Dr. M. Bays
Dr. T. Zou

Universität Münster
Wintersemester 2022/23

Geometric Group Theory I
Exercise Sheet 8

Let T be a tree, we define the *diameter of* T, which takes value in $\mathbb{N} \cup \{\infty\}$, to be the supremum of the length of a reduced path in T. A vertex $x \in T^0$ is called *terminal* if there is only one edge $e \in T^1$ with $\omega(e) = x$.

Exercise 1. Let T be a finite tree. Let P be the set of all terminal vertices of T.

a) Show that the diameter of T is finite and P is non-empty.

b) Let $T - P$ be the graph defined by $(T-P)^0 := T^0 \setminus P$ and
$$(T-P)^1 := T^1 \setminus \{e \in T^1 : \alpha(e) \in P \text{ or } \omega(e) \in P\}.$$
Show that if T has diameter $n \geq 2$, then $T - P$ is a tree of diameter $n - 2$.

c) Suppose T has diameter n. Let S be the group of automorphisms of T. Show that if n is even, then S fixes a vertex (namely, there exists $x_0 \in T^0$ such that $s(x_0) = x_0$ for all $s \in S$). And if n is odd, then S stabilizes a pair of edge $e, \bar{e}$ (namely, $s(e) \in \{e, \bar{e}\}$ and $s(\bar{e}) \in \{e, \bar{e}\}$ for all $s \in S$).
Hint: Induction on the diameter n.

(4 Points)

Let T be a tree, and $X \subseteq T^0$, we define $\widetilde{X}$ be the subtree generated by X, i.e. the smallest subtree of T which contains X.

Exercise 2.

a) Let Γ be a group acting on a tree T. Let $x \in T^0$ and $\widetilde{\Gamma x}$ be the subtree of T generated by the orbit of x under the action of Γ. Show that $\widetilde{\Gamma x}$ is invariant under the action of Γ. And if Γx is finite, then $\widetilde{\Gamma x}$ is finite.

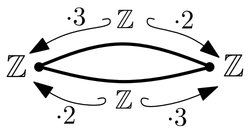
b) Let $\Gamma \leq G := G_1 *_A G_2$ be a finite subgroup. Show that Γ is contained in a conjugate of G_1 or G_2.
Hint: Let G be acting on a tree T such that ${}_G\backslash T$ is a segment with the lift $x \xrightarrow{e} y$ such that $G_x = G_1$, $G_y = G_2$ and $G_e = A$ (see Theorem 6.5). Consider the action of Γ on $\widetilde{\Gamma x}$ and use Exercise 1 item c) to conclude $\Gamma \in G_a$ for some vertex $a \in \widetilde{\Gamma x}$ (note that the action of Γ is non-inversive).

(4 Points)

Exercise 3. Let $G := S_3 *_{\mathbb{Z}/2\mathbb{Z}} S_3 =: G_1 *_{\mathbb{Z}/2\mathbb{Z}} G_2$. Sketch the tree of G constructed in the Theorem 6.5 with at least 7 cosets of G_1 and 7 cosets of G_2. Indicate the action of the subgroup $\mathbb{Z}/2\mathbb{Z}$.

(4 Points)

Exercise 4. Describe the fundamental group of the following graph of groups



with respect to the left-hand point, where $\mathbb{Z} \xhookrightarrow{\cdot 3} \mathbb{Z}$ is the homomorphism which sends x to $3x$, and similarly for $\mathbb{Z} \xhookrightarrow{\cdot 2} \mathbb{Z}$.

(4 Points)

*Submission by **Wednesday** morning 11:00, 07.12.2022, in Briefkasten 161.*
The exercise sheets should be solved and submitted in pairs.
Tutorial: Fridays 12:00-14:00, in room SR1d.
If you have questions about the problem sheet, please write to Tingxiang: tingxiangzou@gmail.com.