Parameters in algebraically closed fields

a short and English version of a preprint with a long title[1]

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A celebrated result of Boris Zil'ber is that an infinite simple algebraic group G , over an algebraically closed field K , is an ω_1-categorical structure.

The remarkable thing about the proof given by Zil'ber is that it contains no algebraic geometry at all: in the original paper, it was proved for any simple group interpretable in an ω_1-categorical structure, and later extended to simple groups of finite Morley rank.

A first aim of this abstract is to describe the elementary theory of G , its elementary extensions and its elementary restrictions, using a minimum of algebraic informations. For that, we have to understand the connections between the automorphisms of the field K and the automorphisms of the group G .

As a preliminary, we review the basic model-theoretic properties of an algebraically closed field :

- K eliminates the imaginary elements

- any structure definable[3] in K is pseudo locally finite (based on compactness, elimination of imaginaries, and Galois theory of finite fields)

- K eliminates the quantifiers (a theorem of Chevalley, that fanatic logicians attribute to Tarski) ; the definable subsets of the cartesian powers of K are therefore the finite boolean combinations of Zariski-closed sets: they are called constructible sets in Geometry; for us, a constructible map will be a map with a constructible graph, <u>not</u> a map for which the reverse image of a constructible set is constructible

- any constructible group is constructibly isomorphic to an algebraic group (based on Weil's Theorem on group chunks, plus a lemma of Hrushovski in characteristic p); since I have no time to explain what are morphisms and varieties, it will be enough for us to know that a simple constructible group is constructibly isomorphic to a Zariski-closed subgroup of some $Gl_n(K)$; one assumes that Zil'ber, in his original paper, meant that sort of groups while speaking of "algebraic simple groups"; in the early eighties, it was not clear that the Algebraicity Conjecture had no counter-example definable in K !

- any infinite constructible field is constructibly isomorphic to the base field K (needs the preceding point)

- in a simple algebraic group G , a copy of the base field K is definable; one has to know that the Borel subgroups of G are not nilpotent, a very plain fact

[1] The preprint, and its bibliography, is attached to this abstract.

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[3] By "definable", I mean "definable with parameters", unless the converse is explicitely stated ; I do not distinguish definability from interpretability.

for a geometer; motivated model-theorists, remaining at the constructible level, can obtain it by observing that a bad group cannot be pseudo locally finite

- any simple infinite constructible group G is constructibly isomorphic to a Zariski-closed subgroup of some $Gl_n(K)$ defined by polynomial equations with integer coefficients; in other words, G has a constructible copy which is defined without parameters; for this result, a paper of Borel is quoted by Aleksandr Borovik, but I believe that it is connected to works of Chevalley; it is also a consequence of the classification by Simon Thomas of the simple pseudo locally finite groups of finite Morley rank, which needs the classification of finite simple groups.

Once we know all that, we can describe precisely the structure of an algebraic simple group; by contrast, on the structure of algebraic simple groups of finite Morley rank, Zil'ber's original theorem says exactly nothing. The connections between the group and the field are established via the Theorem of Borel and Tits.

It is a second paper by Zil'ber that attracted my attention on the model-theoretic interpretation of this theorem; my version is a translation of the notion of L-internality by Hrushovski :

Theorem 1 (Borel-Tits, model-theoretic version). *Let* G *be a simple algebraic group over an algebraically closed field* K *defined by a formula* $G(\ )$, L *be an infinite field definable in the group* G *, and* σ *be a field-isomorphism from* K *to* L *definable in* K *. Then the group-isomorphism* σ^* *between* $G = G(K)$ *and* $G(L)$ *induced by* σ *is definable in the group* G .

We see that the group manages to say, by a sentence $\varphi(\bar{a})$ with parameters in G , that "I am isomorphic to this matrix group defined without parameters over this field L "; we shall see that the field can be defined without parameters, but parameters $\bar{a}$ are needed for the isomorphism. Elementary extensions correspond to extensions of L , but a restriction of L does not make sense if it does not contains the tuple of parameters $\bar{a}$. Since the formula defining G has no parameters, it is clear that the prime model G_0 of the theory of G is associated to the prime field, that is the field of algebraic numbers, and that we can replace $\bar{a}$ by an isolated tuple of parameters, but I would like to know more on the different embeddings of G_0 in G , and know the answer to the following question, which is positive for $G = PSL_2(K)$.

Question. *If* G *is a simple algebraic group over an algebraically closed field, are any two embeddings of the prime model of its theory conjugated by an inner automorphism of* G *?*

Corollary 2. *Let* G *be a simple algebraic group over an algebraically closed field* K *. Then every subset of a cartesian power of* G *which is definable in the sense of the field* K *is definable (with parameters) in the sense of the group* G .

Corollary 3 (Borel-Tits for isomorphisms). *Let* σ *be an isomorphism between two simple algebraic groups* G_1 *and* G_2 *over algebraically closed fields* K_1 *and* K_2 *; then* σ *is the composition of an isomorphism induced by an isomorphism between* K_1 *and* K_2 *and an isomorphism definable in* G_2 .

Corollary 4 (Borel-Tits for automorphisms). *Let* σ *be an automorphism of a simple algebraic groups* $G = G(K)$ *over an algebraically closed fields* K *, and* L *be a copy of* K *definable in* G *; let* τ *be the field isomorphism between* L *and* $L' = \tau(L)$ *induced by* σ *; then* $\sigma = \sigma_1 \circ \tau^* \circ \sigma_2$ *where* σ_1 *and* σ_2 *are definable in* G *and* τ^* *is the isomorphism between* $G(L)$ *and* $G(L')$ *induced by* τ .

Proposition 5. *If* G *is a simple algebraic group over an algebraically closed field* K *, a copy of* K *is definable without parameters in* G *(I mean in* G^{eq} *).*

Proof. Easy in characteristisc 0 , because there is only one constructible isomorphism betwen K and L . More subtle in characteristic p : one has to show that the Frobenius automorphism, and its powers, cannot disturb the family of fields defined in the Borel subgroups (it needs the conjugacy of the Borel subgroups). **End**

If L is defined without parameters in G , Corollary 4 implies that an automorphism of G is constructible, that is, definable in G , if and only if it induces a constructible automorphism of L .

Corollary 6. *Let* σ *be an automorphism of a simple algebraic group over an algebraically closed field.*
(i) *In characteristic* 0 *, if* σ *has a finite order,* σ^2 *is constructible; if* (G,σ) *is superstable,* σ *is constructible.*
(ii) *In characteristic* p *, if* σ *has a finite order, it is constructible; if* σ *belongs to a superstable group of automorphisms of* G *, it is constructible.*

Proof. By Artin's Theorem, a non-identical automorphism of finite order of an algebraically closed field is the conjugacy relative to one of its real closed subfields. It is easy to see that a field of finite Morley rank has no non-identical definable automorphism in characteristic 0 , and that any definable group of automorphisms of it is reduced to the identity in characteristic p ; I let you guess who is the guy who extended the results to superstability. **End**

We finally present Corollary 6, and in fact the whole affair, in its true context. We say that an infinite constructible structure S is *autonomous* if it satisfies the conclusion of Corollary 2: everything in S which is constructible, that is definable in the sense of K , is definable in the structure S . Therefore simple algebraic groups are our paradigm of structures of the kind.

A copy of K is definable in S : by elimination of imaginaries, we can assume that S is a constructible subset of a cartesian power of K ; one of the projections must be infinite, providing a copy of K deprived of a finite subset. And everything that we have said on simple groups, except the definability without parameters of the field in characteristic p , is valid for autonomous constructible structures, since the proofs rest on a model theoretic general nonsense. I will try to convince you, in my conclusion, that Corollary 6 is nevertheless valid for them (see the preprint for a full proof).

In characteristic 0 , we can define without parameters in S^{eq} a copy of the base field, but in characteristic p it may happen that, without parameters, we can only define a *multifield,* that is a finite set $(L_1, ...L_n)$ of copies of the base field, with a family of isomorphisms between them. We shall content ourselves to consider the following bifield.

B is the union of two copies $L_1 \cup L_2$ of the base field; the language of B consists in the equivalence relation $x \in L_1 \leftrightarrow y \in L_2$, in the graphs of the two field operations (the ternary relations $x +_1 y = z \vee x +_2 y = z$, $x \times_1 y = z \vee x \times_2 y = z$), and in an automorphism $\tau = (\tau_1,\tau_2)$ of S , where τ_1 is an isomorphism from L_1 to L_2 and τ_2 an isomorphism from L_2 to L_1 , such that $\tau_2(\tau_1(x)) = x^p$; this implies that $\tau_1(\tau_2(y)) = y^p$. One can easily find a construction of B , which is autonomous because, after fixing one parameter, it is nothing but a duplication of the base field.

Suppose that the autonomous constructible structure S interprets B without parameters and consider the hypothesis of Corollary 6; if the automorphism σ^* of B induced by σ fixes the two fields, it must act as the identity on them, and σ is constructible by Corollary 4. If it exchanges the two fields, σ^* is an involution; but this case is impossible, because B has no involutive automorphism, for the reason that the Frobenius automorphism of an algebraically closed field is not the square of a field automorphism (this is due to the fact that the group of automorphisms of a finite field is cyclic and generated by the Frobenius).

In the general case of a bifield, the constructible $\tau_2(\tau_1(x)) = x\hat{}p^n$ is a power, positive or negative, of the Frobenius ; if the exponent n is odd, it has no involutive automorphism ; if n is even, it has exactly one, $\theta = (\theta_1,\theta_2)$ where θ_1 and θ_2 are inverse each of the other ; the quotient of B by the equivalence relation $x = y \vee x = \theta(y)$ is a third field L_3 definable without parameters.