \\ F(x; \kappa) & \text{if } x \in (1/2, 1) \end{cases}$$

and $F(x) = F(\{x\}) + \lfloor x \rfloor$ on $\mathbb{R}$. Here $F(x; \kappa)$ is the map defined in Example 2.2. This shift-periodic map has a local maximum with value 1/2. Observe that $F([0, 1/2]) = [0, 1/2]$ and that the Lebesgue measure of the subset of $[0, 1]$ on which $F_r^n(x) \in [0, 1/2]$ converges to 1 as $n \to \infty$. But $F_r^n(x) \in [0, 1/2]$ implies that for all $m \geq n$ $\lfloor F^n(x) \rfloor = \lfloor F^m(x) \rfloor$. The behaviour of sequences $\lfloor x_n \rfloor$ for x_n defined as in equation (1.1) is then completely different from what we have observed before. For instance, if we choose the initial value to be x_0 from an uniform distribution on $[0, 1]$ then the sequence will be eventually constant with probability 1 and the same is true for many other initial distributions.

In Section 2, we already discussed that sequences generated by some shift-periodic maps without integer spikes still display random-walk like properties. By studying Example 2.1 more closely, we will now demonstrate how this statement can be made more precise in a limit for some maps and also highlight where difficulties arise when considering these more general-shift periodic maps. Many of these findings could be extended to a wider range of maps, in particular those for which only a small subset of $[0, 1]$ is mapped outside the unit interval.

4.1 Invariant density

Recall from Section 3 that we decomposed sequences (x_n) defined in equation (1.1) as $x_n = \lfloor x_n \rfloor + F_r^n(x_0)$. In order to study the behaviour of $\lfloor x_n \rfloor$ we first find the invariant density with respect to F_r. While this was relatively straightforward in the case of maps with integer spikes, by using topological conjugacy to a set of maps with a particularly nice invariant density, this approach cannot be used for more general shift-periodic maps. We instead apply more general results on invariant densities for piecewise linear maps due to Góra [14] to maps $F(x; \varepsilon, \delta)$ defined in Example 2.1.

Lemma 4.1. *Consider shift-periodic map $F(x; \varepsilon, \delta)$ defined in Example 2.1, where $\varepsilon > 0$ and $\delta > 0$. For parameters $A \in [0, 1]$ and $B \in \mathbb{R} \setminus \{0\}$ define the indicator function $\mathbb{1} : [0, 1] \to \{0, 1\}$ by*

$$\mathbb{1}(x; A, B) = \begin{cases} 1 & \textit{if} \quad x \in [0, A], B > 0 \ \textit{ or } \ x \in [A, 1], B < 0; \\ 0 & \textit{otherwise.} \end{cases}$$

We define the cumulative derivative $\beta(x, n)$ iteratively by

$$\beta(x, n) = \beta(x, n-1) \cdot F_r'(F_r^{n-1}(x; \varepsilon, \delta); \varepsilon, \delta), \quad \textit{for } n \geq 2, \quad \textit{and} \quad \beta(x, 1) = F_r'(x; \varepsilon, \delta),$$

where we leave function $\beta(x, n)$ undefined when the derivatives do not exist. This is the case only for finitely many points in $(0, 1)$ for each n. We further define two cumulative derivatives at $c_1 = 1/4$ and $c_2 = 3/4$ by

$$\beta^L(c_i, n) = \lim_{x \to c_i^+} \beta(x, n) \quad \textit{and} \quad \beta^R(c_i, n) = \lim_{x \to c_i^-} \beta(x, n) \quad \textit{for } i = 1, 2.$$

The invariant density of $F(x;\varepsilon,\delta)$ is given by

$$
\begin{aligned}
f_i(x;\varepsilon,\delta) &= \frac{1}{K}\left(1+\sum_{j=1}^{2} D_j^L \sum_{n=1}^{\infty} \frac{\mathbb{1}(x;F_r^n(c_j),-\beta^L(c_j,n))}{|\beta^L(c_j,n)|}\right. \\
&\qquad \left. + \sum_{j=1}^{2} D_j^R \sum_{n=1}^{\infty} \frac{\mathbb{1}(x;F_r^n(c_j),\beta^R(c_j,n))}{|\beta^R(c_j,n)|}\right), \qquad (4.1)
\end{aligned}
$$

where K is a normalisation constant, chosen so that f_i integrates to 1 over $[0,1]$ and D_1^L, D_2^L, D_1^R, D_2^R are constants dependent on ε, δ with $D_i^R \to 1$ and $D_i^L \to 1$ as $\varepsilon,\delta \to 0$, for $i \in \{1,2\}$.

Proof. This is just an application of Góra's results on invariant densities of eventually expanding maps in [14]. Here $D = (D_1^L, D_1^R, D_1^R, D_2^R)$ is the solution of $(-S^T + I)D^T = (1,1,1,1)^T$ where $S = (S_{i,j})$ is a matrix with entries dependent on $F_r^n(c)$ and $\beta(c,n)$, $c \in \{c_1^L, c_1^R, c_2^L, c_2^R\}$, converging to 0 as the parameter ε and δ of $F_r^n(x;\varepsilon,\delta)$ converge to 0. For more details on S see page 7 of [14]. □

That way, when we choose X to be distributed according to the distribution corresponding to $f_i(x;\varepsilon,\delta)$, all $F^n(X;\varepsilon,\delta)$ will have the same distribution and random variables $X_n = F_r(X;\varepsilon,\delta)$ will be identically distributed.
For small choices of parameters ε, δ, say ε, $\delta < 10^{-6}$, numerical calculations tell us that

$$f_i(x;\varepsilon,\varepsilon) \approx 1 + 1/4^n \quad \text{for } x \in I_n$$

where I_n are defined by

$$
\begin{aligned}
I_1 &= (0, F(1/4;\varepsilon,\varepsilon)) \cup (1 - F(1/4;\varepsilon,\varepsilon), 1) \\
I_n &= (F^n(1/4;\varepsilon,\varepsilon), F^{n+1}(1/4;\varepsilon,\varepsilon)) \cup (1 - F^{n+1}(1/4;\varepsilon,\varepsilon), 1 - F^n(1/4;\varepsilon,\varepsilon)) \text{ for } n \geq 2.
\end{aligned}
$$

This approximation is accurate up to three decimal digits. In Figure 1 we used larger values of parameters, $\varepsilon = \delta = 10^{-2}$. Table 4.1 gives the invariant density $f_i(x;10^{-2},10^{-2})$ on intervals I up to three decimal digits.

For $\varepsilon,\delta = 0$ map $F(x;\varepsilon,\delta)$ is linear between grid lines, so of the type discussed in Section 3, and has invariant density equal 1. So one might expect that $f_i(x;\varepsilon,\delta) \to 1$ as $\varepsilon,\delta \to 0$. This convergence is not uniform on $[0,1]$, but we can make the important observation below:

Corollary 4.1. *Let $f_i(x;\varepsilon,\delta)$ be the invariant density of $F(x;\varepsilon,\delta)$, described in Lemma 4.1. For any $d > 0$ we have $f_i(x;\varepsilon,\delta) \to 1$ as $\varepsilon,\delta \to 0$ uniformly on $[d, 1-d]$.*

Proof. Let $d > 0$. For any fixed $n \in \mathbb{N}$ we have that $F_r^n(c_1) \to 0$ and $F_r^n(c_2) \to 1$ as $\varepsilon,\delta \to 0$, so that the length of the interval on which $\mathbb{1}(x;F_r^n(c_j),-\beta^L(c_j,n)) \neq 0$ or $\mathbb{1}(x;F_r^n(c_j),\beta^R(c_j,n)) \neq 0$ also goes to zero, for $j = 1,2$. Noting that $|\beta^L(c_j,n)| \geq 2^n$ and $|\beta^R(c_j,n)| \geq 2^n$, then the integral of the weighted sum of four infinite sums in equation (4.1) over $[0,1]$ goes to 0 as $\varepsilon,\delta \to 0$. Adding 1 to this integral, we get K. So we have $K \to 1$. By the same argument, for sufficiently small $\varepsilon > 0$ and $\delta > 0$ we have $F_r^n(c_1) < d$ and $F_r^n(c_2) > 1 - d$ for all $n \in \{1,2,\ldots,N\}$, so that we achieve bound

$$\frac{1}{K} \leq f_i(x;\varepsilon,\delta) \leq \frac{1}{K}\left(1 + \frac{D_1^L + D_2^L + D_1^R + D_2^R}{2^N}\right)$$

valid on $[d, 1-d]$ for small ε and δ. But N was chosen arbitrarily. Recall from Lemma 4.1 that also $D_i^R, D_i^L \to 1$ for $i = 1,2$. Combining these results, we find that $f_i(x;\varepsilon,\delta) \to 1$ uniformly on $[d, 1-d]$.

□

We briefly also note that this invariant density additionally gives us a description of how the fractional parts $F_r^n(x_0)$ of sequence x_n behave long-term, by a simple application of Birkhoff's Ergodic Theorem.

I	$f_i(x)$	I	$f_i(x)$	I	$f_i(x)$
$(0, \varepsilon/4)$	1.959	$(0.206, 0.354)$	0.986	$(0.794, 0.839)$	0.984
$(0.0025, 0.01)$	1.224			$(0.839, 0.96)$	0.995
$(0.01, 0.04)$	1.041	$(0.354, 0.646)$	0.988	$(0.96, 0.99)$	1.041
$(0.04, 0.161)$	0.995			$(0.99, 0.9975)$	1.224
$(0.161, 0.206)$	0.984	$(0.646, 0.794)$	0.986	$(1-\varepsilon/4, 1)$	1.959

Table 1: The values of $f_i(x; \varepsilon, \varepsilon)$ with $\varepsilon = 10^{-2}$ up to three decimal digits.

4.2 From maps to continuous-time random walks

Now consider X distributed according to the invariant distribution of $F_r(x; \varepsilon, \delta)$ on $[0, 1]$ and $X_n = \lfloor F^n(X; \varepsilon, \delta) \rfloor$. For now we say a jump occurs when $X_n \neq X_{n+1}$. By Corollary 4.1 the invariant density of $F_r(x; \varepsilon, \delta)$ is close to 1 at the spikes 1/4 and 3/4 of the map for small parameters ε and δ. Further, direct calculation shows that the length of the subset of $[0, 1]$ mapped outside the unit interval by $F(x; \varepsilon, \delta)$ is

$$\ell(\varepsilon) + \ell(\delta) \quad \text{where} \quad \ell(x) = \frac{x(3 + x)}{2(x + 2)(x + 4)}. \tag{4.2}$$

This calculation suggests that the probability of a jump is about $3(\varepsilon + \delta)/16$. However, successive jump probabilities are not independent, since for small parameters successive jumps are impossible. We fix this issue by introducing a scaling in time, together with a scaling of the parameters. This scaling gives us behaviour resembling a continuous-time random walk, defined below.

Definition 4.1. *Consider a continuous-time stochastic process $Y(t), t \geq 0$ with $Y(0) = 0$ which takes values in $\mathbb{Z}$ and is right-continuous. Let $T_0 = 0$. For $j \geq 1$ define the time of the j-th jump by*

$$T_j = \min\{t \in (T_{j-1}, \infty) : Y(t) \neq Y(T_{j-1})\}.$$

Suppose $T_1, T_2, \ldots$ are independent. Then we say $Y(t)$ is a continuous-time random walk.

Theorem 4.1. *Let $\delta, \varepsilon > 0$. For $m \in \mathbb{N}$ let X_m be distributed according to the invariant distribution, $f_i(x; \varepsilon/m; \delta/m)$, with respect to $F(x; \varepsilon/m, \delta/m)$. Define*

$$Y_m(t) = \left\lfloor F^{\lfloor mt \rfloor}(X_m; \varepsilon/m, \delta/m) \right\rfloor.$$

Let $T_{m,1}, T_{m,2}, \ldots$ denote the jump times of $Y_m(t)$, that is,

$$T_{m,j} = \min\{t \in (T_{m,j-1}, \infty) : Y_m(t) \neq Y_m(T_{m,j-1})\}$$

where $T_{m,0} = 0$. Then for any $k \geq 0$ we have

$$\mathbb{P}\left(T_{m,k+1} \leq \frac{\lfloor mt_k \rfloor}{m} + \tau \;\middle|\; T_{m,k} = \frac{\lfloor mt_k \rfloor}{m}, \ldots, T_{m,1} = \frac{\lfloor mt_1 \rfloor}{m} \right) \to 1 - \exp(-\gamma\tau)$$

as $m \to \infty$, where

$$\gamma = \frac{3(\delta + \varepsilon)}{16}. \tag{4.3}$$

Let $Y(t)$ be a continuous-time random walk with waiting times, $T_j - T_{j-1}$, exponentially distributed with mean $1/\gamma$. Then Theorem 4.1 says that for $Y_m(t)$ the probability of the $(k+1)$-th jump occurring at time $\lfloor mt_k \rfloor/m + \tau$, given that the first k jumps occurred at times $\lfloor mt_1 \rfloor/m$, $\lfloor mt_2 \rfloor/m$, $\ldots$, $\lfloor mt_k \rfloor/m$, converges to the probability of the $(k+1)$-th jump of $Y(t)$ occurring at time $t_k + \tau$, given that the first k jumps occurred at t_1, t_2, $\ldots$, t_k.

4.3 Conditionally invariant distribution

While we extensively discussed invariant distributions of $F_r(x;\varepsilon,\delta)$ above, to prove Theorem 4.1 it will actually be more convenient to work with conditional distributions on $[0,1]$ which are invariant with respect to $F(x;\varepsilon,\delta)$ conditioned on the event that the iterative sequence stays in $[0,1]$. For a probability density f of a distribution on $[0,1]$, the density after application of $F(x;\varepsilon,\delta)$, conditional on not mapping outside $[0,1]$, is given by the Frobenius-Perron operator [15]

$$\mathcal{P}_{\varepsilon,\delta}(f)(t) = \begin{cases} \dfrac{1}{C}\left(\dfrac{f\big(F_1^{-1}(t)\big)}{4+\varepsilon} + \dfrac{f\big(F_3^{-1}(t)\big)}{2+\delta} + \dfrac{f\big(F_4^{-1}(t)\big)}{4+\delta}\right) & \text{if } t\in(0,1/2),\\ \dfrac{1}{C}\left(\dfrac{f\big(F_1^{-1}(t)\big)}{4+\varepsilon} + \dfrac{f\big(F_2^{-1}(t)\big)}{2+\varepsilon} + \dfrac{f\big(F_4^{-1}(t)\big)}{4+\delta}\right) & \text{if } t\in(1/2,1), \end{cases} \tag{4.4}$$

where F_1^{-1}, F_2^{-1}, F_3^{-1}, F_4^{-1} denote the inverses of $F(x;\varepsilon,\delta)$ restricted to $(0,1/4)$, $(1/4,1/2)$, $(1/2,3/4)$, $(3/4,1)$, respectively, and normalisation constant C is chosen so that $\mathcal{P}_{\varepsilon,\delta}$ integrates to 1 over the unit interval.

Lemma 4.2. *There exists a unique density $f_c(x;\varepsilon,\delta)$ with $f_c(x;\varepsilon,\delta)=\nu$ on interval $(0,1/2)$ and $f_c(x;\varepsilon,\delta)=2-\nu$ on $(1/2,1)$ such that $\mathcal{P}_{\varepsilon,\delta}(f_c)=f_c$. We will subsequently call this the conditionally invariant density. It satisfies $f_c(x;\varepsilon,\delta)\to 1$ as $\varepsilon,\delta\to 0$.*

Proof. Suppose f is a density which is constant equal ν on $(0,1/2)$ and $1-\nu$ on $(1/2,1)$. It satisfies $\mathcal{P}_{\varepsilon,\delta}(f)=f$ if and only if ν satisfies the following equation

$$\nu\left(\frac{1}{2}\left(\frac{\nu}{4+\varepsilon}+\frac{2-x}{2+\delta}+\frac{2-\nu}{4+\delta}\right)+\frac{1}{2}\left(\frac{\nu}{4+\varepsilon}+\frac{\nu}{2+\varepsilon}+\frac{2-\nu}{4+\delta}\right)\right)=\left(\frac{\nu}{4+\varepsilon}+\frac{2-\nu}{2+\delta}+\frac{2-\nu}{4+\delta}\right).$$

This equation is obtained from the first line of equation (4.4), the left corresponds to normalisation constant C multiplied by ν. Solving this quadratic equation, we obtain for all $\varepsilon,\delta\geq 0$ a unique solution ν with both $\nu\geq 0$ and $2-\nu\geq 0$, and also the unique conditionally invariant density $f(x;\varepsilon,\delta)$ described in Lemma 4.2. This solution linearises to

$$f_c(x;\varepsilon,\delta)\approx\begin{cases}1+\varepsilon/12-\delta/12 & \text{if } x\in(0,1/2),\\ 1-\varepsilon/12+\delta/12 & \text{if } x\in(1/2,1),\end{cases}$$

for small ε,δ. In particular $f_c(x;\varepsilon,\delta)\to 1$ as $\varepsilon,\delta\to 0$. □

4.4 Convergence to the conditionally invariant density

In this subsection we make a first step towards proving Theorem 4.1. We show for some initial densities k that $\mathcal{P}^n_{\varepsilon,\delta}(k)$ does not only converge to the corresponding conditionally invariant density f_c, but that there is an upper bound on the convergence speed which works for all $\varepsilon>0$ and $\delta>0$. Pianigiani and Yorke extensively studied existence of and convergence to conditionally invariant densities for expanding maps in [15]. While their approach does not give us the desired bound on convergence speed, one of their results, the lemma stated below, will be very useful in our proof.

Lemma 4.3 (Pianigiani-Yorke)**.** *Let $\mathcal{P}_F$ be the Frobenius-Perron operator corresponding to a map F on $[0,1]$. Suppose f, g are Lebesgue integrable over $[0,1]$ with $\int_0^1 f(x)\,\mathrm{d}x=\int_0^1 g(x)\,\mathrm{d}x=1$, $\inf\limits_{[0,1]} f(x)>0$, $\sup_n\|\mathcal{P}_F^n(g)\|_\infty<\infty$ and $\sup_n\|\mathcal{P}_F^n(1)\|_\infty<\infty$. Then there exists some L such that for $n\geq 1$*

$$\|\mathcal{P}_F^n(f)-\mathcal{P}_F^n(g)\|_\infty\leq L\|f-g\|_\infty$$

is satisfied. More precisely, we may take

$$L=\frac{1}{\inf\limits_{[0,1]} f}\left(\sup_n\|\mathcal{P}_F^n(1)\|_\infty+\sup_n\|\mathcal{P}_F^n(g)\|_\infty\right).$$

Proof. We can apply [15, Proposition 1]. It requires $f, g \in K = \{f \in C([0,1]) : \sup_{[0,1]} f(x) < \infty, \inf_0^1 f(x)\mathrm{d}x > 0, \int_0^1 f(x)\mathrm{d}x = 1\}$, but the proof also works for the assumptions in Lemma 4.3. Note that here $\inf_{[0,1]} f = \inf\{s \in \mathbb{R} : \lambda(\{x \in [0,1] : f(x) < s\}) = 0\}$. □

We now have all the necessary tools to prove Theorem 4.1. The key idea is to approximate densities by piecewise constant densities.

Lemma 4.4. *Let $\mu > 0$. Then there exists $\omega > 0$ and $N \in \mathbb{N}$ such that for any piecewise constant density $k : [0,1] \to [0,2]$ with*

$$k(t) = \begin{cases} x & \text{if } t \in (0, 1/2), \\ 2 - x & \text{if } t \in (1/2, 1), \end{cases} \tag{4.5}$$

whenever $n \geq N$, $x \in [0,2]$ and $0 < \varepsilon, \delta < \omega$ then

$$||\mathcal{P}^n_{\varepsilon,\delta}(k) - f_c(\,\cdot\,; \varepsilon, \delta)||_\infty < \mu,$$

where $f_c(\,\cdot\,; \varepsilon, \delta) : [0,1] \to \mathbb{R}$ is the conditionally invariant density.

Proof. By [15, Theorem 3], densities $\mathcal{P}^n_{\varepsilon,\delta}(k)$ converge to invariant density f_c as $n \to \infty$. Lemma 4.4 says that this convergence is uniform over all choices of k. Now let k be an arbitrary function satisfying the conditions of the Lemma. Density $\mathcal{P}^n_{\varepsilon,\delta}(k)$ is constant for each $n \in \mathbb{N}$ on both $(0, 1/2)$ and $(1/2, 1)$. First we bound ratio

$$r(x, \varepsilon, \delta) = \frac{k - \mathcal{P}_{\varepsilon,\delta}(k)}{\mathcal{P}_{\varepsilon,\delta}(k) - \mathcal{P}^2_{\varepsilon,\delta}(k)},$$

where x is the value of k appearing in equation (4.5). Note that on $(0, 1/2)$

$$\mathcal{P}_{\varepsilon,\delta}(k) = \frac{1}{c_1(x)}\left(\frac{x}{4+\varepsilon} + \frac{2-x}{2+\delta} + \frac{2-x}{4+\delta}\right) = \frac{a_1(x)}{c_1(x)};$$

where $c_1(x)$ is the normalisation constant C from formula (4.4). Moreover, we obtain $\mathcal{P}_{\varepsilon,\delta}(k) = 2 - a_1(x)/c_1(x)$ on $(1/2, 1)$ from normalisation. Further, on $(0, 1/2)$,

$$\mathcal{P}^2_{\varepsilon,\delta}(k) = \frac{1}{c_2(x)}\left(\frac{a_1(x)/c_1(x)}{4+\varepsilon} + \frac{2 - a_1(x)/c_1(x)}{2+\delta} + \frac{2 - a_1(x)/c_1(x)}{4+\delta}\right) = \frac{a_2(x)}{c_1(x)c_2(x)},$$

where $c_2(x)$ is again the normalisation constant C from formula (4.4) and $a_2(x)$ is a linear polynomial equal to $c_1(x)c_2(x)\mathcal{P}^2_{\varepsilon,\delta}(k)$. Note that for fixed parameters ε and δ denominators $c_1(x)$ and $c_1(x)c_2(x)$ can also be written as linear polynomials of x and are non-zero. With this notation r can be expressed as

$$r(x, \varepsilon, \delta) = \frac{c_1^2(x)c_2(x)x - c_1(x)c_2(x)a_1(x)}{c_1(x)c_2(x)a_1(x) - c_1(x)a_2(x)},$$

a quotient of two polynomials. As a quotient of a cubic and quadratic polynomial, an explicit calculation shows that denominator and enumerator have the same positive root and that this root has multiplicity 1 and is equal to the value of the conditionally invariant density ν from Lemma 4.2. So r can be extended to a continuous function in x, ε and δ for $x \geq 0, \varepsilon \geq 0, \delta \geq 0$. For $\varepsilon = \delta = 0$ a calculation gives $r(x, 0, 0) = -2$. By continuity we can choose some $\omega > 0$ such that $\varepsilon, \delta < \omega$ and $x \in [0,2]$ implies $|r(x, \varepsilon, \delta)| > 3/2$. By repeatedly applying this result,

$$|\mathcal{P}^n_{\varepsilon,\delta}(k)(y) - \mathcal{P}^{n+1}_{\varepsilon,\delta}(k)(y)| < (2/3)^n |k(y) - \mathcal{P}_{\varepsilon,\delta}(k)(y)| \leq 2(2/3)^n, \text{ for } y \in (0, 1/2) \cup (1/2, 1)$$

But as each $\mathcal{P}^n_{\varepsilon,\delta}(k)$ is constant on $(0, 1/2)$ and equal to $2 - \mathcal{P}^n_{\varepsilon,\delta}(k)(1/4)$ on $(1/2 - 1)$, it follows that

$$||\mathcal{P}^n_{\varepsilon,\delta}(k) - f_c(\,\cdot\,; \varepsilon, \delta)||_\infty \leq 6(2/3)^n$$

for each n. □

Densities with three constant pieces are more convenient for approximating a density conditional on a jump having just occurred, something we will look at in the later parts of the proof of Theorem 4.1. So we now focus our attention on such densities.

Definition 4.2. *For $S \geq 0$, we define set K_S of piecewise constant densities $k : [0,1] \to [0,\infty)$ with $\int_0^1 k(t)\,\mathrm{d}t = 1$, which can be written in one of the following two forms*

$$k(t) = \begin{cases} k_1 & \text{if } t \in (0,b), \\ k_2 & \text{if } t \in (b,1/2), \\ k_3 & \text{if } t \in (1/2,1), \end{cases} \qquad \text{where} \quad |k_1 - k_2| \leq S \quad \text{for } b \in (0,1/2), \tag{4.6}$$

or

$$k(t) = \begin{cases} k_1 & \text{if } t \in (0,1/2), \\ k_2 & \text{if } t \in (1/2,b), \\ k_3 & \text{if } t \in (b,1), \end{cases} \qquad \text{where} \quad |k_2 - k_3| \leq S \quad \text{for } b \in (1/2,1), \tag{4.7}$$

where k_1, k_2, k_3 are non-negative constants. We define further map $\Psi : K_S \to [0,\infty)$ with

$$\Psi(k) = \begin{cases} |k_1 - k_2| & \text{for } b \in (0,1/2) \\ |k_2 - k_3| & \text{for } b \in (1/2,1), \end{cases} \tag{4.8}$$

where b is as defined in equations (4.6)–(4.7).

Lemma 4.5. *Let $S > 0$. Then there exist $\omega > 0$ and $B \in [0,1)$ such that for $0 < \varepsilon, \delta < \omega$ and any $k \in K_S$ we have $\mathcal{P}_{\varepsilon,\delta}(k) = k' \in K_S$ with $\Psi(k') \leq B\,\Psi(k)$.*

Proof. Let $k \in K_S$. Let b be as defined in equations (4.6)–(4.7). First, assume $b(4+\varepsilon) < 1/2$. A direct calculation leads to

$$\mathcal{P}_{\varepsilon,\delta}(k)(t) = \begin{cases} k_1' &= \dfrac{1}{C}\left(\dfrac{k_1}{4+\varepsilon} + \dfrac{k_3}{2+\delta} + \dfrac{k_3}{4+\delta}\right) & \text{if } t \in (0, b(4+\varepsilon)), \\ k_2' &= \dfrac{1}{C}\left(\dfrac{k_2}{4+\varepsilon} + \dfrac{k_3}{2+\delta} + \dfrac{k_3}{4+\delta}\right) & \text{if } t \in (b(4+\varepsilon), 1/2), \\ k_3' &= \dfrac{1}{C}\left(\dfrac{k_2}{4+\varepsilon} + \dfrac{k_2}{2+\varepsilon} + \dfrac{k_3}{4+\delta}\right) & \text{if } t \in (1/2, 1), \end{cases}$$

where C is the normalisation constant. The difference between the values of $\mathcal{P}_{\varepsilon,\delta}(k)$ on $(0,1/2)$ is given by

$$|k_1' - k_2'| = \frac{1}{C}\frac{|k_1 - k_2|}{4+\varepsilon}.$$

Proceeding in the same way for all other possible choices of b we get

$$\Psi(k') \leq \Psi(k)/(2C). \tag{4.9}$$

Now we want to bound C below. Say $b > 1/2$. If $k_2 > m$, then $k_3 > m - S$ and as the density is non-negative, $\int_0^1 k(t)\,\mathrm{d}t > (m-S)/2$. But as $\int_0^1 k(t)\,\mathrm{d}t = 1$ we get a contradiction for $m \geq 2 + S$. Similarly if instead $k_3 > m$. We also need $k_1 \leq 2$. For k constant on $(1/2,1)$ we proceed in the same way. So all functions in K_S are bounded above by $2+S$. Let $\ell(\varepsilon) + \ell(\delta)$ be the Lebesgue measure of the subset of $[0,1]$ which is mapped outside the unit interval by $F(t;\varepsilon,\delta)$, as in equation (4.2). Then $\ell(\varepsilon) + \ell(\delta) \to 0$ as $\varepsilon, \delta \to 0$. For sufficiently small parameters, say $0 < \varepsilon, \delta < \omega$, we will have normalisation constant $C \geq 1 - (\ell(\varepsilon) + \ell(\delta))(2+S) \geq 2/3$. Then substituting into inequality (4.9), we obtain inequality (4.9). □

Next we note how Lemma 4.3 can be applied to densities of this piecewise constant form.

Corollary 4.2. *Let $S \geq 0$ and $s > 0$. Let K_S be as described in Lemma 4.5. There exists $L > 0$ and $\omega > 0$ such that for any $0 < \varepsilon, \delta < \omega$, $g \in K_S$, density f with $\inf_{[0,1]} f > s$ and $n \geq 1$*

$$||\mathcal{P}^n_{\varepsilon,\delta}(f) - \mathcal{P}^n_{\varepsilon,\delta}(g)||_\infty \leq L||f - g||_\infty.$$

Proof. By Lemma 4.5 there exists $\omega > 0$ such that $0 < \varepsilon, \delta < \omega$ and $g \in K_S$ imply $\mathcal{P}^n_{\varepsilon,\delta}(g) \in K_S$ for each $n \geq 1$. Using the last paragraph of the proof of Lemma 4.5, then $\sup_n ||\mathcal{P}^n_{\varepsilon,\delta}(g)||_\infty \leq 2+S$. Also, since $\mathcal{P}_{\varepsilon,\delta}(1)$ is constant on both $(0, 1/2)$ and $(1/2, 1)$, we have $||\mathcal{P}^n_{\varepsilon,\delta}(1)||_\infty \leq 2$. By applying Lemma 4.3, we find that for $0 < \varepsilon, \delta < \omega$, density f with $\inf_{[0,1]} f > s$ satisfies

$$||\mathcal{P}^n_{\varepsilon,\delta}(f) - \mathcal{P}^n_{\varepsilon,\delta}(g)||_\infty \leq L||f - g||_\infty,$$

where $L = (4 + S)/s$. □

Lemma 4.6. *Let $\mu > 0$ and $S > 0$. There exist $\omega > 0$ and $N \in \mathbb{N}$ such that for all $n \geq N$, $0 < \varepsilon, \delta < \omega$ and $k \in K_S$*

$$||\mathcal{P}^n_{\varepsilon,\delta}(k) - f_{c_{\varepsilon,\delta}}||_\infty < \mu.$$

Proof. First, use Lemma 4.5 to observe that for any $S_1 \in (0, S)$ there exist $\omega_1 > 0$ and $N_1 \in \mathbb{N}$ such that for any $k \in K_S$ and $0 < \varepsilon, \delta < \omega_1$ we have $\mathcal{P}^{N_1}_{\varepsilon,\delta}(k) \in K_{S_1}$. But S_1 can be chosen small enough that there is some $s > 0$ such that for any $k \in K_S$ and $N_2 \geq N_1 + 1$ we have $\inf_{[0,1]} \mathcal{P}^{N_2}_{\varepsilon,\delta}(k) > s$. Apply Corollary 4.2 with $f = \mathcal{P}^{N_2}_{\varepsilon,\delta}(k)$ to find some $L > 0$ and $\omega_2 > 0$ such that when $0 < \varepsilon, \delta < \omega_2$, $N_2 \geq N_1 + 1$ and $g \in K_S$ then

$$\left|\left|\mathcal{P}^{N_2+n}_{\varepsilon,\delta}(k) - \mathcal{P}^n_{\varepsilon,\delta}(g)\right|\right|_\infty \leq L\left|\left|\mathcal{P}^{N_2}_{\varepsilon,\delta}(k) - g\right|\right|_\infty. \tag{4.10}$$

Also by Lemma 4.5 we can choose N_2 such that for any $k \in K_S$ and $0 < \varepsilon, \delta < \min\{\omega_1, \omega_2\}$, there exists a piecewise constant density $g \in K_S$ such that g is constant on $(0, 1/2)$, g is constant on $(1/2, 1)$ and $||\mathcal{P}^{N_2}_{\varepsilon,\delta}(k) - g||_\infty < \mu/(2L)$. Then by equation (4.10), we get

$$\left|\left|\mathcal{P}^{N_2+n}_{\varepsilon,\delta}(k) - \mathcal{P}^n_{\varepsilon,\delta}(g)\right|\right|_\infty \leq \frac{\mu}{2}$$

for $n \geq 1$. By Lemma 4.4 we also find $N_3 \in \mathbb{N}$ and $\omega_3 > 0$ such that for $0 < \varepsilon, \delta < \omega_3$ and $n \geq N_3$

$$||\mathcal{P}^n_{\varepsilon,\delta}(g) - f_{c_{\varepsilon,\delta}}|| < \frac{\mu}{2}.$$

Now take $N = N_2 + N_3$ and $\omega = \min\{\omega_1, \omega_2, \omega_3\}$ to complete the proof. □

4.5 Proof of Theorem 4.1

Recall from Corollary 4.1 that invariant density $f_{i,m}(x)$ converges uniformly to 1 on $[d, 1-d]$ as $m \to \infty$ for any $d > 0$. In our proof we need this convergence to be extended to all of $[0, 1]$. Therefore we first prove an alternate version of Theorem 4.1, in which we do not start with the invariant density f_i, but a related density f'_i and process Y'_m, defined below.

Definition 4.3. *Fix some $0 < d < 1/2$ and define*

$$f'_i(x; \varepsilon/m, \delta/m) = \begin{cases} f_i(x; \varepsilon/m, \delta/m) & \text{if} \quad x \in [d, 1-d] \\ k_m & \text{elsewhere,} \end{cases} \tag{4.11}$$

where k_m is chosen so that $f'_i(x, \varepsilon/m, \delta/m)$ integrates to 1 over $[0, 1]$. Define

$$Y'_m(t) = \left\lfloor F^{\lfloor mt \rfloor}(X'_m; \varepsilon/m, \delta/m) \right\rfloor,$$

where X'_m is distributed according to $f'_i(x; \varepsilon/m, \delta/m)$. Let $T'_{m,1}, T'_{m,2}, \ldots$ denote the jump times of $Y'_m(t)$, that is,

$$T'_{m,j} = \min\{t \in [T'_{m,j-1}, \infty) : Y'_m(t) \neq Y'_m(T'_{m,j-1})\}$$

where $T'_{m,0} = 0$.

We now describe the behaviour of the first jump $T'_{m,1}$ for process $Y'_m(t)$. To simplify our notation, we will subsequently write

$$\begin{aligned} f_{c,m}(x) &= f_c(x; \varepsilon/m, \delta/m), & F_m(x) &= F(x; \varepsilon/m, \delta/m), \\ f_{i,m}(x) &= f_i(x; \varepsilon/m, \delta/m), & \mathcal{P}_m &= \mathcal{P}_{\varepsilon/m,\delta/m}, \\ f'_{i,m}(x) &= f'_i(x; \varepsilon/m, \delta/m). \end{aligned}$$

Lemma 4.7. *Let $\delta, \varepsilon > 0$. Let $0 < 1/2 < d$. For $m \in \mathbb{N}$ let $Y'_m(t)$ and $T'_{m,1}$ be as described in Definition 4.11. Then for $\tau > 0$*

$$\mathbb{P}(T'_{m,1} \leq \tau) \to 1 - \exp(-\gamma\tau), \qquad \text{as } m \to \infty,$$

where γ is given by equation (4.3).

Proof. Apply Corollary 4.2 with $s = 1/2$, $S = 0$ to find $L > 0$ such that for large enough m, say $m \geq M_1$, and densities f with $\inf_{[0,1]} f > 1/2$

$$||\mathcal{P}^n_m(f) - \mathcal{P}^n_m(1)||_\infty \leq L||f - 1||_\infty. \tag{4.12}$$

Choose $\mu > 0$ small enough so that $\mu(1 + 2d) < d$. Then for large enough m, say $m \geq M_2 \geq M_1$, we have $|f_{i,m}(x) - 1| < \mu$ on $[d, 1-d]$ and by considering bounds on k_m, the value of $f'_{i,m}$ on $(0,d) \cup (1-d, 1)$, we get $||f'_{i,m} - 1||_\infty \leq \mu(1 + 1/(2d))$. Since $\mu(1 + 1/(2d) < 1/2$, we have $f'_{i,m}$ bounded below by $1/2$. Then using equation (4.12) we have

$$||\mathcal{P}^n_m(f'_{i,m}) - \mathcal{P}^n_m(1)||_\infty \leq L\mu\left(1 + \frac{1}{2d}\right), \tag{4.13}$$

when $n \geq 1$ and $m \geq M_2$. By Lemma 4.4 there also exists some $M_3 \geq M_2$ and $N \in \mathbb{N}$ such that when $m \geq M_3$ and $n \geq N$ we have $||\mathcal{P}^n_m(1) - f_{c,m}||_\infty < \mu$. Write $B(d) = 1 + L(1 + 1/(2d))$ and observe, using equation (4.13), that whenever $m \geq M_3$ and $n \geq N$

$$||\mathcal{P}^n_m(f'_{i,m}) - f_{c,m}||_\infty \leq \mu B(d). \tag{4.14}$$

Let $\tau > 0$. Then pick $m > \max\{M_3, N/\tau\}$ and consider the probability that no jump occurs until time τ for $Y'_m(t)$,

$$\mathbb{P}(T'_{m,1} > \tau) = \prod_{n=1}^{\lfloor m\tau \rfloor} \mathbb{P}\left(F^n_m(X'_m) \in [0,1) \mid F^1_m(X'_m), \ldots, F^{n-1}_m(X'_m) \in [0,1)\right).$$

Write A_m for the subset of $[0,1]$ mapped outside the unit interval by F_m. As in equation (4.2), denote by $\ell(\varepsilon/m) + \ell(\delta/m)$ the length of A_m. Write $f_{c,m} = \nu_m$ on $(0, 1/2)$ and $f_{c,m} = 2 - \nu_m$ on $(1/2, 2)$. Using equation (4.14), we get the following upper estimate

$$\prod_{n=N+1}^{\lfloor m\tau \rfloor} \mathbb{P}\left(F^n_m(X'_m) \in [0,1) \mid F^1_m(X'_m), \ldots, F^{n-1}_m(X'_m) \in [0,1)\right) = \prod_{n=N+1}^{\lfloor m\tau \rfloor} \int_{[0,1]\setminus A_m} \mathcal{P}^{n-1}_m(f'_{i,m})(x)\,\mathrm{d}x$$

$$\leq \left(1 - \left(\nu_m - \mu B(d)\right)\ell\left(\frac{\varepsilon}{m}\right) - \left(2 - \nu_m - \mu B(d)\right)\ell\left(\frac{\delta}{m}\right)\right)^{\lfloor m\tau \rfloor - N}.$$

Since $m\,\ell(\varepsilon/m) + m\,\ell(\delta/m) \to \gamma$ as $m \to \infty$, where γ is given by equation (4.3), and $\nu_m \to 1$ as $m \to \infty$ by Lemma 4.2, the upper bound converges to $\exp[-\gamma(1 - \mu B(d))\tau]$ as $m \to \infty$. By a similar argument we have a lower bound converging to $\exp[-\gamma(1 + \mu B(d))\tau]$ as $m \to \infty$.

Let A^N_m be the subset of $[0,1]$, which is mapped outside $[0,1]$ within at most N applications of F_m. The Lebesgue measure of A^N_m goes to 0 as $m \to \infty$. From equation (4.1) we notice that $\sup_m ||f_{i,m}||_\infty < \infty$ and also $\sup_m ||f'_{i,m}||_\infty < \infty$. So by integrating over A^N_m, we obtain

$$\prod_{n=1}^{N} \mathbb{P}\left(F^n_m(X'_m) \in [0,1) \mid F^1_m(X'_m), \ldots, F^{n-1}_m(X'_m) \in [0,1)\right) = 1 - \int_{A^N_m} f'_{i,m}(x)\,\mathrm{d}x \to 1$$

as $m \to \infty$. So $\mathbb{P}(T'_{m,1} > \tau)$ is bounded above by a product converging to $\exp[-\gamma(1 - \mu B(d))\tau]$ and below by a product converging to $\exp[-\gamma(1 + \mu B(d))\tau]$ as $m \to \infty$. But $\mu > 0$ was arbitrary, so $\mathbb{P}(T'_{m,1} \leq \tau) \to 1 - \exp(-\gamma\tau)$ as $m \to \infty$.

□

Now we will prove an alternate version of Theorem 4.1 for $Y_m'(t)$. Lemma 4.7 established such a statement already for the first jump. The key component in the general proof will be the following lemma, which helps in describing how the densities develop after a jump, conditional on no further jump occurring, provided we start off close to the conditionally invariant density $f_{c,m}$.

Lemma 4.8. *Let*

$$A_m^1 = \left(\frac{1}{4} - \frac{\varepsilon/m}{4(4+\varepsilon/m)}, \frac{1}{4}\right)$$

denote the subset of $(0, 1/4)$ *mapped outside of* $[0, 1]$ *by* F_m*. Take* $0 < \mu < 1/4$ *and let* g_m *be a density with* $||g_m - f_{c,m}||_\infty < \mu$*. Let* V_m *be distributed according to that density and* $g^\star$ *denote the density corresponding to the distribution of* $F_r(V_m; \varepsilon/m, \delta/m)$*, conditional on* $V_m \in A_m^1$*. Then there exists* $B > 0$*,* $M \in \mathbb{N}$ *and* $N \in \mathbb{N}$ *such that for all* $m \geq M$ *and* $n \geq N + \lceil \log(m/\varepsilon)/\log(4 + \varepsilon/m)\rceil$ *and for any sequence of* g_m *satisfying above properties we have*

$$||\mathcal{P}_m^n(g_m^\star) - f_{c,m}||_\infty < \mu B. \tag{4.15}$$

Proof. Since $V_m \in A_m^1$, we have $F_r(V_m; \varepsilon/m, \delta/m) \in [0, \varepsilon/(4m)]$. On $(\varepsilon/(4m), 1)$ we then have $g_m^\star = 0$, while we have

$$g_m^\star(x) = \frac{1}{c_0}\left(g_m\left(\frac{1+x}{4+\varepsilon/m}\right)\right), \quad \text{for } x \in \left(0, \frac{\varepsilon}{4m}\right), \tag{4.16}$$

where c_0 is a constant so that $g_m^\star$ integrates to 1 over $[0, 1]$. Let u_m denote the smallest integer such that $F_r^{u_m+1}(1/4; \varepsilon/m, \delta/m) = (4+\varepsilon/m)^{u_m}\varepsilon/(4m) \geq 1/4$. Then $u_m = \lceil \log(m/\varepsilon)/\log(4+\varepsilon/m)\rceil$. From equation (4.4) we deduce that for $n \leq u_m$, density $\mathcal{P}_m^n(g_m^\star)$ is obtained from $g_m^\star$ via a scaling of the form

$$\mathcal{P}_m^n(g_m^\star)(x) = \frac{1}{c_n}\, g_m^\star\left(\frac{x}{(4+\varepsilon/m)^n}\right) = \frac{1}{c_0 c_n}\left(g_m\left(\frac{1 + x/(4+\varepsilon/m)^n}{4+\varepsilon/m}\right)\right), \tag{4.17}$$

where c_n is a constant dependent on m, such that $\mathcal{P}_m^n(g_m^\star)(x)$ integrates to 1 over $[0, 1]$. By Lemma 4.2, there exists $M_1 \in \mathbb{N}$ such that $m \geq M_1$ implies $1/2 < f_{c,m} < 3/2$. Since $\mu < 1/4$, we obtain a bound $g_m(y) \geq 1/4$ and estimate

$$\mathcal{P}_m^{u_m}(g_m^\star)(x) \geq \frac{1}{4c_0 c_{u_m}} \quad \text{for } x \in I_{u_m} = \left(0, \frac{\varepsilon}{4m}\left(4 + \frac{\varepsilon}{m}\right)^{u_m}\right). \tag{4.18}$$

Using $(4+\varepsilon/m)^{u_m}\varepsilon/(4m) \geq 1/4$ and equation (4.18), we get $c_0 c_{u_m} \geq 1/16$, since $\mathcal{P}_m^{u_m}(g_m^\star)$ must integrate to 1 over $[0, 1]$. But recall that $||g_m - f_{c,m}||_\infty < \mu$ and $f_{c,m}$ constant on $(0, 1/2)$. Combining this with equation (4.17) gives us that the values of $\mathcal{P}_m^{u_m}(g_m^\star)$ on I_{u_m} are contained in a subinterval of $(0, \infty)$ of length 32μ. But applying equation (4.4) again, we see that for $\mathcal{P}_m^{u_m+1}(g_m^\star)$ the unit interval can be split into three subintervals $(0, b_1)$, (b_1, b_2) and $(b_2, 1)$, where $b_1 = 1/2$ or $b_2 = 1/2$, on each of which the values of $\mathcal{P}_m^{u_m+1}(g_m^\star)$ are contained in an subinterval of $(0, \infty)$ of length $32\mu/C$. Here C is the normalisation constant from equation (4.4). From $f_{c,m} < 3/2$ and $\mu < 1/4$ we get $g_m(y) < 2$ and $\mathcal{P}_m^{u_m}(g_m^\star)(x) \leq 2/(c_0 c_{u_m}) \leq 32$, since we deduced earlier from equation (4.18) that $c_0 c_{u_m} \geq 1/16$. For m large enough, say $m \geq M_2 > M_1$, we will have $\ell(\varepsilon/m) + \ell(\delta/m) < 1/64$, where ℓ is defined as in equation (4.2). Considering the integral of $\mathcal{P}_m^{u_m}(g_m^\star)$ over the subset of $[0, 1]$ mapped outside the unit interval by F_m, we obtain a lower bound of $1 - 32/64 = 1/2$ on C. So the values of $\mathcal{P}_m^{u_m+1}(g_m^\star)$ on each of $(0, b_1)$, (b_1, b_2), $(b_2, 1)$ are contained in intervals of length 64μ. Using (4.4), calculations show that we can find a bound, independent of choice of g_m with $||g_m - f_{c,m}||_\infty < 1/4$, on the range of values of $\mathcal{P}_m^{u_m+1}(g_m^\star)$ over all of $(0, 1/2)$ and $(1/2, 1)$ respectively. Call this bound S. We now may choose a piecewise constant map $k_m \in K_S$, as defined in Definition 4.2, so that

$$||\mathcal{P}_m^{u_m}(g_m^\star) - k||_\infty < 32\mu.$$

Recalling that $f_{c,m} > 1/2$, we get $g_m > 1/4$, and equation (4.17) gives us lower bound of 1/4 on $\mathcal{P}_m^{u_m}(g_m^\star)$ and by equation (4.4) a lower bound on $\mathcal{P}_m^{u_m+1}(g_m^\star)$, which could be taken for example as 1/20. We then apply Corollary 4.2 with $f = \mathcal{P}_m^{u_m+1}(g_m^\star)$, $g = k_m$, $s = 1/20$ and to find L such that for $n \geq 1$ we have

$$||\mathcal{P}_m^{u_m+n}(g_m^\star) - \mathcal{P}_m^n(k_m)|| < \mu 32L, \tag{4.19}$$

regardless of choice of g_m. By Lemma 4.6 there exists N such that for all $m \geq M_2$ and for $n \geq N$ we have $||\mathcal{P}_m^n(k_m) - f_{c,m}||_\infty < \mu$. Take $B = 32L + 1$ and $M = M_2$ to obtain equation (4.15). ☐

Lemma 4.9. *Let A_m denote the subset of $[0,1]$ mapped outside of $[0,1]$ by F_m. Take $0 < \mu < 1/4$ and let g_m be a density with $||g_m - f_{c,m}||_\infty < \mu$. Let V_m be distributed according to that density and $g^\star$ denote the density corresponding to the distribution of $F_r(V_m; \varepsilon/m, \delta/m)$, conditional on $V_m \in A_m$. Then there exists $B > 0$, $M \in \mathbb{N}$ and $N \in \mathbb{N}$ such that for all $m \geq M$ and $n \geq N + \lceil \log(m/\varepsilon)/\log(4 + \varepsilon/m) \rceil$ and for any sequence of g_m satisfying above properties we have*

$$||\mathcal{P}_m^n(g_m^\star) - f_{c,m}||_\infty < \mu B. \tag{4.20}$$

Proof. We partition on events $V_m \in (0, 1/4)$, $V_m \in (1/4, 1/2)$, $V_m \in (1/2, 3/4)$ and $V_m \in (3/4, 1)$, then apply the same arguments as for $V_m \in (0, 1/4)$ in Lemma 4.8. ☐

Lemma 4.10. *Let $\delta, \varepsilon > 0$. Let $0 < 1/2 < d$. For $m \in \mathbb{N}$ let $Y'_m(t)$ and $T'_{m,j}$, $j = 1, 2, \ldots,$ be as described in Definition 4.3. Then for any $k \geq 1$ and $0 < t_1 < \cdots < t_k$ we have*

$$\mathbb{P}\left(T'_{m,k+1} \leq \frac{\lfloor mt_k \rfloor}{m} + \tau \,\middle|\, T'_{m,k} = \frac{\lfloor mt_k \rfloor}{m}, \ldots, T'_{m,1} = \frac{\lfloor mt_1 \rfloor}{m}\right) \to 1 - \exp(-\gamma\tau)$$

as $m \to \infty$, where γ is given in equation (4.3).

Proof. We first describe how the density corresponding to $F_r^{\lfloor mt \rfloor}(X'_m; \varepsilon/m, \delta/m)$ develops, conditional on $T'_{m,k} = \lfloor mt_k \rfloor/m, \ldots, T'_{m,1} = \lfloor mt_1 \rfloor/m$. Let $\mu > 0$. Using equation (4.14), there exist $N_1 \in \mathbb{N}$ and $M_1 \in \mathbb{N}$ so that $n \geq N_1$ and $m \geq M_1$ implies $||\mathcal{P}_m^n(f'_{i,m}) - f_{c,m}||_\infty \leq \mu B(d)$. For large enough m we will have $\lfloor mt_1 \rfloor > N_1$. Choosing μ small enough, we will have $\mu B(d) < 1/4$ and so can apply Lemma 4.9 with $g_m = \mathcal{P}_m^{\lfloor mt_1 \rfloor - 1}(f'_{i,m})$. Write $g_m^{(1)} = g_m^\star$ for the density after the jump at time $\lfloor mt_1 \rfloor/m$. There exists $N \in \mathbb{N}$ and $B > 0$ such that for large enough m and $n \geq N + \lceil \log(m/\varepsilon)/\log(4 + \varepsilon/m) \rceil = N + S(m)$

$$\left|\left|\mathcal{P}_m^n\big(g_m^{(1)}\big) - f_{c,m}\right|\right|_\infty < \mu B(d)B.$$

Between time $\lfloor t_j m \rfloor/m$ and time $\lfloor t_{j+1} m \rfloor/m$, the density of $F_r^{\lfloor mt \rfloor}(X'_m; \varepsilon/m, \delta/m)$ develops as given by applying operator $\mathcal{P}_m$. Provided that $\mu < B^{-k}/4$ and m is large enough so that $N + S(m) < \lfloor mt_{j+1} \rfloor - \lfloor mt_j \rfloor$ for $j = 1, 2, \ldots, k$, we can iteratively apply Lemma 4.9 with $g_m = \mathcal{P}_m^{\lfloor mt_{j+1} \rfloor - \lfloor mt_j \rfloor - 1}\big(g_m^{(j)}\big)$, where $g_m^{(j)}$ is the density after the j-th jump has occurred, at time $\lfloor mt_j \rfloor/m$. Then for large enough m and $n \geq N + S(m)$ we in particular find

$$\left|\left|\mathcal{P}_m^n\big(g_m^{(k)}\big) - f_{c,m}\right|\right|_\infty \leq \mu B(d)B^k. \tag{4.21}$$

But $\mathcal{P}_m^n\big(g_m^{(k)}\big)$ describes the densities of $F_r^{\lfloor mt \rfloor}(X'_m; \varepsilon/m, \delta/m)$ conditional on $T'_{m,k} = \lfloor mt_k \rfloor/m, \ldots,$ $T'_{m,1} = \lfloor mt_1 \rfloor/m$ and no further jump occurring. Choose $\tau > 0$. Then

$$\mathbb{P}\left(T_m^k > t_k + \tau \,\middle|\, T_m^k = \frac{\lfloor mt_k \rfloor}{m}, \ldots, T_m^1 = \frac{\lfloor mt_1 \rfloor}{m}\right) = \prod_{n=1}^{\lfloor m\tau \rfloor} \int_{[0,1]\setminus A_m} \mathcal{P}_m^{n-1}\big(g_m^{(k)}\big)(x)\,\mathrm{d}x,$$

where A_m is defined as in Lemma 4.9. Since equation (4.21) holds, we can use the same arguments as in the proof of Lemma 4.7 to find lower and upper bounds on

$$\prod_{n=S(m)+1}^{\lfloor m\tau \rfloor} \int_{[0,1]\setminus A_m} \mathcal{P}_m^{n-1}\big(g_m^{(k)}\big)(x)\,\mathrm{d}x$$

converging to $\exp[-\gamma(1+\mu B(d)B)\tau]$ and $\exp[-\gamma(1-\mu B(d)B)\tau]$, respectively, as $m \to \infty$. Using equation (4.19) from Lemma 4.8, there are $l_m \in K_S$ and $L > 0$ such that

$$\left|\left|\mathcal{P}_m^{u_m+n}\big(g_m^{(k)}\big) - \mathcal{P}_m^n(l_m)\right|\right|_\infty < 32\mu B(d)B^k L,$$

for $n \geq 1$, implying that $\mathcal{P}_m^n\big(g_m^{(k)}\big)$ stays close to a piecewise constant function in K_S as soon as jumps are possible. This tells us that an upper bound b on the densities $\mathcal{P}_m^n\big(g_m^{(k)}\big)$ can be found, valid for all $n \geq u_m$ and all m large enough. But then

$$(1 - b(\ell(\varepsilon/m) + \ell(\delta/m)))^{S(m)} \leq \prod_{n=1}^{S(m)} \int_{[0,1]\setminus A_m} \mathcal{P}_m^{n-1}\big(g_m^{(k)}\big)(x)\,\mathrm{d}x \leq 1.$$

The expression on the left converges to 1 as $m \to \infty$ since $S(m)$ grows like log. But combining this with our earlier bounds with limits $\exp[-\gamma(1 \pm \mu B(d)B)\tau]$ and letting $\mu \to 0$, we find that

$$\mathbb{P}\left(T'_{m,k+1} \leq \frac{\lfloor mt_k \rfloor}{m} + \tau \;\Big|\; T'_{m,k} = \frac{\lfloor mt_k \rfloor}{m}, \ldots, T'_{m,1} = \frac{\lfloor mt_1 \rfloor}{m}\right) \to 1 - \exp(-\gamma\tau) \quad \text{as } m \to \infty.$$

□

Theorem 4.1 is now simply a corollary of Lemma 4.10. Let $Y_m(t)$ and $Y'_m(t)$ be as defined in Theorem 4.1 and Definition 4.3 respectively, recalling that Y'_m depends on a choice of $0 < d < 1/2$. Write E and E' for events $\big(T_{m,k} = \lfloor mt_k \rfloor/m, \ldots, T_{m,1} = \lfloor mt_1 \rfloor/m\big)$ and $\big(T'_{m,k} = \lfloor mt_k \rfloor/m, \ldots, T'_{m,1} = \lfloor mt_1 \rfloor/m\big)$, respectively. By definition of initial distributions of Y_m and Y'_m, X_m and X'_m, with underlying densities $f_{i,m}$ and $f'_{i,m}$, we have that

$$\mathbb{P}\left(T_{m,k+1} \leq \frac{\lfloor mt_k \rfloor}{m} + \tau \Big| E, X_m \in [d, 1-d]\right) = \mathbb{P}\left(T'_{m,k+1} \leq \frac{\lfloor mt_k \rfloor}{m} + \tau \Big| E', X'_m \in [d, 1-d]\right).$$

We have already noted in the proof of Lemma 4.7, that $\sup_m ||f_{i,m}||_\infty < \infty$ and $\sup_m ||f'_{i,m}||_\infty < \infty$. Conditioning on the events $X_m, X'_m \in [d, 1-d]$ and $X_m, X'_m \notin [d, 1-d]$, we get

$$\begin{aligned}&\left|\mathbb{P}\Big(T_{m,k+1} \leq \frac{\lfloor mt_k \rfloor}{m} + \tau \Big| E\Big) - \mathbb{P}\Big(T'_{m,k+1} \leq \frac{\lfloor mt_k \rfloor}{m} + \tau \Big| E'\Big)\right| \\ &\qquad \leq \mathbb{P}(X_m \notin [d, 1-d]) + \mathbb{P}(X'_m \notin [d, 1-d]) \;\leq\; 2d\left(\sup_m ||f_{i,m}||_\infty + \sup_m ||f'_{i,m}||_\infty\right).\end{aligned}$$

Since $0 < d < 1/2$ was chosen arbitrarily, we can then apply Lemma 4.10 and let $d \to 0$ to conclude the proof.

□

5 Conclusion

Our discussion has shown how sequences (1.1) generated from shift-periodic maps can give rise to a large variety of stochastic processes. The results on integer spikes in Section 3 describe well the behaviour of (x_n) for a subclass of shift-periodic maps, both in unscaled form with discrete-time random walks, and in scaled form, with Lévy motions and continuous-time random walks.

While the treatment of maps with non-integer spikes is more difficult and we restricted our attention in Section 4 to the specific example of maps given in Example 2.1, the ideas can easily be extended to a variety of other maps. We could, for instance, replace $F(x; \varepsilon/m, \delta/m)$ in Theorem 4.1 by a sequence of shift-periodic maps $F_m : \mathbb{R} \to \mathbb{R}^\infty$ such that the conditionally invariant density of F_m on interval $[0,1]$ converges uniformly to 1, as $m \to \infty$, and they satisfy $\lambda\{x \in [0,1] : F_m(x) \notin [0,1]\} \to 0$ and $m\lambda\{x \in [0,1] : F_m(x) \notin [0,1]\} \to \gamma$, as $m \to \infty$. Only minor amendments to the proof would again show us that we obtain behaviour like that of a continuous-time random walk in a limit, this time with waiting times distributed according to an exponential distribution with mean $1/\gamma$.

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