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REMARKS ON CHACON'S BITING LEMMA

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ABSTRACT. Chacon’s Biting Lemma states roughly that any bounded sequence in \(L^1\) possesses a subsequence converging weakly in \(L^1\) outside a decreasing family \(E_k\) of measurable sets with vanishingly small measure. A simple new proof of this result is presented that makes explicit which sets \(E_k\) need to be removed. The proof extends immediately to the case when the functions take values in a reflexive Banach space. The limit function is identified via the Young measure and approximations. The description of concentration provided by the lemma is discussed via a simple example.

1. INTRODUCTION AND MAIN RESULT

The purpose of this note is to give an elementary proof of the following result.

**Lemma.** Let \((\Omega, \mathcal{F}, \mu)\) be a finite positive measure space, \(X\) a reflexive Banach space, and let \(\{f^{(j)}\}\) be a bounded sequence in \(L^1(\Omega; X)\), i.e.

\[
\sup_j \int_{\Omega} \|f^{(j)}\|_X \, d\mu = C_0 < \infty.
\]

Then there exist a function \(f \in L^1(\Omega; X)\), a subsequence \(\{f^{(\nu)}\}\) of \(\{f^{(j)}\}\), and a nonincreasing sequence of sets \(E_k \in \mathcal{F}\) with \(\lim_{k \to \infty} \mu(E_k) = 0\), such that

\[
f^{(\nu)} \rightharpoonup f \quad \text{weakly in } L^1(\Omega \setminus E_k; X)
\]

as \(\nu \to \infty\) for every fixed \(k\).

In the above \(L^1(\Omega; X)\) denotes the Banach space of (equivalence classes of) strongly measurable mappings \(g: \Omega \to X\) with finite norm

\[
\|g\|_1 = \int_{\Omega} \|g\|_X \, d\mu.
\]

Since \(X\) is reflexive, the dual \(L^1(\Omega; X)^*\) of \(L^1(\Omega; X)\) can be identified with the space \(L^{\infty}(\Omega; X^*)\) of strongly measurable mappings \(h: \Omega \to X^*\) such that

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\[ \|h\|_\infty = \text{ess sup}_\Omega \|h\|_X < \infty \text{ (cf. Diestel and Uhl [6, pp. 98, 76], A. and C. Ionescu Tulcea [10, p. 95]).} \]

For \( X = \mathbb{R} \) the lemma is stated and proved in Brooks and Chacon [5]; another proof, due to Thomsen and Plachky, appears in Plachky [14, pp. 201–202], and is reproduced in Balder [2] but, as pointed out by M. Valadier and E. Balder after the publication of [2], the argument seems to be incomplete. The extension to the case where \( X \) is a (separable) reflexive Banach space has been independently given by Balder [3]. This extension is not difficult to obtain and is not the main goal of the present paper.

The result is a useful tool in some variational problems where there is only an \( L^1 \) bound on minimizing sequences. One such use has recently been made by Lin [11] in a study of the pure traction problem of nonlinear thermoelasticity; he observed that for \( X = \mathbb{R} \) the lemma could easily be deduced from a related lemma of Acerbi and Fusco [1].

Our purpose is providing yet another proof of the lemma here is that our proof is based on different principles and seems to us simpler and more constructive; in particular it makes rather explicit which sets \( E_k \) need to be removed from \( \Omega \) to recover the weak \( L^1 \) convergence. The only nontrivial result necessary for the proof is the Dunford-Pettis criterion for weak compactness in \( L^1 \). Provided an appropriate Banach space valued version of this criterion is used, the proof for the case when \( X \) is a reflexive Banach space is no harder than that for \( X = \mathbb{R} \).

To illustrate some features of the lemma, consider now the case \( X = \mathbb{R} \). Since \( \|f^{(\nu)}\|_1 \leq C_0 \) it follows that, up to the extraction of a further subsequence, \( f^{(\nu)} \) converges weak* to some limit \( \beta \) say, in the sense of measures. In general there is no connection between \( f \) and \( \beta \), even if \( \beta \) is an \( L^1 \) function (see Example 2, page 661, which also shows that the sets \( E_k \) cannot in general be chosen to be closed). The difference between \( f \) and \( \beta \) measures the amount of concentration in the sequence (cf. P.-L. Lions [12, 13]), provided the \( f^{(j)} \) are nonnegative (for general \( f^{(j)} \) there is the possibility of cancellation of positive and negative concentrations, so that a suitable measure of the amount of concentration is obtained by considering \( |f^{(j)}| \) in place of \( f^{(j)} \)).

2. Proof of the Lemma

Let \( \{f^{(j)}\} \) satisfy (1) and for \( l \geq 0 \) define

\[ \varphi_j(l) = \int_{\{\|f^{(j)}\|_1 \geq l\}} \|f^{(j)}\|_X \, d\mu. \]

Then

(i) \( \varphi_j(0) = \|f^{(j)}\|_1 \leq C_0 \);

(ii) for each \( j \), \( \varphi_j(\cdot) \) is nonincreasing and upper semicontinuous (the upper semicontinuity follows, for example, by considering the points \( x_0 \) where
\[ \|f^{(j)}(x_0)\|_X \geq l \] and the points \( x_1 \) where \( \|f^{(j)}(x_1)\|_X < l \), and applying Lebesgue's Dominated Convergence Theorem;

(iii) \( \varphi_j(l) \to 0 \) as \( l \to \infty \), for each fixed \( j \).

By these properties and the Helly Selection Theorem we can extract a subsequence, again denoted \( f^{(j)} \), such that

\[ \alpha(l) \overset{\text{def}}{=} \lim_{j \to \infty} \varphi_j(l) \]

exists for all \( l \geq 0 \). Clearly \( \alpha(\cdot) \) is nonincreasing. Let \( \alpha_\infty = \lim_{l \to \infty} \alpha(l) \).

**Case 1.** \( \alpha_\infty = 0 \). In this case the subsequence \( \{f^{(j)}\} \) is sequentially weakly relatively compact in \( L^1(\Omega; X) \). In fact given \( \varepsilon > 0 \) we can choose \( l_0 \) sufficiently large so that \( \alpha(l_0) < \varepsilon \), then \( j_0 \) sufficiently large so that \( \varphi_j(l_0) < \varepsilon \) for all \( j \geq j_0 \), and then \( l_1 > l_0 \) sufficiently large so that \( \varphi_j(l_1) < \varepsilon \) for all \( j < j_0 \). Thus \( \varphi_j(l_1) < \varepsilon \) for all \( j \), so that by a Banach space valued version of the Dunford-Pettis Theorem (A. and C. Ionescu Tulcea [10, p. 117], Diestel & Uhl [6, pp. 101, 76]; the reader interested only in the case \( X = \mathbb{R} \) can consult, for example, Edwards [9, p. 274]) there exists a further subsequence \( \{f^{(j')}\} \) which converges weakly in \( L^1(\Omega; X) \) to some \( f \in L^1(\Omega; X) \), so that the conclusion of the lemma holds with all the sets \( E_k \) empty.

**Case 2.** \( \alpha_\infty > 0 \). In this case we claim that there exists a subsequence \( l_j \to \infty \) such that \( \varphi_j(l_j) \to \alpha_\infty \). Indeed, we can define \( l_j = \sup\{l \geq 0 : \varphi_j(l) \geq \alpha_\infty - l^{-1}\} \). The supremum is attained because \( \varphi_j(l) \to 0 \) as \( l \to \infty \) and \( \varphi_j \) is upper semicontinuous. If \( \{l_j\} \) contained a bounded subsequence \( \{l_{j'}\} \) then we would have \( \varphi_{j'}(l_{j'}) < \alpha_\infty - (l')^{-1} \) for any \( l' > \sup l_{j'} \); letting \( y \) tend to \( \infty \) gives a contradiction since \( \alpha(\cdot) \) is nonincreasing. Hence \( l_j \to \infty \). Also, for any \( m \geq 0 \),

\[ \alpha_\infty - l_j^{-1} \leq \varphi_j(l_j) \leq \varphi_j(m) \quad \text{for } j \text{ sufficiently large.} \]

Hence \( \alpha_\infty \leq \liminf_{j \to \infty} \varphi_j(l_j) \leq \limsup_{j \to \infty} \varphi_j(l_j) \leq \alpha(m) \), and letting \( m \to \infty \) gives \( \varphi_j(l_j) \to \alpha_\infty \).

We next claim that

\[ \lim_{m \to \infty} \sup_{j \geq 1} \int_{\{m \leq \|f^{(j)}\|_X < l_j\}} \|f^{(j)}\|_X \, d\mu = 0. \]

To see this, note firstly that

\[ S(m) \overset{\text{def}}{=} \sup_{j \geq 1} \int_{\{m \leq \|f^{(j)}\|_X < l_j\}} \|f^{(j)}\|_X \, d\mu \]

is nonincreasing, and secondly that

\[ S(m) = \sup_{j \geq 1, \, l_j > m} [\varphi_j(m) - \varphi_j(l_j)]. \]

Given any \( \varepsilon > 0 \), there exists \( m_1 \) such that \( \alpha(m_1) < \alpha_\infty + \varepsilon \). Then there exists \( j_0 \) such that if \( j \geq j_0 \) then \( \varphi_j(m_1) \leq \alpha(m_1) + \varepsilon \) and \( \varphi_j(l_j) \geq \alpha_\infty - \varepsilon \), and
hence
\[ \varphi_j(m_1) - \varphi_j(l_j) \leq \alpha(m_1) + \varepsilon - \alpha_\infty + \varepsilon \leq 3\varepsilon. \]
Choosing \( m_2 \) such that \( m_2 \geq m_1 \) and \( m_2 \geq \max_{j \leq m_1} l_j \) we deduce that
\[ S(m_2) \leq 3\varepsilon, \]
which proves (1). Given \( \delta > 0 \), choose a new subsequence, again denoted \( \{f^{(j)}\} \), such that
\[ (\sum_j l_j^{-1}) C_0 \leq \delta. \]
Let \( E = \bigcup_j \{ \|f^{(j)}\|_X \geq l_j \} \). Then since
\[ l_j \mu(\{ \|f^{(j)}\|_X \geq l_j \}) \leq \int_\{ \|f^{(j)}\|_X \geq l_j \} \|f^{(j)}\|_X d\mu \leq C_0, \]
we have \( \mu(E) \leq \delta \), and
\[ \lim_{m \to \infty} \sup_{j \geq 1} \int_{\{ \|f^{(j)}\|_X \geq m \} \cap E} \|f^{(j)}\|_X d\mu \leq \lim_{m \to \infty} \sup_{j \geq 1} \int_{m \leq \|f^{(j)}\|_X} \|f^{(j)}\|_X d\mu = 0. \]
Hence by the Banach space valued Dunford-Pettis Theorem \( \{f^{(j)}\} \) is sequentially weakly relatively compact in \( L^1(\Omega \setminus E; X) \). Repeating this procedure for \( \delta = k^{-1}, \; k = 1, 2, \ldots, \) and taking successive subsequences, we obtain a diagonal subsequence \( \{f^{(\nu)}\} \), a nonincreasing sequence \( E_k \) of \( \mu \)-measurable sets with \( \lim_{k \to \infty} \mu(E_k) = 0 \), and a strongly \( \mu \)-measurable function \( f : \Omega \to X \), such that \( f^{(\nu)} \to f \) weakly in \( L^1(\Omega \setminus E_k; X) \) for every \( k \). Since each \( E_k \) differs from a set in \( \mathcal{F} \) by a set of measure zero we can suppose that \( E_k \in \mathcal{F} \) for each \( k \). Finally, we have that
\[ \int_{\Omega \setminus E_k} \|f\|_X d\mu \leq \lim_{\nu \to \infty} \int_{\Omega \setminus E_k} \|f^{(\nu)}\|_X d\mu \leq C_0, \]
so that letting \( k \to \infty \) we deduce that \( f \in L^1(\Omega; X) \). This completes the proof.

**Remarks.** 1. The use of Helly’s Theorem in the proof is not essential; it suffices to extract a subsequence of the \( \varphi_j \) which converges for each positive integer \( l \).

2. The function \( f \) is unique in the sense that if there exists a subsequence \( \{f^{(\nu)}\} \) and two families \( E_k, \tilde{E}_k \) of measurable sets as in the lemma such that, for each \( k \), \( f^{(\nu)} \to f \) in \( L^1(\Omega \setminus E_k; X) \) and \( f^{(\nu)} \to \tilde{f} \) in \( L^1(\Omega \setminus \tilde{E}_k; X) \), then \( f = \tilde{f} \). This follows since, by a suitable choice of test function, \( f = \tilde{f} \) a.e. in \( \Omega \setminus (E_k \cup \tilde{E}_k) \) for each \( k \).

3. **Identification of \( f \) via the Young measure and approximations**

In this section we show how the function \( f \) in the lemma can be identified in terms of the Young measure and various approximation procedures such as truncation. For simplicity we restrict attention to the case \( X = \mathbb{R}^m \), \( \mu = n \)-dimensional Lebesgue measure, \( \Omega \subset \mathbb{R}^d \) \( \mu \)-measurable with \( \mu(\Omega) < \infty \).

Since \( \sup_{\nu} \|f^{(\nu)}\|_1 < \infty \) there exists a family \( (\nu_x)_{x \in \Omega} \) of probability measures on \( \mathbb{R}^m \) (the Young measure), depending measurably on \( x \), and a further
subsequence, again denoted \( \{f^{(\cdot)}\} \), with the following property (cf. Ball [4]): if \( g : \mathbb{R}^n \to \mathbb{R} \) is continuous, if \( A \subset \Omega \) is \( \mu \)-measurable, and if
\[
g(f^{(\cdot)}) \rightharpoonup z \quad \text{weakly in } L^1(A; \mathbb{R}),
\]
then \( g(\cdot) \in L^1(\mathbb{R}^n; \mu_A) \) for a.e. \( x \in A \) (where the exceptional set possibly depends on \( g \)) and
\[
z(x) = \int_{\mathbb{R}^n} g(\lambda) \, d\mu_x(\lambda) \overset{\text{def}}{=} \langle \nu_x, g \rangle \quad \text{a.e. } x \in A.
\]
Applying this with \( A = \Omega \setminus E_k \) and \( g(\lambda) = \lambda_i \), where \( \lambda = (\lambda_1, \ldots, \lambda_m) \), we deduce that the \( f \) defined in the lemma is given by
\[
f(x) = \langle \nu_x, \lambda \rangle = \int_{\mathbb{R}^m} \lambda_i \, d\nu_x(\lambda) \quad \text{a.e. } x \in \Omega.
\]
We now suppose that continuous functions \( g_k : \mathbb{R}^n \to \mathbb{R}^n \), \( k = 1, 2, \ldots \), are given satisfying the conditions:
\begin{itemize}
  \item[(i)] \( g_k(\lambda) \to \lambda \) as \( k \to \infty \), for each fixed \( \lambda \in \mathbb{R}^n \),
  \item[(ii)] \( |g_k(\lambda)| \leq C_1(1 + |\lambda|) \) for all \( k \), all \( \lambda \in \mathbb{R}^n \), where \( C_1 \) is a constant,
  \item[(iii)] \( \lim_{|\lambda| \to \infty} |\lambda|^{-1} |g_k(\lambda)| = 0 \) for each \( k \).
\end{itemize}
The conditions (i)–(iii) hold in the following important cases:
\begin{itemize}
  \item[(a)] (truncation at level \( k \))
    \[ g_k(\lambda) = \psi(k^{-1}|\lambda|) \lambda, \]
    where
    \[
    \psi(t) = \begin{cases}
    1 & \text{if } 0 \leq t < 1, \\
    t^{-1} & \text{if } t \geq 1.
    \end{cases}
    \]
  \item[(b)] (approximation by \( 1/p \)-th powers)
    \[ g_k(\lambda) = \begin{cases}
    |\lambda|^{-1+p_k^{-1}} \lambda & \text{if } \lambda \neq 0, \\
    0 & \text{if } \lambda = 0,
    \end{cases} \]
    where \( p_k > 1 \), \( \lim_{k \to \infty} p_k = 1 \).
\end{itemize}
Then we have the
\begin{proposition}
  For each fixed \( k \) there exists \( f_k \in L^1(\Omega; \mathbb{R}^n) \) such that as \( \nu \to \infty \)
  \[ g_k(f^{(\nu)}) \rightharpoonup f_k \quad \text{weakly in } L^1(\Omega; \mathbb{R}^n). \]
  \( As k \to \infty \),
  \[ f_k \to f \quad \text{strongly in } L^1(\Omega; \mathbb{R}^n). \]
\end{proposition}
\begin{proof}
  Fix \( k \). By (iii),
  \[ \lim_{l \to \infty} \int_{\{|f^{(\nu)}| > l\}} |g_k(f^{(\nu)}(x))| \, dx = 0 \]
  for all \( \nu \), \( k \), and \( l \).
\end{proof}
uniformly in \( \nu \). Thus by the Dunford-Pettis Theorem \( g_k(f^{(\nu)}) \) is sequentially weakly relatively compact in \( L^1(\Omega; \mathbb{R}^m) \). Hence by the properties of the Young measure given above,

\[
g_k(f^{(\nu)}) \rightharpoonup f_k \quad \text{in} \quad L^1(\Omega; \mathbb{R}^m),
\]
as \( \nu \to \infty \), where

\[
f_k(x) = \int_{\mathbb{R}^m} g_k(\lambda) \, d\nu_x(\lambda) \quad \text{a.e.} \quad x \in \Omega.
\]

To prove the \( L^1(\Omega; \mathbb{R}^m) \) convergence of \( f_k \) to \( f \), we use the dominated convergence theorem. We note first that the function \( F \) defined by

\[
F(x) = \int_{\mathbb{R}^m} |\lambda| \, d\nu_x(\lambda)
\]
belongs to \( L^1(\Omega; \mathbb{R}) \); this follows by applying the lemma to the sequence \( \{|f^{(\nu)}|\} \) and using the properties of the Young measure given above with \( g(\lambda) = |\lambda| \). From (ii), (3) we deduce that

\[
|f_k(x)| \leq C_1(1 + F(x)) \quad \text{for a.e.} \quad x \in \Omega.
\]
It thus suffices to show that \( f_k(x) \to f(x) \) for a.e. \( x \in \Omega \). But this follows from (i), (ii), (3) by a preliminary application of the dominated convergence theorem to the sequence \( \{g_k(\cdot)\} \) in \( L^1(\mathbb{R}^m; \nu_x) \) for \( x \) fixed; indeed the upper bound \( C_1(1 + |\lambda|) \) in (ii) belongs to \( L^1(\mathbb{R}^m; \nu_x) \) since \( F(x) \) is finite for a.e. \( x \in \Omega \).

**Remark 3.** It is easily shown that in the cases (a), (b) above the convergence of \( g_k(f^{(\nu)}) \) to \( f_k \) holds weak* in \( L^\infty(\Omega; \mathbb{R}^m) \) and weakly in \( L^m(\Omega; \mathbb{R}^m) \), respectively.

**4. Examples, and Discussion about Concentrations**

**Example 1.** The following statement is false: given any bounded sequence \( \{f^{(l)}\} \) in \( L^1(\Omega; \mathbb{R}) \) and any \( \delta > 0 \), there exists a subset \( E \subset \Omega \) with \( \mu(E) < \delta \) such that \( \{f^{(l)}\} \) is sequentially weakly relatively compact in \( L^1(\Omega \setminus E; \mathbb{R}) \). Consider the case \( \mathbb{R}, \Omega = (0, 1) \) with Lebesgue measure, and the sequence \( \{f^{j,k}\} \), \( j, k = 1, 2, \ldots, j \neq k \), defined by

\[
f^{j,k}(x) = \begin{cases} q_k^{-1} & \text{if } x \in (q_j - q_k, q_j + q_k), \\ 0 & \text{otherwise}, \end{cases}
\]

where \( \{q_j\} \) is an enumeration of the rationals in \( (0, \infty) \). Note that \( \int_\Omega |f^{j,k}| \, dx \leq 2 \). Let \( A \subset (0, 1) \) have positive Lebesgue measure. We show that, for arbitrary \( l > 0 \), there exist an infinite number of pairs of \( j, k \) such that

\[
\int_{\{f^{j,k}| \geq l\} \cap A} |f^{j,k}| \, dx \geq 1.
\]
In fact, let $x_0 \in (0, 1)$ be a point of density of $A$. Then there exists $r \in (0, l^{-1})$ such that $\text{meas}\{(x_0 - r, x_0 + r) \cap A\} \geq 2r \cdot \frac{3}{2}$. Let $q_j \rightarrow x_0$, $q_k \rightarrow r$, where $q_j$, $q_k$ are rationals and $q_k < l^{-1}$. Then

$$\int_{\{|f^{j,k}| \geq l\} \cap A} |f^{j,k}| \, dx = q_k^{-1} \text{meas}(\langle q_j - q_k, q_j + q_k \rangle \cap A)$$

$$\rightarrow r^{-1} \text{meas}(\langle x_0 - r, x_0 + r \rangle \cap A) \geq \frac{3}{2},$$

as $j, k \rightarrow \infty$.

**Example 2.** We take $X = \mathbb{R}$, $\Omega = (0, 1)$ with Lebesgue measure, and define for $j = 2, 3, \ldots$,

$$f^{(j)}(x) = \begin{cases} j^{2/2} & \text{for } x \in (k(j+1)^{-1} - j^{-3}, k(j+1)^{-1} + j^{-3}), \\ 0 & \text{otherwise.} \end{cases}$$

Then $\|f^{(j)}\|_1 = 1$ for each $j$, and it is easily proved that $f^{(j)} \rightharpoonup 1$ in the sense of measures. We now identify the function $f$ and a possible choice of the sets $E_k$ of the lemma. We take

(4) $$E_k = \bigcup_{j \geq k} \{f^{(j)} \neq 0\},$$

corresponding to the choice $l_j = j^{2/2}$ in the proof of the lemma. Then, since $\text{meas}\{f^{(j)} \neq 0\} = 2j^{-2}$,

$$\text{meas} E_k \leq \sum_{j \geq k} 2j^{-2} \rightarrow 0 \quad \text{as } k \rightarrow \infty,$$

and if $x \in \Omega \setminus E_k$, $f^{(j)}(x) = 0$ for all $j \geq k$ (so that in particular $f^{(j)} \rightarrow 0$ a.e. in $(0, 1)$). Hence $f = 0$. In this example we do not need to extract a subsequence.

Since $f$ is unique (see Remark 2, page 658) and since $\lim_{j \rightarrow \infty} \int_I f^{(j)} \, dx = \text{meas} I$ for any open interval $I \subset (0, 1)$, it follows that the sets $E_k$ cannot be chosen to be closed.

In Example 2, the weak* limit of $f^{(j)}$ in the sense of measures (or, more precisely, the difference $1 - 0$ between the weak* limit of $f^{(j)}$ and the $f$ of the lemma) sees the concentrations of $f^{(j)}$ as being in the limit smeared out uniformly throughout $\Omega$. The same is true of the generalized Young measure of DiPerna and Majda [7], which in this example is constant in $\Omega$. The lemma, on the other hand, shows that in general the concentration takes place on progressively smaller and smaller sets. In Example 2 there is even a set of points, whose complement is of arbitrarily small measure, at which the $f^{(j)}$ are for large enough $j$ identically zero, and it does not seem satisfactory to describe these points as being points of concentration.
An attempt to give a precise meaning to concentration sets, in a context different from but related to ours, has been made by DiPerna and Majda [8] for the purpose of applications to the Euler equations of fluid mechanics. They consider, for example, the case of a sequence \( v^{(j)} \) converging weakly in \( L^2(\Omega; \mathbb{R}) \) to \( v \), say, where \( \Omega \subset \mathbb{R}^n \) is open, and define the associated ‘reduced defect measure’ \( \theta \) as the outer measure

\[
\theta(B) = \limsup_{j \to \infty} \int_B \left| v^{(j)} - v \right|^2 \, dx,
\]

for any Borel subset \( B \) of \( \Omega \). They then define the ‘concentration sets’ for \( \theta \) as the Borel sets \( E \) for which \( \Omega \setminus E \) is a countable union of null sets of \( \theta \). Thus they are interested in detecting on which sets an \( L^2 \) weakly convergent sequence converges strongly, while in this paper our goal has been to isolate the sets where a bounded sequence in \( L^1 \) is not weakly convergent in \( L^1 \). We may nevertheless try to apply these definitions to Example 2 by setting \( v^{(j)} = (f^{(j)})^{1/2} \), \( v = 0 \). However, the conclusion is unfortunately that any Borel set \( E \) (including the empty set) is a concentration set. To prove this we set \( G = \cap_{k \geq 2} E_k \), where the \( E_k \) are given by (4). Then \( G \) is a Borel set of Lebesgue measure zero, and by the definition of \( \theta \) is hence a null set for \( \theta \). Then, since \( \Omega \setminus E_k \) is a null set for \( \theta \), the equation

\[
\Omega = G \cup \bigcup_{k \geq 2} (\Omega \setminus E_k)
\]

shows that the empty set is a concentration set, and it is easily proved that any Borel set containing a concentration set is itself a concentration set.

These remarks suggest that the tools presently available do not give as complete a description of concentrations as one might desire.

Added in proof. We are grateful to M. Valadier for having pointed out to us the paper of M. Slaby, \textit{Strong convergence of vector-valued pramarts and sub-pramarts}, Probability and Mathematics, \textbf{5} (1985) 187-196, who proves a result essentially equivalent to the Biting lemma by an argument similar to ours. Some ingredients of the argument also appear in P. L. Lions [12 Lemma 1.1].

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