23.—Weak Continuity Properties of Mappings and Semigroups.* By
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SYNOPSIS

The relationship between weak and sequential weak continuity for mappings between Banach spaces
and semigroups on Banach space is studied.

1. INTRODUCTION

In this paper we discuss weak and sequential weak continuity for mappings between
Banach spaces, and for semigroups on Banach space. To facilitate discussion we
make the following definitions. Let $X, Y$ be Banach spaces and $f : X \to Y$. We use
$\to$ and $\rightharpoonup$ to denote strong and weak convergence of sequences respectively.

Definition 1.1

(a) $f$ is (strongly) continuous iff $x_n \to x$ implies $f(x_n) \to f(x)$.

(b) $f$ is weakly continuous iff $f$ is continuous with respect to the weak topologies
on $X$ and $Y$.

(c) $f$ is sequentially weakly continuous iff $x_n \rightharpoonup x$ implies $f(x_n) \rightharpoonup f(x)$.

It is clear that (b) implies (c). If $f$ is linear then (a), (b) and (c) are equivalent, as
can be seen from the proof of the equivalence of (a) and (b) given in Dunford and
Schwartz [8, p. 422]. In section 2 we present examples to show that (i) for arbitrary
non-zero $Y$ there exists functions $f$ satisfying (a) and (c) but not (b), if and only if
$X$ is finite-dimensional; (ii) if $H$ denotes infinite-dimensional separable real Hilbert
space and $X = Y = H$ then there exist functions $f$ satisfying the remaining possibilities
$(a)$, not $(b)$, not $(c)$, $(a)$, not $(b)$, $(b)$ and $(c)$ and $(a)$, not $(b)$, $(c)$. Although the distinc-
tion between properties $(a)$, $(b)$, $(c)$ underlies the hypotheses of many theorems in
non-linear functional analysis, there seem to be no references in the literature to the
relevant counter-examples—this is particularly unfortunate in case (i) above, since
it is common practice to define ‘weak continuity’ to be our ‘sequential weak con-
tinuity’. We cannot expect analogous results to (ii) for arbitrary Banach spaces
$X, Y$ since in the space $l_1$ strong and weak convergence of sequences coincide (Dunford
and Schwartz [8, p. 296]).

In section 3 we consider semigroups. Let $\mathbb{R}^+$ denote the non-negative reals. By a
semigroup on a topological space $S$ we mean a family of maps $T(t) : S \to S, t \in \mathbb{R}^+$
satisfying (i) $T(0) =$ identity and (ii) $T(s + t) = T(s)T(t), s, t \in \mathbb{R}^+$. Among abstract
formulations of semigroup theory various choices of continuity axioms are current.

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Non-linear Mappings between Banach Spaces

Throughout the rest of this paper, X and Y are Banach spaces with dual spaces $X^*$, $Y^*$, respectively. We denote the norms in both X and Y by $\| \cdot \|$. 

Definition 2.1
A function $g : X \rightarrow R^+$ belongs to $S(X)$ iff
(i) $g(0) = 0$,
(ii) $x_{n} \rightarrow x$ implies $g(x_{n}) \rightarrow g(x)$,
and there exists a subset $E$ of $X(0)$ such that
(iii) for any finite subset $G$ of $X^*$ the set $E_{n}(x \in E : x^*(x) = 0 \text{ for all } x^* \in G)$ is non-empty, and
(iv) for any $x \in X$, $e \in E$, $\lim_{n \rightarrow \infty} g(x + en) = \infty$.

Examples of functions $g \in S(X)$ arise naturally as follows (the proofs are straightforward).

Examples
2.1. Let $X, X_1$ be Sobolev spaces with $X$ compactly embedded in $X_1$ and let $E = X(0)$. Let $\sigma > 0$ and define $g(x) = \| x \|_{L_{2}}^2$.
2.2. Let $H$ be separable Hilbert space with orthonormal basis $(e_{n})$. Then the map $h : H \rightarrow H$ defined by $h = \sum_{n}^\infty \sum_{n}^\infty \sum_{n}^\infty \sum_{n}^\infty e_{n}$ maps the closed unit ball of $H$ into the compact Hilbert cube. Let $X = E_{\sigma}(0) = H, \sigma > 0$, and define $g : H \rightarrow R^+$ by $g(x) = \| x \|_{H^2}^2, x \in H$.
2.1 holds and $\mathcal{S}(T) = \{u \in \mathbb{R}^n : \phi(T(u)) = 0\}$. Let $x = ax + le$, where $a$ and $l$ are real constants to be chosen. Choose $a > 0$ small enough so that for each $i = 1, \ldots, n$, $\phi(x_i) < a$. Then $x \in \mathcal{B}$. But

$$|\phi(x_i)| = |\phi(x'_i)| = |\phi(f(x_i)) + ax_i(x_i) + \varepsilon y_i(Lx_i)| \geq ax_i(x_i) - \|x_i\|^1/|s(x)|,$$

which tends to $\infty$ as $l \to \infty$. Hence for large enough $l$, $f(x) \notin \mathcal{B}$. This contradiction completes the proof.

**Corollary 2.3.** Let $Y$ be non-zero. There exist functions $f : X \to Y$ satisfying (a), (c) and (b) if $X$ is infinite-dimensional. Proof. Suppose $X$ is infinite-dimensional. Let $\theta \in X$, $\theta \neq 0$ and $y_0 \in Y$, $y_0 \neq 0$. Define $L_x = (x,y_0)$, which is linear and continuous. Let $f = 0$ and $g$ be as in Lemma 2.1. The result now follows from the theorem.

**Corollary 2.4.** If $X$ is separable and infinite-dimensional, and if $X = Y$, then the function $f$ in Corollary 2.3 can be chosen to be $-1$ and onto.

Proof. Let $x^* \in X^*$ span $X^*$, $\|y\| = 1$, $y_0 = 0$, as in Lemma 2.1. The details are easy to verify.

**Examples.**

2.3. Let $\Omega$ be the open interval $(0, 1)$. Let $X = Y = H^1(\Omega)$ (for the definition of this space see Lions and Magenes [9]). Let $f(u) = |u|^{1/2}, y > 0$, where

$$\|y\| = \int_\Omega y^2 dx.$$

Let $L_x = (x,y_0)$. Then $x \neq 0$. Then (see example 2.1) since the embedding of $H^1(\Omega)$ in $L^2(\Omega)$ is compact the hypotheses of Theorem 2.2 are satisfied. Hence $f : H^1(\Omega) \to H^1(\Omega)$ defined by $f(u) = |u|^{1/2}$ satisfies (a), (c) and (b). Note that $f$ is $1-1$ and onto.

2.4. Let $\Omega = (0, 1)$, $\Sigma = H^1(\Omega) \times L^2(\Omega)$, $f = 0$, $X = H^1(\Omega)$, $Y = L^2(\Omega)$,

$L_x = u_x$, $g(0) = 1, 1$. Then by Theorem 2.2 the function $f : H^1(\Omega) \to L^2(\Omega)$ defined by $f(u) = |u^2| x u_x$, $\phi(x) = \phi(\theta, x)$ satisfies (a), (c) and (b). Then $F : X \to X$ defined by $F(x, \phi(x)) = (u, f(u, \phi))$ satisfies (a), (b), (c) and (d).

The proof of the following lemma is left to the reader.

**Lemma 2.5.** Let $H$ be separable Hilbert space with orthonormal basis $\{e_i\}$. Define

$$\psi : \mathbb{R}^n \to \mathbb{R}^n, \psi(n) = e_i, n = 1, 2, \ldots, \psi(x) \in \mathbb{R}^n \text{ linear in each interval } \left[ \frac{1}{n} - \frac{1}{n} \right].$$

Then $\psi(x) = e_i$ for $x \in [1, \omega]$. Define $f_0 : H \to H$ by $f_0(x) = \phi(x, e_i), x \in H$. Then $f_0$ satisfies (a), (b), (c) with $X = Y = H$.

**Corollary 2.6.** There exist functions $f : H \to H$ satisfying (a), (b), (c).

Proof. The reader will easily find an $f$ satisfying (a), (b), (c). For the second example let $f_0$ be as in Lemma 2.5 and note that $f_0$ is bounded and does not satisfy (a). Let $g$ be as in Lemma 2.1 and $L = \text{Identity}$. The result follows from Theorem 2.2.

### 3. Non-linear Semigroups on Banach Space

It is easy to use Example 2.3 to construct a semigroup $T(t)i \in R^+$ on $H^1(\Omega)$ such that for each $t > 0$ the map $u \mapsto T(ut)$ satisfies (a) and (b), and is $1-1$ onto. We seek such a semigroup of the form $T(t) = u^{-1/2}$. The semigroup property requires that $g(t)$ satisfies the equation

$$g(t+1) = g(t) + f(\phi(t+1)) \quad t \in R^+.$$

Hence the semigroup (which is in fact a group) has the form

$$T(t)u = u^{-1/2} = u,$$

for some $k \geq 0$, $(T(t))t \in R^+$ is generated by the equation

$$\frac{dE}{dt} = \lambda \int_\Omega \phi(x) dx.$$

Note that (2.2) defines a semigroup on $L^2(\Omega)$ such that for each $t > 0 T(t)$ satisfies (a) but not (c).

Is it true in general that semigroups $T(t)i \in R^+$ of physical origin are such that the map $u \mapsto T(ut)$ is weakly continuous for each $t \in R^+$? We conjecture that this is not so. In [1-3] a non-linear beam equation was considered whose only non-linear term was that given in Example 2.4. In order to prove that the equation generated a semigroup of sequentially weakly continuous maps it was necessary to use a continuity property of $F$ similar to, but stronger than, sequential weak continuity. Since $F$ is not weakly continuous it is plausible to suppose that the same holds for the generated semigroup, although we have been unable to verify this.

In order to recover continuity for a semigroup of sequentially weakly continuous maps we proceed as follows.

**Definition 3.1.** (see Drudonou [7])

The bounded weak topology (BW) for $X$ is the finest topology coinciding with the weak topology of $X$ on every closed ball $B_r = \{x \in X : |x| \leq r\}$.

**Lemma 3.1.** Let $X$ be reflexive. Then $f : X \to Y$ is sequentially weakly continuous if $f$ is continuous when $X, Y$ are given their BW topologies.

Proof. If $f$ is continuous with respect to the BW topologies, then $f$ is sequentially weakly continuous, since convergence of sequences in the weak and bounded weak topologies coincides. Conversely, let $E$ be a BW closed set in $X$. Let $r > 0$ and let $x$ belong to the weak closure of $f^{-1}(E) \cap B_r$. Then $f$ is sequentially weakly continuous.

**Remark.** The lemma may be false if $X$ is not reflexive (e.g. let $f : X \to \mathbb{R}$ be defined by $f(x) = x, x \in X$).

By the above lemma, if $T(t)i \in X \times X$ satisfies (c) and if $X$ is reflexive, then $T(t)$ is continuous with respect to the BW topology on $X$, as required. For further information concerning weak continuity properties of non-linear semigroups see [4]. Finally...
we remark that the result in [11] referred to above enables one to (i) drop the separability assumption in the 'weak' invariance principles of Slemrod [10] and Ball [2] and (ii) prove that any sequentially weakly continuous map \( f \) from a reflexive Banach space \( X \) to a Banach space \( Y \) which satisfies \( \| f(x) \| \to 0 \) as \( \| x \| \to \infty \) is identically zero. To prove (ii) note that for any \( x \in X \) and \( R > 0 \) there exists a sequence \( \{ x_n \} \subseteq X \) with \( \| x_n - x \| = R \) each \( n \) such that \( x_n \to x \). The result follows by letting \( R \to \infty \) in the inequality \( \| f(x) \| \leq \liminf_{n \to \infty} \| f(x_n) \| \).

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References to Literature


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