MINIMIZERS AND THE EULER-LAGRANGE EQUATIONS

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Consider the problem of minimizing an integral of the form
\[ I(u) = \int_{\Omega} f(x, u(x), \nabla u(x)) \, dx \]
subject to given boundary conditions, where \( \Omega \subset \mathbb{R}^m \) is a bounded open set and the competing functions \( u : \Omega \to \mathbb{R}^m \). Frequently it is possible to use the direct method of the calculus of variations to establish the existence of a minimizer \( u \) in an appropriate Sobolev space \( W^{1,p}(\Omega; \mathbb{R}^m) \).

Then formally we expect that \( u \) satisfies the weak form of the Euler-Lagrange equations
\[ \int_{\Omega} \left( -\frac{\partial f}{\partial u} \varphi \right) dx + \int_{\partial \Omega} g \varphi \, ds = 0 \quad \text{for all } \varphi \in C^0_0(\Omega; \mathbb{R}^m), \tag{1} \]
but a search of the literature reveals that in general the theorems guaranteeing this make stronger growth assumptions on \( f \) than are necessary to prove existence. That this is not just a technical difficulty can be seen from one-dimensional examples due to Niezg and myself that are announced in [6]. One of these examples concerns the problem of minimizing
\[ I(u) = \int_{-1}^{1} \left( (x-u)^2 (u')^2 + \varepsilon (u')^2 \right) \, dx \tag{2} \]
subject to \( u(-1)=k, u(1)=k \), where \( \varepsilon \geq 14 \) is an integer, \( \varepsilon > 0 \) and \( 0 < k < 1 \). (Here \( m=n=1 \) and the prime denotes \( \frac{d}{dx} \).) Note that the integrand \( f(x,u,u') \) in (2) is smooth and regular (i.e., \( f_u, f_{uu}' > 0 \)) so that the Euler-Lagrange equation can be reduced to the form \( u'' = g(x,u,u') \). Given \( k \), let \( \varepsilon > 0 \) be sufficiently small. Then I attains an absolute minimum on the set \( \mathcal{A} = \{ v \in W^{1,1}(-1,1) : v(-1) = v(1) = \pm k \} \) and any minimizer \( u \) satisfies \( u(0)=0 \), \( u'(0)=\pm \epsilon \). Furthermore \( f_u \notin L_{1,0}^{1,1}(-1,1) \) and hence (1) does not hold. Also, we have that
\[ \inf_{v \in W^{1,1}(-1,1)} I(v) > I(u) \quad \text{(the Lavrentiev phenomenon).} \tag{3} \]

I will now sketch the most important part of the proof, which establishes (3), that \( u(0)=0 \), and that if \( 0 \leq \mu < 1 \) then \( |u(x)| \geq k|x|^{2/3} \)

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for all \( t \in [-1, 1] \), provided \( \epsilon > 0 \) is sufficiently small. The argument is an adaptation of Mania [9] (cf. Cesari [8, p. 514]). Further details can be found in Ball and Hizel [7]. Let \( v \) be any element of \( \mathcal{V} \). Then \( v(x_0) = 0 \) for some \( x_0 \in (-1, 1) \) and by symmetry we can suppose that \( x_0 \geq 0 \). Suppose further either that \( x_0 \neq 0 \) or \( x_0 = 0 \) and \( 0 < v(0) < \mu k_2^{2/3} \) for some \( k \in (0, 1) \). Let \( u < v \). In either case there exists an interval \( (x_1, x_2) \), \( 0 < x_1 < x_2 < 1 \), on which \( \mu k_2^{2/3} < v(x) \leq \mu k_2^{2/3} \) and such that \( v(x_1) = \mu k_2^{2/3}, \ v(x_2) = \mu k_2^{2/3} \). On this interval \( (x^4 - v^4)^2 > x^8 (1 - v(k) \))^2 \), and hence
\[
I(v) \geq \left( 1 - v(k) \right)^2 \int_{x_1}^{x_2} \frac{x^8 (v')^2}{x^8 (v')^2} dx.
\]
Putting \( y = x^2 \), where \( \theta = 2r - 9 \frac{x^2}{2r - 1} \), we get, using Jensen's inequality
\[
\int_{x_1}^{x_2} x^8 (v')^2 \frac{dx}{x^8 (v')^2} = \theta \int_0^\theta x^8 (v')^2 \frac{dy}{v'} = \frac{1}{2r - 1} (x_2^8 - x_1^8)^{2r - 1} = \frac{1}{2r - 1} (x_2^8 - x_1^8)^{2r - 1} = h(x_1, x_2).
\]
It is easily verified that if \( r \geq 14 \) then \( h(x_1, x_2) > 0 \), and it follows that \( I(v) \geq \eta > 0 \) for all \( v \) as above, \( \eta \) being independent of \( \epsilon \). Now let \( \tilde{v}(x) = \frac{|x|}{2} \text{sign } x \) for \( |x| \leq k/2 \), \( \tilde{v}(x) = k \) for \( x > k/2 \), \( \tilde{v}(x) = -k \) for \( x < -k/2 \). Then \( \tilde{v} \in \mathcal{V} \) and
\[
I(\tilde{v}) = 2\epsilon \int_{0}^{k/\sqrt{2}} (x^2 - 1/3)^2 dx,
\]
which is less than \( \eta \) if \( \epsilon \) is sufficiently small. Thus \( u(0) = 0, |u(x)| \leq \mu k|x|^{2/3} \) for any minimizer \( u \), and (3) holds. As far as we are aware the examples in [6, 7] are the first in which the singular set in Tonelli's partial regularity theorem [10, p. 359] has been shown to be nonempty.

I now turn to nonlinear elassticities, which in fact motivated the work in [6, 7]. Consider a simple mixed boundary value problem in which it is required to minimize
\[
I(u) = \int_{\Omega} W(\nabla u(x)) dx
\]
on the set \( \mathcal{W} = \{ u \in W^{1,1}(\Omega; \mathbb{R}^n) : I(u) < \infty, u|_{\partial \Omega} = u_0 \} \) in the sense of trace). Here \( \Omega \subset \mathbb{R}^n \) is a strongly Lipschitz bounded open set, \( \partial \Omega \subset \partial \Omega \) has positive \( n-1 \) dimensional measure, and \( W : H^1(\Omega; \mathbb{R}^n) \) is the stored-energy function of a homogeneous material. We suppose that \( W \in C^1(\mathbb{R}^n) \), where \( N_\mathbb{R}^n = \{ A \in \mathbb{R}^n : \det A > 0 \} \), that \( W(A) = \infty \) if \( \det A \leq 0 \), \( W(A) = \infty \) as \( \det A \to 0^+ \), and that for some \( \epsilon_0 > 0 \)
\[
\frac{2W(\nabla u(x))}{\epsilon} \leq \text{const. } (W(A) + 1)
\]
for all \( A \in \mathcal{W} \) with \( |A - 1| < \epsilon_0 \). Let \( u \in \mathcal{W} \) be \( C^1 \) with \( \nabla u \) uniformly bounded and \( u \vert_{\partial \Omega} = u_0 \) \( u(x) = u(x) \). Then it is not hard to show that \( u \in \mathcal{W} \) and that
\[
\frac{d}{d\epsilon} I(u\epsilon) \bigg|_{\epsilon = 0} = \int_{\Omega} \frac{2W(\nabla u\epsilon)}{\epsilon} \nabla u\epsilon \cdot \nabla u\epsilon \bigg|_{\Omega \times (0, 1)} dx = 0.
\]
Under further hypotheses (c.f. [2]) \( u \) is invertible and (5) can then be recognized as a weak form of the Cauchy equilibrium equations
\[
\frac{3}{\mu} \Gamma_1 = 0,
\]
where \( \Gamma_1 \) is the Cauchy stress tensor. If instead we define for \( \epsilon > 0 \),
\[
u(x) = u(z) , x = z + \epsilon v(z),
\]
and make an analogous hypothesis to (4), we obtain the weak form of the equation
\[
\int_{\Omega} \frac{2W(\nabla u\epsilon)}{\epsilon} \nabla u\epsilon \cdot \nabla u\epsilon \bigg|_{\Omega \times (0, 1)} dx = 0.
\]
Details of these results will appear in [3]. To obtain the weak form
\[
\int_{\Omega} \frac{2W(\nabla \psi)}{\epsilon} \nabla \psi \cdot \nabla \psi \bigg|_{\Omega \times (0, 1)} dx = 0
\]
on of the equilibrium equations one would need to show that \( I(u\epsilon) \) is differentiable with respect to \( \epsilon \), with the obvious derivative, for a large class of variations \( u\epsilon(x) = u(x) + \epsilon v(x) \), and it is not clear how to do this under any realistic hypotheses on \( W \). The one-dimensional examples suggest that infinite values of \( \nu_0(x) \) or \( \nu_0(x)^{-1} \) may occur in minimizers; this could be the source of the difficulty, and may also be relevant to the onset of fracture.

Finally I remark that the Laverentz phenomenon severely restricts the class of numerical methods capable of detecting singular minimizers; see [4].

References