On Zilber's Trichotomy

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- Zilber's solution of the non-finite axiomatizability of totally categorical theories.

Zilber's proof uses, crucially, a weak trichotomy involving the geometry of strongly minimal sets.

A quick reminder

Strongly minimal sets and their geometries

- A definable set S in a saturated structure $\mathcal{M}$ is strongly minimal if every definable set subset of S is either finite or co-finite.
- A strongly minimal structure satisfies the Exchange Principle $(a \in \mathrm{acl}(Ab) \setminus \mathrm{acl}(A) \Rightarrow b \in \mathrm{acl}(Aa)))$.
- So $(S, \mathrm{acl}(\cdot))$ is a pre geometry.
- A pre-geometry is disintegrated if $\mathrm{cl}(A) = \bigcup_{a \in A} \mathrm{cl}(a)$.
- A pre-geometry is locally modular if it is either trivial or the pre geometry of a linear space (affine, linear or projective) over a division ring.

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Continuations include: Non-finite axiomizability of totally categorical theories, the theory of smoothly approximable structures, locally modular regular types...

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This conjecture would imply:

The algebraicity conjecture

A simple non-abelian group of finite Morley Rank is an algebraic group.

Hrushovski's factory for counter examples

Hrushovski's constructions

- There is a continuum of non-locally modular geometries of *ab initio* strongly minimal sets where no group is interpretable.
- (Essentially) any two strongly minimal structure can be fused into a strongly minimal structure having both as reducts.

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New questions –the road not taken

1. Pillay conjectured: For all n there exists a strongly minimal set whose geometry is n-ample non-$(n+1)$-ample.
2. How to classify the hoard of geometries of *ab initio* strongly minimal structures?
3. Is there any hope of an (intrinsic) classification of the geometries of "fused" strongly minimal theories.

But not all is lost

Special cases

The trichotomy is true in many natural settings:

1. It is true in DCF_0, in ACFA, in SCF.
2. Original proofs built on the Trichotomy in Zariski Geometries.

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- In ACFA the results go beyond the stable context, and even go through Robinson theories.

The o-minimal Trichotomy Theorem completely detaches the trichotomy from the stable setting.

The Restricted Trichotomy Conjecture

Definition

Let $\mathcal{M}$ be a structure. An $\mathcal{M}$-relic is a structure $\mathcal{N}$ whose universe is definable in $\mathcal{M}^{eq}$ and whose atomic sets are $\mathcal{M}$-definable,

Some restricted conjectures:

The Trichotomy was conjectured to hold of *strongly minimal* relics of the following structures:

- ACF (Zilber, ∼1985)
- O-minimal (Peterzil, 2005).
- ACVF (Kowalski-Randriambololona, 2016)

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3. Consider a strongly minimal family $\mathcal{C}$ of curves through (a, a). Show that tangency of a curve in $\mathcal{C}$ to a curve in $\mathcal{C} \circ \mathcal{C}$ at the point (a, a) is definable (up to small enough errors).

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Zilber's strategy is the only game in town. Definability of tangency is the key problem.

Some History

Preliminary results on ACF-relics

1. The conjecture holds for certain polynomial reducts of ACF (Martin).
2. The conjecture holds for $(\mathbb{C}, +, X)$ (Marker-Pillay).
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A major breakthrough:

Theorem (Castle)

The conjecture holds of ACF_0*-relics.*

From algebraic geometry to topology

Why "A major breakthrough"?

- Rabinovich's result as well as H.-Sustretov use algebro-geometric intersection-theoretic tools.
- Such tools are ill equipped to deal with non-geometric objects.
- Proofs are hard, and don't generalise.

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Castle's main idea

- Multiple intersections can be detected locally and topologically.
- In ACF_0 tangency and multiple intersections are the same.
- Non-locally modular ACF_0-relics can define just enough of the Euclidean topology to detect tangency.

O-minimal influences

Castle's main technical result is a (hard) adaptation of

Theorem (Eleftheriou-H.-Peterzil)

Let $\mathcal{M}$ be a strongly minimal non-locally modular 2-dimensional o-minimal relic expanding a group. Let $S \subseteq M^2$ be a plane curve. Then the (o-minimal) frontier of S is finite and $\mathcal{M}$-algebraic over $[S]$.

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- Castle's proof is, at its core, topological with few (but crucial) touches of analytic geometry.
- Nothing of the sort exists in ACF_p.

Another topological setting

Two similar results

1. The proof of Marker-Pillay for $(\mathbb{C}, +, X)$ follows very closely Zilber's strategy, using the the argument principle.
2. The exact same proof works for $(\mathcal{K}, +, X)$ for $\mathcal{K}$ an algebraically closed valued field of char. 0.

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An idea

Prove Zilber's Trichotomy for ACVF_p to conclude the result for ACF_p.

An axiomatic approach to relics

The slogan

1. Zilber's Trichotomy tends to be true in tame topological structures.
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1. Partition known proofs of the Trichotomy into self contained segments.
2. A first such segment is the *detection of closures* argument.
3. Formulate an axiomatic framework covering each segment of the proof.

Hausdorff Geometric Structures with enough open maps

Definition

An $\aleph_1$-saturated *geometric* structure $\mathcal{K}$ is a Hausdorff Geometirc Structure if:

1. It is equipped with a Hausdorff topology τ (that need not be definable).
2. If $X \subseteq K^n$ is definable and $a \in \overline{\mathrm{Fr}(X)}$ then $\dim(a/[X]) \leq \dim(X)$.
3. For any definable X and countable parameter set B the set of B-generic points of X is dense.
4. Finite correspondences are generically locally graphs of homeomorphisms.

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- A technical condition aimed to assure, roughly, that non-transverse intersections can only occur for a good reason.
- Transverse intersections are identified by openness of the projection of the family of intersections onto the parameter space.
- verifying “enough open maps” directly seems hard.
- It is straightforward, e.g., if the structure admits a well behaved (abstract) notion of smoothness and an associated (abstract) notion of differential manifolds satisfying an appropriate version of Sard’s Lemma.

Examples

Many natural examples

- $(\mathbb{C}, +, \cdot)$ with the Euclidean topology.
- O-minimal expansions of fields.
- 1-h-minimal fields.
- Topological ez-fields (including ACVF).

In all examples, except the first, the topology is definable.

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Theorem (Castle-H.-Ye)

If $(\mathcal{K}, \tau)$ is a Hausdorff Geometric Structure with enough open maps then $\mathcal{K}$ detects multiple intersections.

Relics of wrong dimension

Another of Castle's crucial observations:

Fact

1. *In* ACF_0 *the ramification locus of projections with finite fibres between smooth manifolds is empty or of pure co-dimension 1.*

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2. ACF_0*-relics detect multiple intersections, so they detect ramification, so they are 1-dimensional.*

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2. ACF_0*-relics detect multiple intersections, so they detect ramification, so they are 1-dimensional.*

We axiomatise (a weaker version of) purity of ramification to get:

Theorem (Castle-H.-Ye)

1. *If* $\mathcal{K}$ *is a HGS with enough open maps and purity of ramification then any strongly minimal non-locally modular* $\mathcal{K}$*-relic is 1-dimensional.*
2. *ACVF has purity of ramification.*

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Theorem

let $\mathcal{M}$ by o-minimal.

1. *Strongly minimal non-locally modular $\mathcal{M}$-relics are internal to an $\mathcal{M}$-definable o-minimal field.*

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Theorem

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1. *Strongly minimal non-locally modular $\mathcal{M}$-relics are internal to an $\mathcal{M}$-definable o-minimal field.*
2. *1-dimensional strongly minimal $\mathcal{M}$-relics are locally modular.*

Summary of known results

Theorem (Castle-H.-Ye, H.-Onshuus-Pinzon)

1. *Zilber's Trichotomy holds of definable ACVF-relics.*
2. *Consequently (using EOI in* ACF*) Zilber's Trichotomy holds of* ACF*-relics.*
3. *In residue char. 0, the Trichotomy holds of all relics.*

Theorem

1. *Zilber's Trichotomy holds (vacuously) of definable relics of T-convex expansions of o-minimal fields of dimension* $\neq 2$.
2. *Zilber's Trichotomy holds of 2-dimensional strongly minimal group relics of o-minimal structures.*