MEASURABILITY AND CONTINUITY CONDITIONS FOR NONLINEAR EVOLUTIONARY PROCESSES

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ABSTRACT. This paper generalizes to nonlinear evolutionary processes on a metric space the well-known results connecting measurability and continuity properties with respect to time of linear semigroups of continuous operators on a Banach space.

1. Introduction. Let \( X \) be a topological space. By definition, an evolutionary process on \( X \) is a family of operators \( U(t, s): X \to X \), defined for \( t \in \mathbb{R}^+, s \in \mathbb{R} \) and satisfying (i) \( U(0, s) = \text{identity} \); (ii) \( U(t + \tau, s) = U(t, s + \tau)U(\tau, s) \) for \( t, \tau \in \mathbb{R}^+, s \in \mathbb{R} \). Such processes arise in the mathematical modelling of nonautonomous systems, when \( U(t, s)x \) represents the position (or state) at time \( t + s \) of the point which at time \( s \) was at \( x \). In the special case when the operators \( U(t, s) = T(t) \) are independent of \( s \), the evolutionary process defines a semigroup \( \{T(t)\}, t \in \mathbb{R}^+ \). In the above \( \mathbb{R}^+ \) denotes the nonnegative reals.

In [3] it was shown that in certain situations measurability and continuity properties known to be satisfied for a semigroup could be strengthened using the semigroup properties. We extend this work to evolutionary processes. Our Theorem 1 is, however, new even for semigroups for which it takes the following form.

THEOREM 1'. Let \( \{T(t)\}, t \geq 0 \), be a semigroup on a metric space \( X \). If \( T(t) \) is continuous for each \( t \geq 0 \), and if the map \( t \mapsto T(t)x \) is strongly measurable on \( (0, \infty) \) for each \( x \in X \), then the map \( (t, x) \mapsto T(t)x \) is continuous on \( (0, \infty) \times X \).

Theorem 1' generalizes classical results due to von Neumann [17], Dunford [12] and Phillips [20] for the case when \( X \) is a Banach space and each \( T(t) \) is linear, and improves the result of Phillips (see Crandall and Pazy [8]) for \( X \) Banach and \( \{T(t)\}, t \geq 0 \), a semigroup of nonexpansions. It is a Lebesgue measure counterpart for its category version due to Chernoff and Marsden [6], [7] (see also [3, Theorem 5.1]).

The other theorems of the paper follow in a straightforward way from those in [3]. Included are some counterexamples indicating directions in which the results cannot be improved.

2. Preliminaries. Throughout this section let \( X \) be a metric space with metric \( d \) and denote Lebesgue measure in \( \mathbb{R} \) by \( m \). A function \( f: (0, \infty) \to X \) is said

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to be strongly measurable if there exists a sequence \( \{f_n\} \) of measurable countably-valued functions which converges almost everywhere to \( f \) on \((0, \infty)\), and almost separably-valued if there exists a subset \( E \subseteq (0, \infty) \) of zero measure such that \( f((0, \infty) \setminus E) \) has a countable dense subset. It is easily shown (see Dunford and Schwartz [13, p. 147] for an analogous proof) that \( f \) is strongly measurable if and only if (a) \( f \) is almost separably-valued and (b) \( f^{-1}U \) is Lebesgue measurable for every open \( U \subseteq X \).

We need the following version of Lusin's theorem, our proof of which is adapted from that in Oxtoby [19].

**Lemma 1.** A function \( f: (0, \infty) \to X \) is strongly measurable if and only if given any \( \varepsilon > 0 \) there exists a closed set \( F \) with \( m((0, \infty) \setminus F) < \varepsilon \) such that \( f \) is continuous when restricted to \( F \).

**Proof.** Let \( f \) be strongly measurable. Then there exists \( E \subseteq (0, \infty) \) of zero measure such that \( Z = f((0, \infty) \setminus E) \) has a countable base of open sets \( U_i \cap Z \) (\( i = 1, 2, \ldots \)) with \( U_i \) open in \( X \). For each \( i \) there exists an open set \( G_i \subseteq f^{-1}U_i \) such that \( m(G_i \setminus f^{-1}U_i) < \varepsilon / 2^i + 1 \). Let \( S = \bigcup_{i=0}^{\infty} (G_i \setminus f^{-1}U_i) \) so that \( m(S) < \varepsilon / 2 \). Let \( U \subseteq X \) be open. Then \( U \cap Z = \bigcup_k (U_{i_k} \cap Z) \) and

\[
f^{-1}(U \cap Z) = \bigcup_k (G_{i_k} \setminus S) \cap f^{-1}Z = \bigcup_k G_{i_k} \setminus (S \cup E)
\]

which is open in \((0, \infty) \setminus (S \cup E)\). There exists a closed set \( F \subseteq (0, \infty) \setminus (S \cup E) \) with \( m((0, \infty) \setminus F) < \varepsilon \), and clearly \( f \) restricted to \( F \) is continuous.

Conversely let \( F_i \) be closed sets with \( m((0, \infty) \setminus F_i) < 1/i \) and such that \( f \) is continuous when restricted to each \( F_i \). Let \( F = \bigcup_{i=1}^{\infty} F_i \). Then \( m((0, \infty) \setminus F) = 0 \) and \( f(F) = \bigcup_{i=1}^{\infty} f(F_i) \) has a countable dense subset. Thus \( f \) is almost separably-valued. Let \( U \subseteq X \) be open. For each \( i \) there exists an open set \( G_i \subseteq (0, \infty) \) with \( (f^{-1}U) \cap F_i = G_i \cap F_i \). Hence

\[
f^{-1}U = ((f^{-1}U) \setminus F) \cup \bigcup_{i=1}^{\infty} (G_i \cap F_i),
\]

which is clearly Lebesgue measurable. \( \square \)

3. **Main results.** Throughout this section we suppose that the evolutionary process \( \{U(t, s)\} \) defined on the topological space \( X \) satisfies the hypothesis:

(A) For each \( t \in \mathbb{R}^+ \) the map \( (s, x) \mapsto U(t, s)x \) is (jointly) sequentially continuous from \( \mathbb{R} \times X \) to \( X \).

**Theorem 1.** Let \( X \) be a metric space. Suppose that for each \( s \in \mathbb{R}, x \in X \) the map \( t \mapsto U(t, s)x \) is strongly measurable on \((0, \infty)\). Then the map \( (t, s, x) \mapsto U(t, s)x \) is continuous on \((0, \infty) \times \mathbb{R} \times X \).

**Proof.** We first prove Theorem 1'. Let \( x \in X \). We show that the map \( f(t) = T(t)x \) is continuous on \((0, \infty)\). The result then follows from a theorem of Chernoff and Marsden [6] (see also [3]). Let \( 0 < a < a + \delta < \infty \) and denote by \( I \) and \( J \) the open intervals \((a, a + \delta)\) and \((a + \delta / 3, a + 2\delta / 3)\) respectively. Since \( f \) is strongly measurable, by Lemma 1 there exists \( I \) a closed set \( F_i \) of measure greater than \( \delta - 1/r^2 \) on which the restriction of \( f \) is
The continuity being uniform, there exists $\delta/3 > \eta > 0$ such that $t, t + h \in E_r$ and $|h| < \eta$ imply that $d(f(t + h), f(t)) < 1/r$. Fix $h_r$ with $|h_r| < \eta$. The set $\{t \in E_r \cap J : t + h_r \in E_r\}$ has measure less than $1/r^2$. Therefore $d(f(t + h_r), f(t)) < 1/r$ holds for all $t$ in a subset $E_r \subseteq J$ of measure greater than $\delta/3 - 2/r^2$. Clearly $J \setminus \lim_{r \to \infty} E_r$ has measure zero. Therefore $T(t + h_r)x \to T(t)x$ almost everywhere in $J$. (This argument is due to Auerbach [2].) Let $t \in J$. There exists $t_1 < t$ belonging to $J$ such that $T(t_1 + h_r)x \to T(t_1)x$. Then

$$T(t + h_r)x = T(t - t_1)T(t_1 + h_r)x \to T(t - t_1)T(t_1)x = T(t)x$$

by the assumed continuity of $T(t - t_1)$. Thus $T(t + h_r)x \to T(t)x$ everywhere in $J$. Since from any sequence $\{h_k\}$ tending to zero we may extract a subsequence $\{h_{k_r}\}$ with $|h_{k_r}| < \eta$, it follows that $T(t + h_{k_r})x \to T(t)x$ everywhere in $J$. This completes the proof in the semigroup case.

The proof in the general case follows immediately by applying Theorem 1' to the semigroup $\{S(t)\}, t \in \mathbb{R}^+$, which is defined on the space $\mathbb{R} \times X$ by

$$S(t) \begin{pmatrix} s \\ x \end{pmatrix} \overset{\text{def}}{=} \begin{pmatrix} s + t \\ U(t, s)x \end{pmatrix}.$$ 

**Corollary.** Let $X$ be a subset of a Banach space. Suppose that for each $s \in \mathbb{R}, x \in X$, the map $t \mapsto U(t, s)x$ is weakly continuous from the right on $(0, \infty)$. Then the map $(t, s, x) \mapsto U(t, s)x$ is continuous on $(0, \infty) \times \mathbb{R} \times X$ with respect to the norm topology on $X$.

**Proof.** See [3, Theorem 5.2].

**Remarks.** 1. The proof of Theorem 1 bears some resemblance to that of Banach [4] of the result of Fréchet that every Lebesgue measurable real-valued solution $g$ of the functional equation

$$g(s) + g(t) = g(s + t)$$

is continuous, and thus of the form $g(t) = At$ for some constant $A$. (This is a special case of Theorem 1, since any $f$ satisfying (2) generates a semigroup on $\mathbb{R}$ given by $T(t)\tau = e^{g(t)\tau}$.)

2. In this case $X = \mathbb{R}$, Theorem 1 may also be proved by an argument used by Alexiewicz and Orlicz [1] in their proof of Fréchet's result. With the notation of the above proof, the function $f$ is measurable and thus approximately continuous almost everywhere in $(0, \infty)$; by the semigroup property and the continuity of $T(t)$ for $t > 0$ it follows that $f$ is approximately continuous everywhere in $(0, \infty)$, and hence $f$ is continuous (see Denjoy [11] and Looman [16]). It might be possible to extend this argument to arbitrary metric $X$.

3. There is an obvious modification of Theorem 1 to the case when the evolutionary process is defined only locally in $t$.

4. It is not in general possible to deduce that $(t, s, x) \mapsto U(t, s)x$ is continuous on $[0, \infty) \times \mathbb{R} \times X$, even if $t \mapsto U(t, s)x$ is continuous on $(0, \infty)$ for every $(s, x)$. (See Chernoff [5].)

All the other results in [3] may be generalised in a straightforward way to evolutionary processes using the transformation (1). We give as a sample two
such generalisations and take the opportunity to weaken slightly the hypotheses of the corresponding results in [3, Theorems 5.1, 5.3] (the proofs are very similar).

**Theorem 2.** Let $X$ be arbitrary. Suppose that for each $s \in \mathbb{R}$, $x \in X$, the map $t \mapsto U(t, s)x$ is Baire continuous on $(0, \infty)$ and, when restricted to the complement of some first category set, has second countable range. Then the map $(t, s, x) \mapsto U(t, s)x$ is sequentially continuous on $(0, \infty) \times \mathbb{R} \times X$.

**Theorem 3.** Let $X$ be a subset of a uniformly convex Banach space. Suppose that
(a) for each $s_1, s_2 \in \mathbb{R}$, $x_1, x_2 \in X$, $t_n \to 0$ implies
$$\liminf_{n \to \infty} \|U(t_n, s_1)x_1 - U(t_n, s_2)x_2\| \leq \|x_1 - x_2\|,$$

(b) for each $s \in \mathbb{R}$, $x \in X$, the map $t \mapsto U(t, s)x$ is weakly continuous from the right at $t = 0$.

Then for each $s \in \mathbb{R}$, $x \in X$, the map $t \mapsto U(t, s)x$ is continuous on $[0, \infty)$ with respect to the norm topology on $X$.

We remark that condition (i) in the definition of an evolutionary process is not needed for the validity of Theorems 1 and 2. We remark also that there are useful methods of generating a semigroup from a given process other than by (1). (See Dafermos [9] and the references therein.) However, these methods, while having definite advantages over (1) for stability theory, do not improve our results.

4. **Some counterexamples.** Perhaps the simplest example of an evolutionary process is when $X = \mathbb{R}$ and each operator $U(t, s)$ is linear and defined on $\mathbb{R} \times \mathbb{R}$. Let $\{U(t, s)\}$ have the form

$$U(t, s)r = e^{g(t, s)}r \quad (3)$$

for some function $g: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. $g$ satisfies the functional equation

$$g(t + \tau, s) = g(t, \tau + s) + g(\tau, s), \quad \text{for all } t, \tau, s \in \mathbb{R}. \quad (4)$$

The general solution of (4) is

$$g(t, s) = h(t + s) - h(s), \quad (5)$$

where $h: \mathbb{R} \to \mathbb{R}$ is arbitrary. It is therefore clear that, for example, neither strong measurability nor Baire continuity of $(t, s) \mapsto U(t, s)x$, $x \in X$, suffices to prove continuity of this map when (A) is replaced by an assumption of continuity of $U(t, s)x$ with respect to $x$ alone.

When $\{U(t, s)\} = \{T(t)\}$ is a semigroup then $g(t, s) \equiv g(t)$, where $f$ satisfies Cauchy's equation (2). In 1905 Hamel [14] showed using the axiom of choice that there are discontinuous solutions of (2). Thus even for semigroups of continuous linear operators on a Banach space Theorems 1 and 2 are false without the hypothesis of strong measurability or Baire continuity on the map $t \mapsto T(t)x$. Can this hypothesis be weakened to the requirement of precompactness of $T((\alpha, \beta))x$ for all $\alpha, \beta \in \mathbb{R}^+$? This question is motivated by the
result of Ostrowski [18], who, extending work of Darboux [10] and Sierpiński [21], showed that any solution \( f \) of (2) which is bounded above on a set of positive measure is necessarily continuous (for an alternative proof see Kestelman [15]). The answer is no. For example, define \( \{ T(t) \}, t \in \mathbb{R} \), by

\[
T(t)(\pm \pi/2) = \pm \pi/2,
\]

\[
T(t)\tau = \tan^{-1}[g(t) + \tan \tau], \quad \tau \in (-\pi/2, \pi/2),
\]

where \( g \) is any discontinuous solution of (2). It is easily checked that (6) defines a group of continuous (nonlinear) operators on \([-\pi/2, \pi/2]\) such that each nontrivial orbit is discontinuous (in fact, there will be just one orbit in \((-\pi/2, \pi/2)\) if and only if \( g \) is bijective—such solutions \( g \) to (2) are easy to construct using a Hamel basis of \( \mathbb{R} \) over the rationals). \( \{ T(t) \}, t \in \mathbb{R} \), can be trivially extended to \( \mathbb{R} \). Finally, we remark that \( (S(t)\theta)(\tau) = \theta(T(t)\tau) \) for \( \theta \in C([-\pi/2, \pi/2]) \) defines a group \( \{ S(t) \}, t \in \mathbb{R} \), of linear isometries on \( C([-\pi/2, \pi/2]) \) with the maximum norm such that each nontrivial orbit is discontinuous.

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