# An ultraproduct approach to rigged Hilbert spaces

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Geometry from the model theorist's point of view
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# Quantum mechanics crash course I

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In particular, the position of a particle is described by a function $\psi$ of a position variable $\bar{x} \in \mathbb{R}^n$ such that $|\psi(\bar{x})|^2$ encodes probabilities:

$$\int_{\mathbb{R}^n} |\psi(\bar{x})|^2 d\bar{x} = 1 \quad \text{and} \quad \mathbb{P}(\bar{x} \in E) = \int_E |\psi(\bar{x})|^2 d\bar{x}.$$

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Note, that Dirac did not think of his deltas as being actual vectors.

# Time evolution

A quantum system evolves over time according to the Schrödinger equation

$$i\hbar \frac{d}{dt}\psi = H\psi$$

where $H$ is the *Hamiltonian* of the system, the self-adjoint operator corresponding to the energy of the system (depends on system).

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Knowing the Hamiltonian, one can – in theory – calculate later states of the system

$$\psi_t = K^t \psi_0$$

where

$$K^t = e^{-itH/\hbar}.$$

Here we assume $H$ is time-independent.

# The Feynman propagator

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With eigenstates $|x\rangle$ and $|y\rangle$

$$K(x, y, t) = \langle y|K^t|x\rangle.$$

Without eigenvectors, we look at the *kernel* of the integral representation of $K^t$ (if it exists)

$$(K^t\psi)(y) = \int_{\mathbb{R}} K(x, y, t)\psi(x)dx$$

# A quest for eigenvectors

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- find structures where Dirac deltas exist (and behave like vectors)
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Next, we'll look at a distribution approach to Dirac deltas.

# Rigged Hilbert space

A rigged Hilbert space consists of a Hilbert space $H$ and a subspace $\Phi$ of "test functions", with a finer norm on $\Phi$.
One then has

$$\Phi \subset H \subset \Phi^*$$

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$(\Phi, H, \Phi^*)$ is also called a *Gelfand triple*.

### Example

Let $H = L_2(R)$, and let $\Phi$ the set of *Schwartz functions*, i.e., infinitely differentiable functions $\varphi : \mathbb{R} \to \mathbb{C}$ whose derivatives tend to 0 at infinity faster than any power of $\frac{1}{|x|}$.
Then for every $x \in \mathbb{R}$ the functional $f_x\varphi = \varphi(x)$ acts as a Dirac delta function corresponding to the value $x$.

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## Definition

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## Theorem

*If $A$ is a bounded self-adjoint operator on $H$ with a cyclic vector, then there exists a measure $\mu$ on $\sigma(A)$ and a unitary operator $U : H \to L_2(\mathbb{R}, d\mu)$ such that*

$$UAU^{-1}\varphi(x) = x\varphi(x).$$

Note: in the non-cyclic case we get an orthogonal sum of such $L_2$ spaces.

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- build a spectral representation for a bounded, self-adjoint operator with a cyclic vector,
- build the spectral measure as an ultraproduct of scaled counting measures in finite dimensional spaces,
- look closer at the ultraproduct, to find something resembling a rigged Hilbert space

# Self-adjoint via unitary

## Theorem (Stone)

*There is a one-one correspondence between unitary operators $U$ and self-adjoint operators $A$ with spectrum $\subseteq [0,1]$ and not having 0 in the point spectrum, given by $U = e^{2\pi i A}$.*

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We assume $\sigma(A) \subset [-\frac{\pi}{2}, \frac{\pi}{2}]$, and consider $U = e^{iA}$.

Now $\varphi$ is cyclic also for $U$, in the sense that the vectors $U^k\varphi$, $k \in \mathbb{Z}$, span a dense set of $H$.

# Finding finite dimensional approximations of $(H, U)$

Consider the spanning vectors

$$\cdots \quad U^{-n}\varphi \quad U^{-n+1}\varphi \quad \cdots \quad U^{-1}\varphi \quad \varphi \quad U\varphi \quad \cdots \quad U^{n-1}\varphi \quad U^{n}\varphi \quad \cdots$$

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Define finite dimensional spaces

$$H_N = \overline{\text{span}\{U^k\varphi : -N \leq k \leq N\}}$$

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# Approximations of $U$

Let

$$H_N^- = \overline{\text{span}\{U^k \varphi : -N \leq k < N\}}$$

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Let $U_N$ be built from

- $U$ on $H_N^-$
- a unitary operator mapping $W^+$ to $W^-$

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In each $H_N$

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$$\varphi = \sum_{k=0}^{2N} \xi_N(k) u_N(k)$$

where each $\xi_N(k)$ is a non-negative real, and $\sum_{k=0}^{2N} \xi_N(k)^2 = 1$.

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where each $\xi_N(k)$ is a non-negative real, and $\sum_{k=0}^{2N} \xi_N(k)^2 = 1$.

Note: The spaces $H_N$ extend each other, but the bases do not.

# A first glimpse of the ultraproduct model

Let $\mathcal{U}$ be a non-principal ultrafilter on $\omega$.

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As $\varphi$ is cyclic, there is a natural embedding of $H$ into $H^m$:

## Definition

For $P(X, Y) \in \mathbb{C}[X, Y]$, let $P(U_N, U_N^{-1})$ be natural interpretation as an operator on $H_N$, e.g.,

$$X^2Y(U_N, U_N^{-1}) = U_N \circ U_N \circ U_N^{-1} = U_N.$$

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Then $P(U_N, U_N^{-1})(\varphi)$ makes sense in almost all $H_N$, and thus we can define $G^m : H \to H^m$ by

$$G^m(P(U_N, U_N^{-1})(\varphi)) = (P(U_N, U_N^{-1})(\varphi))_{N<\omega}/\mathcal{U}.$$

# Spectral measure in $H_N$

Remember: in each $H_N$, $\varphi = \sum_{k=0}^{2N} \xi_N(k) u_N(k)$

**Definition**

For each $N < \omega$, define a measure $\mu_N$ for subsets $X \subset \mathbb{C}$:

$$\mu_N(X) = \sum_{k<2N, \lambda_N(k) \in X} \xi_N(k)^2$$

Note that for all $X \subset \mathbb{C}$, $\mu_N(X) \leq 1$, as $\|\varphi\| = 1$.

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2. consider a set of *nice* vertical $(I_r)$ and horizontal $(J_r)$ lines for which for all $\delta > 0$ there is $\varepsilon > 0$ such that their "$\varepsilon$-thickenings" $I_r^\varepsilon$ and $J_R^\varepsilon$ have small "measure"

$$\mu^n(I_r^\varepsilon) < \delta, \quad \mu^n(J_r^\varepsilon) < \delta$$

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$$\mu^n(I_r^\varepsilon) < \delta, \quad \mu^n(J_r^\varepsilon) < \delta$$
3. define an outer measure based on the $\mu^n$-value of boxes bounded by nice lines
$$\mu^*(Y) = \inf \left\{ \sum_{k=0}^{\infty} \mu^n(X_k) \mid X_k \text{ a } \textit{nice box}, Y \subseteq \bigcup_{k<\omega} X_k \right\}$$

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4. by Caratheodory's construction, find a $\sigma$-algebra of sets for which $\mu^*$ is a measure

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Let $D(S)$ be the subspace of $C(S)$ that consists of functions

$$f_P(\lambda) = P(\lambda, \bar{\lambda})$$

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where $P \in \mathbb{C}[X, Y]$ and $\bar{\lambda}$ is the complex conjugate of $\lambda$.
Define $U_D$ and $U_D^*$ by

$$U_D(f_P) = f_{XP} \quad \text{and} \quad U_D^*(f_P) = f_{YP}.$$

Note that $U_D(f_P)(\lambda) = \lambda f_P(\lambda)$, and $U_D^*(f_P)(\lambda) = \bar{\lambda} f_P(\lambda)$.

**Theorem**

1. *The measure* $\mu^*$ *is zero outside the spectrum of* $U$.
2. *There is an isometry mapping* $L_2(S, \mu^*)$ *to* $H$, *and (the extension of)* $U_D$ *to* $U$. *(We find it going via* $H^m$.*)*
3. *We can transfer the measure* $\mu^*$ *from the unit circle to the real line to get (from the isometry above) an isomorphism between* $L_2(\sigma(A), \mu)$ *and* $(H, A)$.

# Dissecting the ultraproduct construction

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   this last part can be split in two: mod out the zeros, then the (other) infinitesimals

# Classical ultraproduct of Hilbert spaces

If we take the classical ultraproduct of the $H_N$ spaces, $H^u = \prod_N H_N/\mathcal{U}$ we get a vector space over $\mathbb{C}^u = \mathbb{C}^\omega/\mathcal{U}$.

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For $q = (q_N)_{N<\omega}/\mathcal{U}$ in $\mathbb{C}^u$, let $|q|^u = (|q_N|)_{N<\omega}/\mathcal{U}$.

If $X$ is the range of this "absolute value", then $\mathbb{R}^u = X \cup \{-r \mid r \in X\}$ is a real closed field containing the reals. In particular, it is linearly ordered and can be used to compare the "norm" $\|v\|^u$ of vectors $v \in H^u$ and rational numbers.

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We den define $\langle u|v \rangle$ to be

- $q \in \mathbb{C}$ such that $\langle u|v \rangle$ is infinitesimally close to $q$, if such a $q$ exists
- $\infty$ otherwise

# Other norms

We look two functions from $H_N$ to $\mathbb{C}$.

## Definition ("$L_1$-norm")

Let $X \subseteq S \subseteq \mathbb{C}$ be a closed set. Define

$$\| \sum_{k<2N+1} a_n u_N(n) \|_\infty^X = \sup\{\xi_N(n)^{-1}|a_n| \mid \lambda_N(n) \in X\},$$

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Considering ultraproducts of these, we get – on part of $H^u$ – seminorms that can be used for the "metric steps" of the ultraproduct construction: throwing out "bad" elements and moding out infinitesimals.

# Distributions

- For $f \in C(S)$, $\|F^u(f)\|_0 \leq \|F^u(f)\|_2 \leq \|F^u(f)\|_\infty$, where $F^u$ is a particular embedding of $C(S)$ into $H^u$ (used also to find the isometry between $L_2(S, \mu^*)$ and $H$).
- Distributions can be found as vectors in the space $H^{m0}$, the metric ultraproduct built from the 0-norm.
- Under extra assumptions, the distributions can be used to calculate Feynman propagators in the physics style.