Dr. M. Bays
Dr. T. Zou

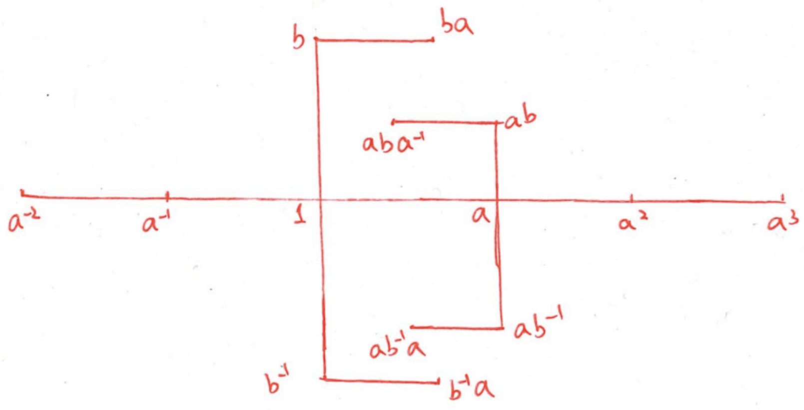
Universität Münster
Wintersemester 2022/23

Geometric Group Theory I
Exercise Sheet 5

Exercise 1. Let $H := \langle\langle a^2, b \rangle\rangle \leq F(a,b) := F(\{a,b\})$ be the normal subgroup generated by a^2 and b in the free group $F(a,b)$. Let $T := \Gamma(F(a,b), \{a,b\})$ be the Cayley graph. Consider the natural action $H \circlearrowleft T$ of H on T.

a) Draw the quotient graph ${}_H\backslash^T$ and calculate $B_1({}_H\backslash^T)$.

b)
- Find a tree T_r of representatives of $H \circlearrowleft T$ which includes the vertex 1.
- Mark the orbit GT_r of T_r under the action of H in the following subgraph Z, namely mark the intersection $Y := GT_r \cap Z$.
- Draw Z/Y, i.e. the graph obtained by contracting the trees in Y.



Subgraph Z of T.

c) What is the rank of H? Find a basis of them.
Hint: Consider $d := [F(a,b) : H]$ and use Schreier's formula, indeed you could read the set of generators from Z/Y.

d) How many index 2 subgroups of $F(a,b)$ are there? Find a basis for each of them.
Hint: Consider a homomorphism $f : F(a,b) \to \mathbb{Z}/2\mathbb{Z}$ and list all possible images of a, b.

(6 Points)

Exercise 2. Show that the following group is trivial:

$$\langle a, b \mid aba^{-1} = b^2, bab^{-1} = a^2 \rangle.$$

(2 Points)

Exercise 3.

a) Let G be a group generated by $t_1, \dots, t_{n-1}$ satisfying the following relations $(\star)$:

$$t_1^2 = \cdots = t_{n-1}^2 = 1;\ (t_1t_2)^3 = \cdots = (t_{n-2}t_{n-1})^3 = 1;$$
$$t_it_j = t_jt_i \text{ for } 1 \leq i, j \leq n-1 \text{ and } j \geq i+2.$$

Let H be the subgroup of G generated by $t_2, \dots, t_{n-1}$. Show that the index $[G : H]$ is at most n and conclude that $|G| \leq n!$.
Hint: Show that the following collection of cosets $H, Ht_1, Ht_1t_2, \dots, Ht_1 \cdots t_{n-1}$ is closed under right multiplication by $t_1, \dots, t_{n-1}$.

b) Show that $S_n \cong \langle t_1, \dots, t_{n-1} \mid (\star) \rangle$.

(4 Points)

Exercise 4. Let X, Y be connected graphs and $x \in X^0, y \in Y^0$. Suppose $f : X \to Y$ is a morphism with $f(x) = y$.

a) Show that the map f_* defined as $f_*(\prod_{i<n} e_i) := \prod_{i<n} f(e_i)$ for any path $(e_0, \dots, e_{n-1})$ from x to x, is a homomorphism from $\pi_1(X, x)$ to $\pi_1(Y, y)$.

b) Prove or give counterexamples of the following statements:

   i) If f is surjective, then f_* is surjective;
   ii) If f is surjective, then f_* is injective;
   iii) If f is injective, then f_* is injective;
   iv) If f is injective, then f_* is surjective.

(4 Points)

*Submission by **Wednesday** morning 11:00, 16.11.2022, in Briefkasten 161.*
The exercise sheets should be solved and submitted in pairs.
Tutorial: Fridays 12:00-14:00, in room SR1d.
If you have questions about the problem sheet, please write to Tingxiang: tingxiangzou@gmail.com.