

MACAULAY2 PROBLEMS

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Remember that `viewHelp` (or `help` if you do not have your browser integrated properly) is a very useful command!

A. Solving systems of polynomial equations. In this problem, we will look at finding all solutions to systems of polynomial equations

$$f_1(x_1, \dots, x_n) = 0, \quad f_2(x_1, \dots, x_n) = 0, \quad \dots \quad f_k(x_1, \dots, x_n) = 0$$

in n variables x_1, \dots, x_n . All the examples we look at will have a *finite* number of solutions. To find these, we will create the ideal

$$I = \langle f_1, \dots, f_k \rangle$$

in the polynomial ring generated by x_1, \dots, x_n and study it. Note that solving the system above is equivalent to finding the common zeros of *all* polynomials in the ideal I ; it is also equivalent to finding the common zeros of any basis (generating set) of the ideal I .

The fact that there is a finite number of solutions corresponds to the fact that the *dimension* of (the solution set of) the ideal I (use `dim`) is 0. Seeing the solutions directly will be hard; but finding a suitable basis (the Gröbner basis) of the ideal I will help us find the solutions more easily. This basis can be generated using the command `gb`; use `gens` to list generators. At certain points, the command `factor` might be useful to factor high-degree polynomials.

- (1) Use Macaulay2 to solve the following system of equations.

$$\begin{aligned}x + 2y + z &= 13 \\2x + y + 2z &= 17 \\x - y + 2z &= 11\end{aligned}$$

- (a) First treat this as a linear algebra problem: create a 3×3 matrix, and try to invert it, using the commands `matrix` and `inverse`. What do you find? Check its `det`. Can you fix the problem within Macaulay2?
- (b) Next, treat this as a problem in polynomial algebra. Create the correct ideal (note you will have to write the equations in the form $\dots = 0$), and compute a Gröbner basis. Can you read off the solutions?

- (2) Find all the solutions to the following system of equations

$$\begin{aligned}x^2 + y^2 &= 5 \\ xy &= 2\end{aligned}$$

- (a) Create the ideal in a polynomial ring; use the ring

$$\mathbf{R} = \mathbf{QQ}[\mathbf{x}, \mathbf{y}, \text{MonomialOrder} \Rightarrow \text{Lex}]$$

to get the right basis later.

- (b) Check that the dimension of the solution set is 0.
 (c) Find the solutions by looking at the Gröbner basis.
 (d) This is a small problem that can of course be done by hand. Try using standard algebra to solve this problem on a piece of paper and compare with the solution from the method above.

- (3) Using a similar method, find all the solutions to the following system of equations

$$\begin{aligned}x^2 + y + z &= 1 \\ x + y^2 + z &= 1 \\ x + y + z^2 &= 1\end{aligned}$$

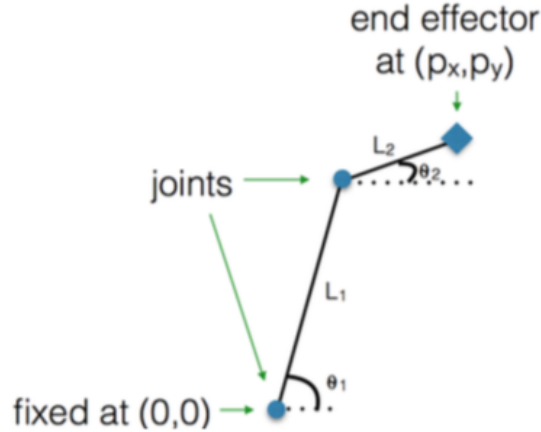
A final comment: try the following as an alternative. Load the numerical polynomial solver package by

```
loadPackage "NumericalAlgebraicGeometry"
```

Use complex coefficients for your rings: $\mathbf{S} = \mathbf{CC}[\mathbf{x}, \mathbf{y}, \dots]$; then list polynomials in a polynomial system $\mathbf{F} = \{ \dots, \dots, \dots \}$; then type `solveSystem F`. Note how for the simpler systems (1)-(2), you get the solutions right away. What happens for (3)? Do you recognise your solutions from before?

B. Basic kinematics of robotic arms.¹

A 2R planar robot is a planar robotic arm with two links and two rotational joints.



The joints can spin freely (let's ignore collisions); the links have fixed lengths L_1 and L_2 ; the bottom joint is fixed at $(0,0)$.

The *forward kinematics problem* is to determine the location of the end effector, given the joint angles θ_1 and θ_2 . The *inverse kinematics problem* is to determine joint angles that will put the end effector at a particular point in the plane.

- (1) Use trigonometry to write down two (non-polynomial) equations that determine p_x and p_y from L_1 , L_2 , θ_1 and θ_2 .
- (2) Replace $\cos \theta_1$ with a new variable c_1 , and use c_2 , s_1 , and s_2 similarly, throwing in the two equations

$$c_1^2 + s_1^2 - 1 = c_2^2 + s_2^2 - 1 = 0,$$

to get a 4-equation, 4-variable polynomial system.

- (3) Let $L_1 = L_2 = 1$ and set $(p_x, p_y) = (1, 1)$. What solution(s) do you expect? What do you get?
- (4) Can you find a point (p_x, p_y) at which there is a single multiplicity 2 solution?
- (5) What happens if (p_x, p_y) falls outside the *workspace* (the region of the plane that can be reached by the end effector)?
- (6) Can you find a point (p_x, p_y) at which there are infinitely solutions?
- (7) Write down the system for the 3R planar and/or the 2R spatial robot and ask yourself similar questions.

¹With thanks to Dan Bates

C. Singularities of affine hypersurfaces. In this problem, we will look at *singularities* of a hypersurface

$$H = \{f(x_1, \dots, x_n) = 0\} \subset \mathbb{A}^n,$$

the locus given by the vanishing of a single polynomial equation f in n variables. The *singular locus* of the hypersurface H is given by the set of points on H where the gradient

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right)$$

vanishes. To compute this locus, we need to create an ideal consisting of f and its partial derivatives, and investigate it. The command `diff` will be useful throughout this problem. Use `dim` to check how big a solution set you are expecting; note that by convention `dim = -1` means that there is no solution.

- (1) Some classical hypersurfaces
 - (a) Determine the set of singularities of

$$S = \{x^2 + y^2 + z^2 = 1\} \subset \mathbb{A}^3.$$

- (b) Determine the set of singularities of

$$C = \{x^2 + y^2 = z^2\} \subset \mathbb{A}^3.$$

- (c) Determine the set of singularities of

$$W = \{x^2 - y^2z = 0\} \subset \mathbb{A}^3.$$

- (2) Singularities of the space of non-invertible matrices
 - (a) Create a 3×3 matrix of variables. You may want to name your variables `x_{1,2}`
If you already have a variable named `x`, then you will need to clear it first using `x = symbol x`.
 - (b) Compute the determinant of your matrix. Remember that when the determinant of A is 0, the matrix A is not invertible
 - (c) Consider the hypersurface H given in the space of matrices by the equation $\det A = 0$. Treating the determinant as a polynomial in the matrix entries, compute partial derivatives (the command `diff` will be useful here).
 - (d) Compute the ideal of partial derivatives.
 - (e) Check that this ideal is the same as the ideal of 2×2 minors of A (try `minors(2,A)`)

- (f) Can you generalise this to $d = 4$ from $d = 3$? To check your generalization, the command `jacobian ideal f` will be useful to generate the ideal of partials of a polynomial f succinctly.

D. The Grassmannian. The Grassmannian $\text{Gr}(d, n)$ is the projective variety that parameterises all d -dimensional subspaces of an n -dimensional vector space.

- (1) Create a 2×4 matrix with generic entries (e.g., x_{ij}).
- (2) Compute the six 2×2 minors of your matrix. The command `gens minors(2,A)` will produce a matrix with these entries.
- (3) Compute a homomorphism from a polynomial ring in six variables to your ring that takes the i th generator to the i th minor on your list.
- (4) Take the kernel of your homomorphism. This is the ideal of the Grassmannian $\text{Gr}(2, 4)$. If you already knew what this variety was, compute the dimension to check that this is correct.
- (5) Now write a function that takes as input your choice of $d < n$ to replace 2 and 4.
- (6) This command actually already exists in Macaulay2! Look at the help for `Grassmannian`. (This command uses the projective convention for the Grassmannian, so to see our example you should type `Grassmannian(1,3)`). How can you use this to test that your function is correct?

Where next?

- (1) Read the tutorials on the M2 webpage, starting with the first one.
- (2) Have a look at the list of packages written by other people that are distributed with Macaulay2. Find one that is close to your research interests, and look at the help. Install it (use `needsPackage` or `installPackage`), and try some commands.