Complex field with quasiminimal structure

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Zilber's quasiminimality conjecture (1997)

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In 2018 Bays and Kirby introduced a method requiring only Countable Closure Property and Exponential-Algebraic Closedness.

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Open Question (Koiran)

Let $\mathcal{F}$ be the set of all unary entire complex functions.
Is $(\mathbb{C}; +, \cdot, \mathcal{F})$ quasiminimal?

However, there are functions $f : \mathbb{C}^2 \to \mathbb{C}^2$ whose image is open but not dense, making $(\mathbb{C}; +, \cdot, f)$ non-quasiminimal.

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- $\mathbb{C}_{\mathrm{AE}} = (\mathbb{C}; +, \cdot, \Gamma_{\mathrm{AE}})$, where

$$\Gamma_{\mathrm{AE}} = \{(x, y) \in \mathbb{C}^2 : y = e^{x+q+2\pi i r} \text{ for some } q, r \in \mathbb{Q}\},$$

  also by proving analogues of CCP and EAC (Kirby, 2019).

Correspondence between two elliptic curves

Let E_1 and E_2 be two complex elliptic curves with exponential maps $\exp_1$ and $\exp_2$, and $\Gamma = \{(\exp_1(z), \exp_2(z)) : z \in \mathbb{C}\} \subseteq E_1 \times E_2$.

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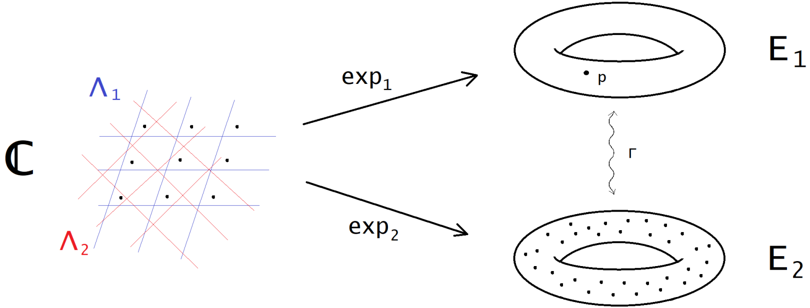

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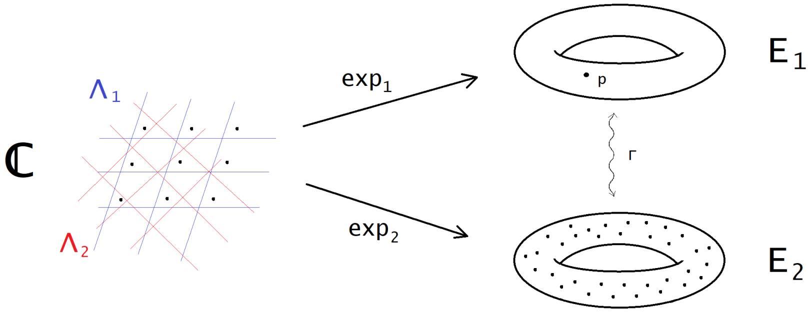

Theorem (AD)

The structure $\mathbb{C}_{\text{corr}} = (\mathbb{C}; +, \cdot, \Gamma)$ is quasiminimal.

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- The proof consists of showing analogues of CCP and EAC, similar to the proof for $\mathbb{C}_{\mathrm{AE}}$.
- In the general case a fiber of Γ is countable and dense, serving as an analogue of $\mathbb{Q} + 2\pi i\mathbb{Q}$.

Theory of a generic function

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  - Existential closedness: for any algebraic variety V over $\bar{c}$ satisfying certain conditions there are distinct $a_1, \ldots, a_n$ with $a_i \notin \bar{c}$ such that $(a_1, \ldots, a_n, g(a_1), \ldots, g(a_n)) \in V$.

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- Zilber showed that T_{gf} is consistent and complete, ω-stable and has Quantifier Elimination in a certain language extension.
- Wilkie and Koiran provided an explicit family of examples of entire generic functions on $\mathbb{C}$, inspired by Liouville numbers.

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- Quasiminimality follows from CCP and Quantifier Elimination.
- Categoricity follows from the uniqueness of prime models over sets.

Summary

1. Zilber's quasiminimality conjecture
2. Replacing exp
3. Examples of quasiminimal structure
4. Correspondence between two elliptic curves
5. Theory of a generic function

Thank you!