Polyhedral geometry and `polymake`

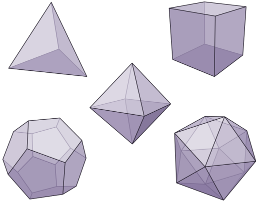
Dominic Bunnett

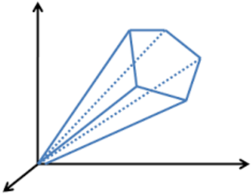
October 6, 2020

Intro

We are all familiar with convex polygons.

- Polyhedral geometry Geometric objects with 'flat sides'.
- For example polyhedral cones and polytopes.
- Polytopes are *d*-dimensional generalisations of 2-dimensional convex polygons and 3-dimensional convex polyhedra.

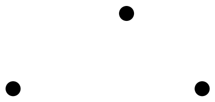


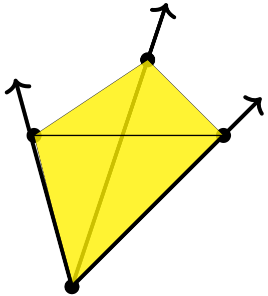


Definitions: Polyhedral Cones

Definition

Let $S = \{u_1, \ldots, u_r\} \subset \mathbb{Z}^n$. Define
$C = \mathrm{Cone}(S) = \{\sum_{i=1}^{r} \lambda_i u_i \mid \lambda_i \geq 0\} \subset \mathbb{R}^n$.





Definition

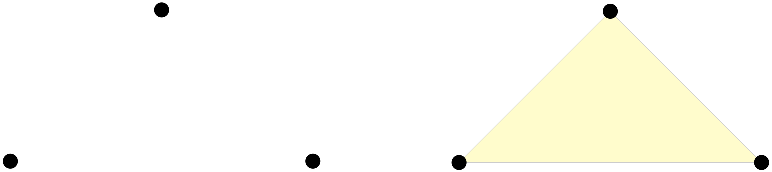
$\dim C = \dim \mathrm{Span}_{\mathbb{R}} S$

Definitions: Polytope

Definition

Let $S = \{u_1, \ldots, u_r\} \subset \mathbb{Z}^n$. Define

$$P = \mathrm{Conv}(S) = \left\{ \sum_{i=1}^{r} \lambda_i u_i \mid \sum_{i=1}^{r} \lambda_i = 1, \quad \lambda_i \geq 0 \right\} \subset \mathbb{R}^n.$$



Definition

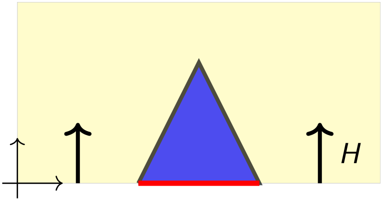
$\dim P =$ *dimension of the smallest affine space containing* P.

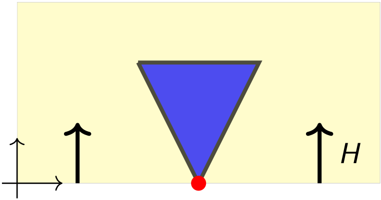
In the example above, P has dimension 2.

Definitions: Faces

Definition

$Q \subset P$ is a face if $Q = P \cap H$, where $H \subset \mathbb{R}^n$ is a hyperplane and P lies on one side of H.





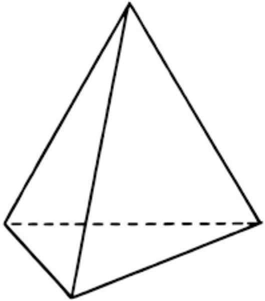
Definition

1. Faces of dimension 0 are called **Vertices**
2. Faces of dimension 1 are called **Edges**
3. Faces of dimension $\dim P - 1$ are called **Facets**

Examples

P a 3-dimensional polytope.

For example:



Let

1. V = *number of vertices*
2. E = *number of edges*
3. F = *number of faces*

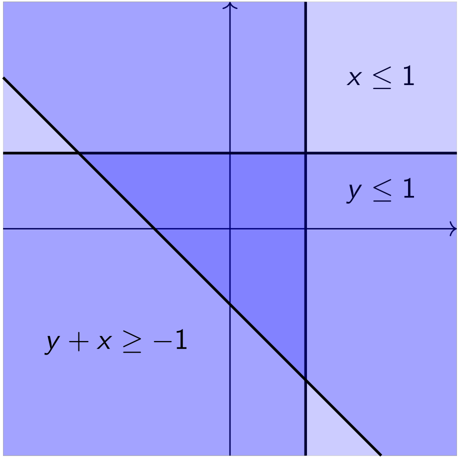
Theorem (Euler's polyhedron formula)

$$V + F - E = 2$$

We can quickly check: $(V = 4) + (F = 4) - (E = 6) = 2$

Inequality description of a polytope

A polytope can also be described by a series of linear inequalities.

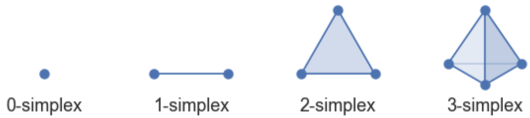


$P = \{(x, y) \in \mathbb{R}^2 \mid y + x \geq -1, y \leq 1, x \leq 1\}.$

Properties

Definition

A polytope $P \subset \mathbb{R}^n$ is a simplex if $V(P) = \dim P + 1$.



Remark

Given $p_0, \ldots, p_n \in \mathbb{R}^n - \{0\}$ we can decide if $P = \text{Conv}(p_0, \ldots, p_n) \subset \mathbb{R}^n$ is a simplex by computing the determinant of the matrix $M = (p_0 \mid \cdots \mid p_n)$.

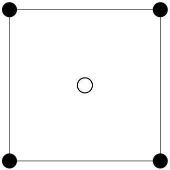
Properties

Definition

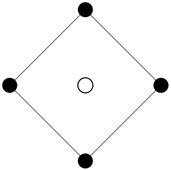
$P \subset \mathbb{R}^n$ a polytope containing the origin. Define P's dual to be

$$P^\circ = \{u \in \mathbb{R}^n \mid m \cdot u \geq -1 \text{ for all } m \in P\}.$$

We say that P is reflexive if P° is a lattice polytope, that is, the vertices of P° are all lattice points.



P



P°

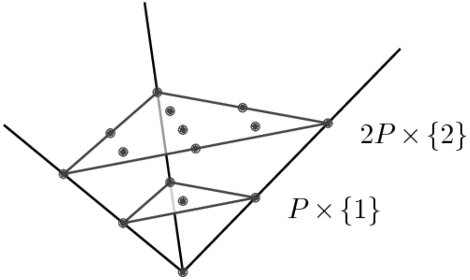
Properties

Definition

Let P be a polytope. Consider the following function $L_P(t) = \#\{tP \cap \mathbb{Z}^n\}$. Then

$$L(P,t) = a_d(P)t^d + a_{d-1}(P)t^{d-1} + \cdots + a_0(P)$$

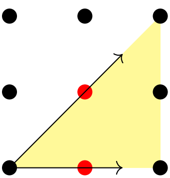
is the **Ehrhart polynomial** where $a_i(P) \in \mathbb{Q}$ and $d = \dim P$.



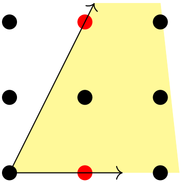
Properties

Definition

Let C be a cone. Suppose that $C = \text{Cone}(S)$, with $S = \{u_1, \ldots, u_r\} \subset \mathbb{Z}^n$. We say that C is smooth if S forms part of a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.



Smooth cone



Not a smooth cone.

Definition

A polytope is smooth if each of the cones generated by the vertices are smooth.

Appearance in Linear Optimization

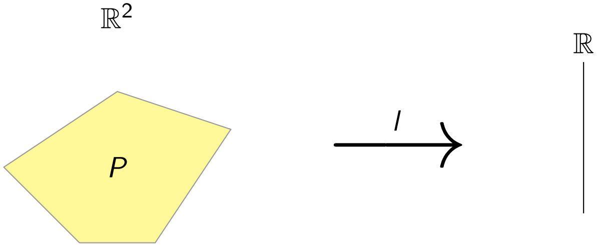
Consider a linear functional $l : \mathbb{R}^n \to \mathbb{R}$.

Question

Given a polytope $P \subset \mathbb{R}^n$, what is

$$\max\{l(p) \mid p \in P\}$$

and which points $p \in P$ attain this value?



Appearance in geometry

$$\mathbb{Z}^n = \{(a_1, \ldots, a_n) \mid a_i \in \mathbb{Z}\} \longleftrightarrow \{x_1^{a_1} \cdots x_n^{a_n}\} \subset \mathbb{C}[x_1, \ldots, x_n]$$

$$P \cap \mathbb{Z}^n \longleftrightarrow \{x_1^{a_1} \cdots x_n^{a_n} \mid (a_1, \ldots, a_n) \in P \cap \mathbb{Z}^n\}$$

Consider $P = \text{Cone}((2,0,0),(0,2,0),(0,0,1))$. Then

$$P \cap \mathbb{Z}^3 = \{(2,0,0),(0,2,0),(0,0,1),(1,1,0)\}.$$

Giving us the monomials: $\{x_1^2, x_1x_2, x_2^2, x_3\}$.

$$\begin{aligned} \Phi_P : \left(\mathbb{C}^*\right)^3 &\longrightarrow \mathbb{C}^4 \\ (t_1, t_2, t_3) &\longmapsto (t_1^2, t_1 t_2, t_2^2, t_3). \end{aligned}$$

We define the variety $X_P = \overline{\text{Im}(\Phi_P)}$.

Now time for polymake

- All names must come with a dollar sign:

```
$P = new Polytope( ) ;
```

- Coordinates for polytopes have an extra 1 in the first position:

```
$P = new Polytope(POINTS=>[[1,0,0],[1,1,0],[1,0,1]]);
```

  This is the polytope $\mathrm{Conv}((0,0),(1,0),(0,1))$.

- Properties are called in the following way:

```
$P->properties
```

- Individual properties:

```
print $P->VERTICES;
```

- $P a polytope, P's **F Vector**

$$(\# \text{ of vertices}, \# \text{ of edges}, \# \text{ of 2-faces}, \dots, \# \text{ of facets})$$

To compute this:

```
$P->F_VECTOR;
```

- Suppose p\subset \mathbb{R}^n$ is a polytope and $l : \mathbb{R}^n \to \mathbb{R}$ a linear functional. Then $L = (L_1, \dots, L_n)$ can be expressed as a vector (taking the canonical basis of the dual vector space).

```
$lp = $p->LP(LINEAR_OBJECTIVE=>new Vector([L]));
```

We can solve this via:

```
print $lp->MAXIMAL_VALUE, ", ", $lp->MAXIMAL_VERTEX;
```