Categoricity of universal covers of Fuchsian groups: the non-arithmetic case
Geometry from the model theorist's point of view

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Covering spaces of varieties arising from Fuchsian groups

Zilber's program: covering spaces

Zilber proposed that central mathematical structures should be canonical. He made this precise as: the infinitary theory should be categorical in uncountable powers.

Today we discuss this problem for coverings of varieties obtained by quotienting hyperbolic space by a Fuchsian group.
Andres Villaveces and I wrote a survey of the work by Zilber, Daw, Harris, Eterovic —Bays, Hart, Pillay, Gavrilovich, Hils; we attempted to explain one particular case: modular curves – the arithmetic case – and sketch the relation with other cases.
Ronnie Nagloo suggested that he and I should study the non-arithmetic case: it might be trivial, easy but enlightening, or completely different.
The second was correct and I sketch it here to give the outline of the model theoretic argument for the 'quasiminimal method'.

Fuchsian groups and commensuration

Definition

1. Two subgroups Γ and Γ' of a group $\tilde{G}$ are said to be *commensurable* if $\Gamma \cap \Gamma'$ is of finite index in each of them.
2. The *commensurator* $G = G_\Gamma = \mathrm{comm}(\Gamma)$ of a subgroup Γ of $\mathrm{PSL}_2(\mathbb{R})$ ($\tilde{G}$) is

$$\{\delta \in \mathrm{PSL}_2(\mathbb{R})\colon \delta\Gamma\delta^{-1} \text{ is commensurable with } \Gamma\}.$$

Definition

A Fuchsian group Γ is a discrete subgroup of $PSL_2(\Re)$.
Γ is arithmetic if has infinite index in its commensurator, which is always $GL_2^{ad}(\mathbb{Q})$.
Γ is non-arithmetic if it has finite index in its commensurator, denoted $G = G_\Gamma$.
$S(\mathcal{C}) = \Gamma \setminus \mathbb{H}$ 'is' a compact Riemann surface and an algebraic variety.

The cover (domain) sort

Just as modules are studied by introducing unary function symbols for the ring elements, the group action is now encoded using the vocabulary τ_G: **unary** function symbols.
Consider the upper half plane $H = \{\gamma \in \mathcal{C} : im(\gamma) > 0\}$ with the group $G = GL^+(\mathbb{Q})/Z(GL^+(\mathbb{Q}))$ (+ means positive determinant) acting on it via fractional linear translations.

Fact: Each $a \in H$ is either special or Hodge generic.

1. Hodge generic: No $g \in G$ fixes a.
2. special: There is a $g_a \in G$ with exactly one fixed point a in H. (The quadratic equation derived from $gz = z$ has two complex conjugate roots.)
   So there are countably many special points and they are each in quadratic extensions of $\mathbb{Q}$.
   Note that for every $g \in G$ either $\forall x f_g(x) \neq x$ or $\exists x f_g(x) = x$ in the theory of $\langle H, f_g : g \in G \rangle$.

The image (field) sort

Definition

For a Fuchsian group Γ, the prototypical structure has S as the variety isomorphic to $\Gamma \setminus \mathbb{H}$.

1. the structure $\langle \mathcal{C}, +, \times; p; S(\mathcal{C}) \rangle$
2. the first order theory

$$\mathrm{Th}(\langle H, \{f_g; g \in G\} \rangle) \cup \mathrm{Th}(\langle \mathcal{C}, +, \times; S(\mathcal{C}) \rangle)$$

The vocabulary τ_f contains a set $\mathcal{R}$ of relations for each Zariski closed set of $S(\mathcal{C})$ defined over F and constants for the elements of $L = E^{ab}(\Sigma)$ (the defining field of S and coordinates of the special points).

$\tau = \tau_G \cup \tau_f \cup \{q\}$.

The *mark* of D on S^ω: The $Z_{\mathbf{g}}$

Notation

With G as fixed above, as each subgroup of G acts on H, we can define (mathematically) for any finite sequence of the form $\mathbf{g} = \langle e, g_2, \ldots, g_n \rangle$ from G (by convention, $g_1 = e$),

1. $\Gamma_{\mathbf{g}} = \Gamma \cap g_2^{-1}\Gamma g_2 \ldots \cap g_n^{-1}\Gamma g_n$.
2. With $p : \mathbb{H} \to S(\mathcal{C})$., $Z_{\mathbf{g}}$ is the set $\{(p(x), p(g_2 x), \ldots, p(g_n x)) \in S(\mathcal{C})^n : x \in \mathbb{H}\}$.

In the finite index case $\hat{\mathbf{g}}$ is a finite list of coset representatives of G/Γ.

Each $Z_{\mathbf{g}}$ is clearly τ-definable. It is crucial that each $Z_{\mathbf{g}} \subset S(CC)^n$ is τ_f-variety definable over F. Discussed below.

Connecting the sorts

The connection axioms

1. FIRST ORDER:
   a. axioms from previous two slides
   b. modularity axioms:
      1. $(\forall x \in H)(q(g_1 x), \ldots q(g_n(x)) \in Z_{\mathbf{g}}$
      2. $(\forall z \in Z_{\mathbf{g}}(\exists x \in H)q((g_1 x), \ldots q(g_n(x)) = z$
   c. 'Special Point axioms' SP_g: For each $g \in G$ that fixes a unique point in D :

$$\forall x, y \in D[(g(x) = x \wedge g(y) = y) \Rightarrow x = y]$$

2. INFINITARY standard fiber axiom **SF**:

$$\forall x \forall y [j(x) = j(y) \rightarrow \bigvee_{g \in G} x = f_g(y)]$$

3. INFINITARY Φ_∞ asserts the (trivial) geometry on $(H, g \in G)$ and the non-modular geometry of $S(F)$ have infinite dimension.

Results

Theorem

If Γ

1. [DH17]/ [Ete22] is arithmetic, or
2. (here) non-arithmetic

the $L_{\omega_1,\omega}$ theory of the prototypical model is categorical in all infinite cardinalities.

We sketch here the proof of the second case which avoids much of the geometric complexity of the earlier result.

Finite Index in Commensurator

Group Theory

Fix $G = \mathrm{comm}(\Gamma)$ with $[G : \Gamma] = n < \omega$. The following lemmas are undoubtedly well-known.

Lemma

1. If $[\Gamma : H] < \omega$ then $\mathrm{comm}_\Gamma(H) = \Gamma$.
2. If $G = \mathrm{comm}_{\tilde{G}}(\Gamma)$, $[G : \Gamma] < \omega$, and $[\Gamma : H] < \omega$ then $\mathrm{comm}_{\tilde{G}}(H) = \mathrm{comm}_G(H) = G$.
   Then Γ and H are commensurable in $\tilde{G}$.

Lemma

If $[G : \Gamma] < \omega$, there are only finitely many subgroups of Γ of the form $\Gamma_\delta = \delta\Gamma\delta^{-1} \cap \Gamma$ with $\delta \in \tilde{G}$.

Denote the intersection of the Γ_δ by $\check{G}$.

Crucially τ-definable variety

Fact: For a covering map p associated with a Fuchsian group, the following is well-known and can be proved via double cosets.
For every coset $\delta\Gamma$ of G/Γ, there is a polynomial $\phi_\delta \in \mathcal{C}[x, y]$ such that $\phi_\delta(p(a), p(b)) = 0$ implies $b = \delta' a$ for some $\delta' \in \delta\Gamma$.

Lemma

The set $Z_{\check{\mathbf{g}}}$ is definable in τ^-, the field language, (wolog name the parameters).

Proof.

The formula defining $\{(p(x), p(g_2 x), \ldots, p(g_n x)) \in S(\mathcal{C})^n : x \in \mathbb{H}\}$ is

$$\bigwedge_{0<i\leq n} \phi_{\check{g}_i}(x_0, x_i).$$

$\square$

[Ete22, Lemma 3.31] and [DH17, Lemma 2.6] prove the definability of $Z_{\mathbf{g}}$ in [Ete22, Lemma 3.31] and [DH17, Lemma 2.6] use the Mumford-Tate module and refer to [Mil17].

Completeness and Quasi-minimality

Small model/ Complete sentence / homogeneity over the emptyset

Definition

A τ-structure *M* is $L_{\omega_1,\omega}$-small if *M* realizes only countably many $L_{\omega_1,\omega}$-types (over the empty set).

An $L_{\omega_1,\omega}$-sentence ϕ is *complete* if for every $L_{\omega_1,\omega}$-sentence ψ,

$\phi \Rightarrow \psi$ or $\phi \Rightarrow \neg\psi$

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If ϕ is small then there is a complete sentence ψ_ϕ such that:

$\phi \wedge \psi_\phi$ has a countable model.

So ψ_ϕ implies ϕ.
All models of a complete sentence are $L_{\infty,\omega}$ -equivalent.

The three properties are (non-trivially) equivalent for an $\aleph_1$-categorical sentence of $L_{\omega_1,\omega}$.

Definition: Almost Quasi-minimal excellent classes

Let $\boldsymbol{K}$ be a class of τ-structures such that $M \in \boldsymbol{K}$ admits a closure relation cl_M mapping $X \subseteq M$ to $\mathrm{cl}_M(X) \subseteq M$ that satisfies:

1. **Basic Conditions**
   1. Each cl_M formally τ-defines a pregeometry on M.
   2. For each $X \subseteq M$, $\mathrm{cl}_M(X) \in \boldsymbol{K}$.
   3. countable closure property (ccp): If $|X| \leq \aleph_0$ then $|\mathrm{cl}(X)| \leq \aleph_0$.
2. **Homogeneity**
   1. A class $\boldsymbol{K}$ of models has $\aleph_0$**-homogeneity over** $\emptyset$ if the models of $\boldsymbol{K}$ are pairwise qf-back and forth equivalent. i.e. $\boldsymbol{K}$ is axiomatized by a complete sentence of $L_{\omega_1,\omega}$.
   2. A class $\boldsymbol{K}$ of models has $\aleph_0$**-homogeneity over models** if for any $G \in \boldsymbol{K}$ with G empty or $G \in \boldsymbol{K}$ countable, any $M, M' \in \boldsymbol{K}$ with $G \leq M$, $G \leq M'$, M is qf-back and forth equivalent with M' over G.
3. $\boldsymbol{K}$ is an *(almost) quasiminimal excellent geometry* if the universe of any model $M \in \boldsymbol{K}$ is in $\mathrm{cl}(X)$ for any maximal cl-independent set $X \subseteq M$. [BHH+14].

Why almost?

Zilber's original notion of quasiminimal was:
Every definable set is countable or cocountable.
Obviously this fails for the two-sorted case.

Just as *almost strongly minimal structure* is $\mathrm{acl}(X)$ for X strongly minimal, we write *almost quasi-minimal structure* for $\mathrm{cl}(X)$ for X quasminimal.

Toward Proving quasiminimality: I

We noted above:

Lemma

The following are equivalent:

1. $\boldsymbol{K}$ satisfies $\aleph_0$-homogeneity over the emptyset.
2. $\boldsymbol{K}$ is defined by a complete sentence of $L_{\omega_1,\omega}$ ($\aleph_0$-homogeniety over $\emptyset$).

So our first goal is to show $\boldsymbol{K} = \mathrm{mod}(T_{SF}^{\infty})$ satisfies $\aleph_0$-homogeneity over the emptyset.

Proof for finite index case

The quantifier-free back and forth scheme: I

$E^{ab}(\Sigma)$ is the extension (F_0 in [Ete22, p 19]) of the reflex field E^{ab} obtained by adding the coordinates of the ($\leq \aleph_0$) special points and closing to a field. All the points are named in τ_f.

Description:The back and forth scheme

Fix two models $\mathbf{q} = \langle D, S(F), q\rangle$ and $\mathbf{q}' = \langle D', S(F'), q'\rangle$ of $T(\mathbf{p})$.

Consider the set of finitely generated substructures of $\mathbf{q}$ and $\mathbf{q}'$ and collection I of partial isomorphisms such that: For each $f \in I$, dom f and rg f are each finitely generated over $E^{ab}(\Sigma)$.

A typical member f of the system I has dom $f = U = U_D \cup U_S$. Since U is finitely generated, U_D consists of the G-orbits of a finite number of $x \in D$; U_S is $S(L_U)$ where L_U is the field generated by $E^{ab}(\Sigma)$ (since the elements of $E^{ab}(\Sigma)$ are named) and the coordinates of the $q(x)$ for $x \in U_D$.

Note that the additional points obtained by including the orbits determine only finitely many new field elements since q is constant on each orbit, so the field remains finitely generated.

The quantifier-free back and forth : II

Define a similar subsystem for $\mathbf{q}'$, labeling by putting primes on corresponding objects. We showed every point of D is either special and so named in the vocabulary, or Hodge generic. Thus, we can ignore the special points in building the back and forth system.

Suppose f is an isomorphism between $U \subseteq \mathbf{q}$ and $U' \subseteq \mathbf{q}'$. Then f restricts to a G-equivariant (elements in the same orbit have the same image) injection of U_D into $U_{D'}$ and a bijection between $S(L_U) \subseteq S(F)$ and $S(L_{U'}) \subseteq S(F')$ that fixes $E^{ab}(\Sigma)$ and that is induced by an embedding σ of L into $S(F')$, that fixes $L = E^{ab}(\Sigma)$.

Our task is to prove this scheme has the back-and-forth property. We need a preliminary Lemma.

Recall that $\check{\mathbf{g}}$ lists representatives for the coset of G/Γ.

Reduction to the field type: I

Lemma

Assume $[G:\Gamma] < \omega$. Let $\mathbf{p} = \langle D, S, p\rangle$ be a model of $T_{SF}(p)$. If $d \in D$ is Hodge generic and $d, e \notin U_D$ and
$r = \mathrm{qtp}_{\tau_f}(q(\check{\mathbf{g}}(d))/L_U) = \mathrm{qtp}_{\tau_f}(q(\check{\mathbf{g}}(e))/L_U)$ then
$\mathrm{qtp}_\tau(d/U) = \mathrm{qtp}_\tau(e/U)$.

Proof: We show that there is a unique quantifier-free type over U of an element of D that restricts to $r = \mathrm{qtp}_{\tau_f}(q(\check{\mathbf{g}}(d))/L_U)$. Since $d \notin U_D$, the quantifier free types in τ_G that can be satisfied by d are

i) $\{x \neq b : b \in U_D\}$
ii) $\{x \neq gx : g \in G - \Gamma\}$.

Case i) For any $b \in U_D$, $x_0 \neq p(b) \in r = \mathrm{qtp}_{\tau_f}(q(\check{\mathbf{g}}(d))/L_U)$
As U_D contains entire orbits, this implies i).

Reduction to the field type: II

Case ii) $\{x \neq gx : g \in G - \Gamma\}$.
Any domain element, a, is Hodge generic if and only if it is fixed by no $g \in G$, so for any g,

$$(q(x), q(gx)) \not\models \bigvee_{g_i \in \check{\mathbf{g}}} \phi_{g_i}$$

is in $\mathrm{qtp}_{\tau_f}(q(\check{\mathbf{g}}(d))/L_{U_D})$.
Since $e \in D_M - U_D$ and $q(\check{\mathbf{g}}(e))$ realizes $\mathrm{qtp}_{\tau_f}(q(\check{\mathbf{g}}(d))/L_U)$, e must be Hodge generic and so e realizes $\mathrm{tp}_{qf}(d/U)$ as required.

$\check{\mathbf{g}}$ vs $\langle g_0, g_1, \ldots \rangle$

In the arithmetic case, the argument for the Hodge generic on the last slide $q(\check{\mathbf{g}}(a))$ is replaced by the infinite sequences $\langle q(\mathbf{g}^n(a)) : \lg(\boldsymbol{a}) = n < \omega \rangle$.

Strong geometric theorems involving galois representation are required to identify a finite subsequence that suffices as $\check{\mathbf{g}}$ does in the argument here.

Definition: *s*-isolation

For $\boldsymbol{a} \in \mathrm{cl}(C)$, $\mathrm{tp}(\boldsymbol{a}/C)$ is *s*-isolated (by C_0) if there is a finite $C_0 \subset C$ such that for any $\boldsymbol{a}' \in \mathrm{cl}(C)$, if $\mathrm{tp}(\boldsymbol{a}'/C_0) = \mathrm{tp}(\boldsymbol{a}/C_0)$, then $\mathrm{tp}(\boldsymbol{a}'/C) = \mathrm{tp}(\boldsymbol{a}/C)$.

Claim: r is s-isolated

Definition

We write $Z_{\check{\mathbf{g}}}(F)$ for the points in $S(F)^n$ satisfying (the formula defining) $Z_{\check{\mathbf{g}}}$ (analogously for F'). For any $d \in D$, let $W_{U,d}$ be the minimal variety containing $q(\check{g}(d))$ and defined over U_L.
By $Mod^1_{\mathbf{g}}$, $W_{U,d}$ is contained in $Z_{\check{\mathbf{g}}}(F) \subseteq S(F)^n$.

Proof of Claim:
r is the field type of $q(\check{g}(d))$ over L_U; it is generated by $W_{U,d}$ and the assertion that $q(\check{g}(d))$ realizes the generic type over L_U within $W_{U,d}$.
This assertion is expressed by a type over the finite set of generators of L_U.

The back and forth exists

Theorem

Suppose that $\mathbf{q}$ and $\mathbf{q}'$ satisfy $T^{\infty}_{SF}(p)$. Then, the *qf*-system defined above is a quantifier-free *back and forth* system between them. Hence, the first order theory $T(\mathbf{p})$ admits elimination of quantifiers ([Pil02, Proposition 29] and is complete.
There is then a unique countable model M of $T^{\infty}_{SF}(p)$ so $T^{\infty}_{SF}(p)$ is axiomatized by a complete sentence of $L_{\omega_1,\omega}$, the Scott sentence of M.

Proof: Our back and forth scheme has an isomorphism f between $U \subseteq \mathbf{q}$ and $U' \subseteq \mathbf{q}'$. To prove the existence of a back and forth, it suffices to show we can extend the f from U by a single point as the 'back' is symmetric. Note f restricts to a G-equivariant injection of U_D into $U_{D'}$ and a bijection between $S(L_U) \subseteq S(F)$ and $S(L_{U'}) \subseteq S(F')$ that fixes $E^{ab}(\Sigma)$.

Easy cases

For $x \in \mathbf{q} - U$, we must find $x' \in U'$ so that $f \cup \{\langle x, x' \rangle\}$ generates an isomorphism between the structures generated by $U \cup \{x\}$ and $U' \cup \{x'\}$.
If $x \in S(F)$, $x = q(d)$ for some $d \in D = D(F)$ so we need only to consider the case $d \in D$.
If $d \in U_D$, d' exists as U'_D is closed under action by G. Since the coordinates of a special point d are in $E^{ab}(\Sigma)$, whose points are all named in $F \cap F'$, for a special point d, d' must equal d.

The difficult case of the back and forth argument

The difficult case is when $d \in (D - U_D)$ is Hodge generic.
Since the formulas in $\mathrm{qtp}_{\tau_f}(\check{\mathbf{g}}(d)/U)$ are over L_U and f is an isomorphism between L_U and $L_{U'}$, $f(r)$ is a consistent type over the finite set of generators of $L_{U'}$.

Suppose the coordinates of $\mathrm{qtp}_{\tau_f}(\check{\mathbf{g}}(d)/U)$ are algebraic over L_U. Since $Mod^1_{\check{\mathbf{g}}}$ is true in $\mathbf{q}$ we saw above that $f(W_{\check{\mathbf{g}}}(F)) \subseteq Z_{\check{\mathbf{g}}}(F')$. But since $Z_{\check{\mathbf{g}}}(F))$ is defined over $E^{ab}(\Sigma)$, $f(Z_{\check{\mathbf{g}}}(F)) = Z_{\check{\mathbf{g}}}(F')$ and $f(W_{\check{\mathbf{g}}}(F)) \subseteq Z_{\check{\mathbf{g}}}(F')$.

Since $\mathbf{q}' \models Mod^2_{\check{\mathbf{g}}}$, , the realizations of r have pre-images in $D_{F'}$ and we finish this case.

Suppose some coordinates of $\mathrm{qtp}_{\tau_f}(\check{\mathbf{g}}(d)/U)$ are transcendental over L_U. Since F' has infinite transcendence degree, we find realizations of those coordinates in F' and repeat the previous argument over the field containing them.

Obstruction in the arithmetic case

In the arithmetic there are infinitely many finite index subgroups of Γ with the form $\Gamma_\delta = \Gamma^\delta \cap \Gamma$. These are indexed by the cosets of $\Gamma \in G$. They, in turn, represents orbits in D of the groups Γ/Γ_δ. If there are infinitely many such, they lead, by taking a finite decreasing sequence of such orbits (indexed by $\mathbf{g}$), to the minimal subvariety containing $p(\mathbf{g}(d))$. Since the space of orbits is profinite, it is uncountable if infinite.
For the arithmetic case this violates FIC1 (consequence of Serre's finite index theorem).
For us, there are only finitely many possibilities.

The arithmetic analogy

The argument for the Hodge generic in the difficult case is the key distinction from the arithmetic case ([DH17, Ete22, BV24]). In those papers, the sequence $\mathbf{g}$ is infinite (all members of G) and strong geometric are required to identify a finite subsequence that suffices as $\check{\mathbf{g}}$ does in the argument here.

QUESTION: Would rearranging the order of presentations in those papers by doing the Galois representation argument first ([DH17, 5.1], [Ete22, 7.1], [BV24, §4.1] avoid the detour through restricting to ω-saturated structures?

Toward Proving quasiminimality II

Definition

For $\boldsymbol{a} \in \mathrm{cl}(C)$, $\mathrm{tp}(\boldsymbol{a}/C)$ is s-isolated (by C_0) if there is a finite $C_0 \subset C$ such that for any $\boldsymbol{a}' \in \mathrm{cl}(C)$, if $\mathrm{tp}(\boldsymbol{a}'/C_0) = \mathrm{tp}(\boldsymbol{a}/C_0)$, then $\mathrm{tp}(\boldsymbol{a}'/C) = \mathrm{tp}(\boldsymbol{a}/C)$.

Lemma [BHH$^+$14, Cor 5.5]

If $\boldsymbol{K}$ satisfies all conditions for 'excellence' except $\aleph_0$-homogeneity over models, the following are equivalent.

1. If $\mathcal{M} \models^{\infty}_{SF}$ and M is a countable closed (in the pre-geometry) subset of $\mathcal{M}$, then $\boldsymbol{a}, \mathbf{b} \in \mathcal{M}$ and $\mathbf{b} \in \mathrm{cl}(M\boldsymbol{a})$ then $\mathrm{tp}(\mathbf{b}/M\boldsymbol{a})$ is s-isolated.
2. $\boldsymbol{K}$ satisfies $\aleph_0$-homogeneity over models.

s-isolation and Categoricity

Lemma

[BHH[+]14, Prop 5.2] Let $\mathcal{M} \models^{\infty}_{SF}$ and M be a countable closed (in the pre-geometry) subset of $\mathcal{M}$. If $\boldsymbol{a}, \mathbf{b} \in \mathcal{M}$ and $\mathbf{b} \in \mathrm{cl}(M\boldsymbol{a})$ then $\mathrm{tp}(\mathbf{b}/M\boldsymbol{a})$ is s-isolated.

Proof.

The argument for s-isolation for types over models arising from finitely generated fields works the same way for arbitrary countable closed set. Namely, it suffices to show that for $d \in D_{\mathcal{M}} \cap \mathrm{cl}(M\boldsymbol{a})$, $\mathrm{qtp}_{\tau_f}(\check{\mathbf{g}}(d))$ is s-solated. Let $\theta(\mathbf{x}, \mathbf{h})$ define the minimal variety satisfied by $\check{\mathbf{g}}(d)$ over $\mathrm{cl}(M\boldsymbol{a})$. As $\mathbf{h} \in \mathrm{acl}(M\boldsymbol{a}) \subset \mathrm{cl}(M\boldsymbol{a})$ we finish.

□

Main Result

Theorem

For $[G : \Gamma] < \omega$, the models of T_{SF}^{∞} form an almost quasi-minimal excellent class; so T_{SF}^{∞} is categorical in all infinite cardinalities.

Proof.

The direction $c \to a$ of [BHH^{+}14, Corollary 5.3] states exactly, since we have $\aleph_0$-homogeneity over $\emptyset$, that T_{SF}^{∞} satisfies $\aleph_0$-homogenety over models and thus is almost quasi-minimal excellent.

□

Contrast with Arithmetic Case

The argument for modular Shimura curves can be done with this pattern.
The essential distinction is that the reduction to a single **g** requires a consequence of the Serre finite index theorem. See [BV24, Lemma 4.4.9]

Definition

First Finite Index Condition (FIC1) The first finite index condition is satisfied by a modular curve $p\colon \mathbb{H} \to S(\mathcal{C})$ if:
For any non-special points $x_1, \ldots x_m \in \mathbb{H}$ in distinct G-orbits and for any field L containing the field over $E^{ab}(\Sigma)$ along with the coordinates of the $p(x_i)$, the image of the induced homomorphism $\rho : \mathrm{Gal}(\overline{L}/L) \to \overline{\Gamma}^m$ has finite index in $\overline{\Gamma}^m$.

A second finite index condition requires even more geometric information that we avoid.

Zilber's Universal Cover Legacy

The following chart organizes the papers which are the major source for this study. It also provides a keyword describing the main method or context used, and the section of this paper where issues around the specific variant are explained.

	topic	paper	method/context	section
1	Complex exponentiation	[Zil05]	quasiminimality	§**??**
2	cov mult group	[Zil06]	quasiminimality	§**??**
3		[BZ11]	quasiminimality	
4	*j*-function	[Har14]	background	§**??**
5	Modular/Shimura Curves	[DH17]	quasiminimality	§**??**
6	Modular/Shimura Curves	[DZ22]	quasiminimality	
6a	non-arithmetic fuchsian groups	[BN24]	quasiminimality	
7	Abelian Varieties	[BGH14]	finite Morley rank groups	§**??**
8	Abelian Varieties	[BHP20]	fmr & notop	§**??**
9	Shimura *varieties*	[Ete22]	notop	§**??**
10	Smooth varieties	[Zil22]	o-quasiminimality	§**??**

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