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Geometric Group Theory I
Exercise Sheet 6

Exercise 1. Let G be a group and $S \subseteq G$ be a set of generators. Let $\Gamma(G, S)$ be the Cayley graph and H be a subgroup of G.

a) Show that there exists a surjective homomorphism $f : \pi_1({}_H\backslash^{\Gamma(G,S)}, H) \to H$.

b) Use a) to show that a finite-index subgroup of a finitely generated group is finitely generated.[1]

(4 Points)

Exercise 2.

a) Let $G = A * B$ be a free product. Let

$$X := \{a^{-1}b^{-1}ab : a \in A \setminus \{1\}, b \in B \setminus \{1\}\}.$$

Show that the subgroup of G generated by X is free with basis X.

b) Suppose $G = A * B$ with A, B non-trivial groups. Let $a \in A \setminus \{1\}$, $b \in B \setminus \{1\}$ and $g := ab$. Show that $C_G(g) := \{h \in G : hg = gh\}$ is the infinite cyclic group $\langle g \rangle$. Conclude that $Z(G) = \{1\}$.

Hint for a) and b): Use the normal forms.

(4 Points)

Exercise 3. Let $A * B$ be a free product. An element $w \in A * B$ with normal form $w = \prod_{i<n} g_i$ is called *cyclically reduced* if either $n \le 1$ or g_0 and g_{n-1} do not belong to the same factor A or B, i.e. $g_0 \in A$ if and only if $g_{n-1} \in B$.

a) Show that any element in $A * B$ is conjugate to a cyclically reduced element.

b) Show that any element in $A * B$ of finite order is conjugate to either an element in A or an element in B.

(4 Points)

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[1]This gives an alternative proof of part of Proposition 4.10 in the lecture notes.

Given groups G and H. Denote $\operatorname{Hom}(G, H)$ to be the group of all homomorphisms from G to H, with the group operation defined as $\alpha\beta : g \mapsto \alpha(g)\beta(g)$.

Exercise 4. Let $G *_A H$ be an amalgamated free product and C be a group. Show that the map $\theta(\alpha) := (\alpha|_G, \alpha|_H)$ defines an isomorphism:

$$\theta : \operatorname{Hom}(G *_A H, C) \to \{(\beta, \gamma) \in \operatorname{Hom}(G, C) \times \operatorname{Hom}(H, C) : \beta|_A = \gamma|_A\}.$$

(4 Points)