A Second Philosophy account of the introduction of Forcing

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Maddy's philosophy of mathematics

- *Naturalism in Mathematics*, Oxford University Press, 1997.
- *Second Philosophy*, Oxford University Press, 2007.
- *Defending the Axioms: On the Philosophical Foundations of Set Theory*, Oxford University Press, 2011. (DA)
- Set-theoretic Foundations, In: Andrés Eduardo Caicedo, James Cummings, Peter Koellner Paul B. Larson (eds.). American Mathematical Society, 2016 (STF)

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Following a descriptive account of Maddy's Second Philosophy, we work internal in the setting of Second Philosophy to describe how set-theoretic methodology changed with the introduction of forcing. We will specifically focus on the use of models of set theory in Cohen's proof. We give an outlook on how this new picture of set-theoretic methodology corresponds with several of Maddy's foundational roles of set theory.

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2. How does SP inquiry into set-theoretic methodology?
    - Case study of actual set-theoretic practice with main focus on means-end relations between methods and goals.
    - Assume that the examples from the case study are typical.

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    - Foundations for what?
    - Changing foundational goals.

Second philosophy

The Second philosopher persona (SP)

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[The SP] begins from common sense, she trusts her perceptions, subject to correction, but her curiosity pushes her beyond these to careful and precise observation, to deliberate experimentation, to the formulation and stringent testing of hypotheses, to devising ever more comprehensive theories, all in the interest of learning more about what the world is like. [...] [S]he is always on the alert to improve her methods of observation, of experimental design, of theory testing, and so on, undertaking to improve her methods as she goes. (SP paper, p. 77)

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> *In this humdrum way [asking questions, formulating hypothesis, testing them etc], by entirely natural steps, our inquirer has come to ask questions typically classified as philosophical. She doesn't do so from some special vantage point outside of science, but as an active participant, entirely from within.*

(DA, p.39)

SP and mathematics

> *[The SP has] good reason to pursue mathematics herself, as part of her investigation of the world, but she also recognizes that it is developed using methods that appear quite different from the sort of observation, experimentation, and theory formation that guide the rest of her research. This raises [the following question:] as part of her continual evaluation and assessment of her methods of investigation, she will want to know how best to carry on this particular type of inquiry.*

(DA, p. 39)

> *[A]ny attempt at a once-and-for-all characterization of our inquirer's methods would run counter to the ever-improving, open-ended nature of her project. So I'm not advocating any meta-philosophical doctrine or principle to the effect that we should 'trust only science'; I'm simply describing this inquirer, counting on you to get the hang of how she would approach the various traditionally philosophical questions we're interested in.*

(DA, p. 40)

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- SP has an interest in understanding mathematical methodology:
    - Mathematics is an important part of the world and of the SP's scientific methods.
    - Mathematical methodology seems to be different from scientific methodology.

Inquire into set-theoretic methodology

SP's procedure

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a. Analyze examples from actual set-theoretic practice via means-end relations: Identifying a mathematical goal in the practice, set-theoretic methods are rational if they are effective means towards this goal.

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a. Analyze examples from actual set-theoretic practice via means-end relations: Identifying a mathematical goal in the practice, set-theoretic methods are rational if they are effective means towards this goal.
b. Argue that the examples chosen in a. are good examples: They should not be heuristic aids, they should be methodologically relevant, part of the evidential structure of the subject and based on shared convictions that actually drive the practice.

Case study 1: Cantor's introduction of set

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Method: Introducing the new entity of "set" (as point sets).

Conclusion: Introducing sets is a rational method because it is an effective means towards a mathematical ends.

Notice: There is no metaphysical claim connected to this. Instead Maddy uses terms like "exists" or even "ontology" not in " any philosophically loaded way: I just mean what the practice asserts to exist, leaving the semantic or metaphysical issues open." (STF, p. 296)

Case study 2: Zermelo's defense of his axiomatization

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What examples should not be: heuristic aids

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Example: Dedekind believes that the natural numbers are "free creations of the human mind" (1888).

> *Given the wide range of views mathematicians tend to hold on these matters, it seems unlikely that the many analysts, algebraists, and set theorists ultimelty led to embrace sets would all agree on a single conception of the nature of mathematical objects in general, or of sets in particular; the Second Philosopher concludes that such remarks should be treated as colorful asides or heuristic aids, but not as part of the evidential structure of the subject.*

(DA, p. 52/53)

The iterative conception

Example (from STF, p. 303): A case where an axiom A is introduced (means) because it complies well with the iterative conception (ends) should not be considered, because the iterative conception is merely “a brilliant heuristic device”. Instead the end should be to “further various mathematical goals of set theory, including its foundational ones.”

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Interpretation 1. This is meant in the same way as the 'heuristic aids'. (Context of the quote)

Interpretation 2. In DA, Maddy seems to mean more: examples should be typical, part of the evidential structure of the subject and based on shared convictions that actually drive the practice.

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Atypical: from a historical perspective; being in tension with already established goals/methods; from the perspective of the community; ...

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The introduction of sets:

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- This means-ends argument "can be tested for plausibility in the eyes of contemporary practitioners" (NM, p.197) and pass as relevant.
- It survived the historical progress of mathematics, i.e. it was neither marginalized nor eliminated.
- It lies at the core of the subject (actually driving the practice).

Case study: Cohen's introduction of forcing

Material

- Cohen, *Set theory and the Continuum Hypothesis*, 1966.
- Cohen, The Discovery of Forcing, 2002. (DF)
- Moore, The Origins of Forcing, 1987.

Goal and method

Goal: Prove the independence of AC and CH.

Method: Cohen's Forcing.

Conclusion: Adopting Cohen's forcing is a rational method because it is an effective means towards the goal of proving the independence of AC and CH.

Goal and method

Goal$_{min}$: Prove the independence of AC and CH.

Method$_{min}$: Cohen's Forcing.

Conclusion$_{min}$: Adopting Cohen's forcing is a rational method because it is an effective means towards the goal of proving the independence of AC and CH.

Goal and method, maximally

Cohen: Essential to developing forcing was "thinking about the existence of various models of set theory as being natural objects in mathematics". (DF, p.1072)

Goal_{max}: Prove the independence of AC and CH.

Method_{max}: Introduce models of set-theory as objects that exist naturally in mathematics.

Conclusion_{max}: Introducing models of set-theory as objects that exist naturally in mathematics is a rational method because it is an effective means towards the goal of proving the independence of AC and CH.

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Method$_{max}$: Introduce models of set-theory as objects that exist naturally in mathematics.

Conclusion$_{max}$ is not valid because it uses a heuristic aid that is disguised as a method.

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- Sets are regarded as new entities “in their own right”,
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- their use encourages one to speak about ‘arbitrary’ sets and
- is independent of the way in which they are represented.

In short: What is new about sets is not that they appear for the first time, but that they are used in a certain way for the first time.

Models of set theory before forcing

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- Restriction to certain sets: Gödel's model of definable sets (1938), von Neumann's model of well-founded sets (1925).
- Models with urelements: Zermelo (1908), Mirimanoff (1917), Fraenkel (1922, 1929), Fraenkel-Mostowski permutation models.
- Löwenheim-Skolem Theorem(s) (1920s).
- Precursors to forcing, for example Skolem (1923).
- Ultraproduct constructions, Scott's Ultrafilter method.
- Work with higher-order models (Gödel, Skolem).

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In both cases the objects are used in a conceptually different way as before, allowing a mathematical "change in perspective" to use the objects in a more generalized way, autonomously from previous, more specific contexts.

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- they are susceptible to mathematical operation *between the mst themselves*.

Again: What is new about mst after forcing is not that they appear for the first time, but that they are *used in a different way* for the first time.

Goal and method of Cohen's introduction of forcing

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Goal$_{opt}$: Prove the independence of AC and CH.

Method$_{opt}$: Introduce the models of set theory in a general and flexible way, that makes them objects of research themselves.

Conclusion$_{opt}$: Introducing models of set theory in the above way is a rational method because it is an effective means towards the goal of proving the independence of AC and CH.

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Allowing extrinsic evidence that is more tailored on nowadays use of forcing, it is possible to expand goals:

Goal_1: Show independence results in set theory.

Goal_2: Build a model that is closed under forcing.

Goal_3: Investigate relations between models of set theory.

...

Foundational goals of set theory and mst

Providing a foundation?

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She argues that set theory in a universist interpretation fits these goal better than category theory or a multiversist interpretation of set theory.

Claim: The extended picture of set theory, that includes the models of set theory as new entities in the way described above, provides a good picture for the foundational goals. In particular, it improves the fit for the goals and/or extends the goals themselves.

The foundational goals

Meta-mathematical Corral Provide a general theory, where mathematics can be corralled into a manageable package, so that general theorems about mathematics can be addressed (such as consistency, provability etc.).

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Generous Arena A single arena (V) "where all the various structures studied in all the various branches [of mathematics] can co-exist side-by-side, where their interrelations can be studied, shared fundamentals isolated and exploited, effective methods exported and imported from one to another, and so on."

The foundational goals

Of course **Shared Standard** *and* **Generous Arena** *depend on the same facts of set- theoretic reduction as* **Meta-mathematical Corral***: that formal proof is a good model of provability by humans and that the axioms of set theory codify the fundamental assumptions of classical mathematics. What separates them are the uses to which these facts are being put: in* **Meta-mathematical Corral**, *'derivable in ZFC' functions as model for 'provable in classical mathematics'; in* **Shared Standard**, *it's used as a benchmark for what counts as a legitimate in- formal proof; in* **Generous Arena**, *V brings all the objects and methods of classical mathematics together for fruitful interaction. As foundational uses, these are distinct. (Maddy, 2016, p. 297)*

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Excluded: Metaphysical Insight, Epistemic Source.

Classical mathematics

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Two versions phrased in terms of practices:

1. Classical mathematical practices vs non-classical mathematical practices (non-standard analysis, intuitionistic mathematics).
2. Mathematical practices vs foundational mathematical practices (certain set-theoretic practices).

Set theory and (meta)-mathematics

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> *[The] branches of modern mathematics are intricately and productively intertwined, from coordinate geometry, to analytic number theory, to algebraic geometry, to topology, to modern descriptive set theory (a confluence of point-set topology and recursion theory), to the kind of far-flung interconnections recently revealed in the proof of Fermat's Last Theorem.*

(Maddy, 2016, p. 297)

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Argument: The use of mst is often where the line between mathematics and meta-mathematics is drawn. But, as we have seen, the introduction of mst into the set-theoretic methodology is rational. And what we accept as a set-theoretic method should be accepted as a mathematical method.

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Argument: The use of mst is often where the line between mathematics and meta-mathematics is drawn. But, as we have seen, the introduction of mst into the set-theoretic methodology is rational. And what we accept as a set-theoretic method should be accepted as a mathematical method.

Proposal: Regard all set-theoretic practices as classical mathematical practices (call that classical*.)

Meta-mathematical Corral*

Meta-mathematical Corral Provide a general theory, where classical mathematics can be corralled into a manageable package, so that general theorems about mathematics can be addressed (such as consistency, provability etc.). Here ‘derivable in ZFC’ functions as model for ‘provable in classical mathematics’.

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Meta-mathematical Corral* Provide a general theory, where classical* mathematics can be corralled into a manageable package, so that general theorems about mathematics can be addressed. Here 'derivable in ZFC + further axioms' functions as model for 'provable in certain models of classical* mathematics'.

Extending the corral

New perspective: Add an additional layer to the corral where questions like consistency, provability etc. can not only be studied on the level of singular models but compared between models

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The reason is that mst are now regarded as objects of research themselves that are susceptible to mathematical operations between them.

Claim: Meta-mathematical Corral* is more desirable as a foundational goal, because it provides a corral for additional mathematical practices *and does not exclude any of the previous practices*.

Changing foundational goals

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General argument: Mathematics and its practice changes all the time. Foundational goals should take this into account.

Even more: Such a change already happened (done by Maddy herself).

Final court of appeal

"Provide decisive answers to questions of ontology and proof: if you want to know whether or not a so-and-so exists, see whether one can be found in V; if you want to know whether or not such-and-such is provable, see whether it can be derived from the axioms of set theory." (STF, p.296)

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But **Final court of appeal** seems too restrictive.

Example: "Is there a definable (projective) well-ordering of the reals?" With **Final court of appeal** we can only say that we don't know; but what we want to say is: "It depends on the axiomatization."

From Final Court to Generous Arena

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So, starting from recognizing a change in set-theoretic practice, we adjust the foundational goal to incorporate this new practice.

Finally, the new practice here is based on the consideration of different mst as the set-theoretic methodology we use to investigate truth in these different axiomatizations.

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- We showed that with these methods it is rational to introduce the models of set theory into set-theoretic methodology.
- We argued that this extended methodology gives rise to a extended, more desirable foundational goal in the case of **Meta-mathematical Corral**.
- We claim that this is actually the continuation of a development, Maddy herself started in developing the goal of **Generous Arena**.
- Outlook: We would like to use this to bridge the gap between "universist practices" and "multiversist practices".

Thank you! Questions?