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Universität Münster
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Geometric Group Theory I
Exercise Sheet 1

Exercise 1. Consider the group $G = \mathrm{GL}_m(\mathbb{R}) \times \mathrm{GL}_n(\mathbb{R})$ and the set $X = \mathrm{Mat}_{m\times n}(\mathbb{R})$.

a) Show the following map:
$$G \times X \to X$$
$$((P, Q), A) \mapsto P \cdot A \cdot Q^{-1}$$
is a group action of G on X.

b) Is this action transitive? Is it faithful? Please justify your answer.

c) Describe the orbit decomposition of X under this action.

(4 Point)

Exercise 2. Consider the group G of rotations of the cube K in $\mathbb{R}^3$, i.e. the group of orientation-preserving isometries of $\mathbb{R}^3$ which preserves K.

a) Show $|G| = 24$ by considering the stabilizer of a vertex under the action of G on the set of vertices of K.

b) Show that $G \cong S_4$ where S_4 is the symmetric group of degree 4.
Hint: Consider the action of G on the set of 4 longest diagonals of K.

(4 Point)

Exercise 3. Let $A_n \leq S_n$ be the alternating group, i.e. the group which consists of even permutations of an n-element set.

a) Show that for $n \geq 3$, the group A_n is generated by 3-cycles.

b) Show that for $n \geq 5$, the group A_n is generated by permutations of the form $(i\ j)(k\ l)$, namely $A_n = \langle (i\ j)(k\ l) : i, j, k, l \leq n \text{ pairwise distinct}\rangle$.

c) Show that for $n \geq 3$, for any $\gamma \in S_n \setminus \{1\}$ there is a transposition in S_n which does not commute with γ.

d) Show that for $n \geq 5$, the group A_n is the only proper non-trivial normal subgroup of S_n.

(4 Point)

Exercise 4. Let $n \geq 3$ and D_n be the *dihedral group*, i.e. the automorphism group of a regular n-gon.

a) Show that:

   (i) $|D_n| = 2n$.

   (ii) There are elements $d, s \in D_n$, such that $\langle d, s \rangle = D_n$, $d^n = 1$, $s^2 = 1$ and $sds = d^{-1}$.

   (iii) $D_n = \langle d \rangle \rtimes \langle s \rangle$. Does $D_n = \langle d \rangle \times \langle s \rangle$ also hold?

b) Describe the center of D_n.

c) Let G be a finite group of order ≥ 6. Show that G is generated by two involutions if and only if G is a dihedral group.
   Hint: Find a cyclic normal subgroup of index 2. Now use part a)(iii).

(4 Point)

Submission until Thursday morning 10:00, 20.10.2022, in Briefkasten 161.
The exercise sheets should be solved and submitted in pairs.
Tutorial: Fridays 12:00-14:00, 21.10.2022 in room SRZ1d.