Abstract

Zilber's trichotomy conjecture has had an extraordinary influence on the course taken by model theory in the last half-century. I will recall the conjecture and its background, the totally categorical case, counterexamples, Zariski geometries, the trichotomy for differential algebra, for difference equations, for o-minimal structures; and speculate about a possible new chapter with globally valued fields.

Zilber's trichotomy conjecture

Ehud Hrushovski

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Minimal sets

Let X be a set defined by some formulas in a structure M. We say X is *minimal* if it is infinite, but for any definable D in any elementary extension, $X \cap D$ or $X \setminus D$ are finite.
An *algebraic function* (definable in M) from Y to X is a definable subset of $Y \times X$, whose projection to Y is onto and $\leq m$-to-one.
The relation: $x \in \mathrm{acl}(x_1, \ldots, x_n)$ has the properties of a pre-geometry; it thus gives rise to a dimension theory on definable subsets of X^n.

Two key definitions

A minimal set has trivial geometry. No definable families of irreducible subsets of X^n, other than ones like $X^{n-1} \times \{b\}$. Equivalently, the algebraic closure geometry is trivial.

X is locally modular. if there are no high-dimensional families of irreducible subsets of X^n; in fact a family of k-dimensional subsets of X^n has dimension $\leq n-k$.

In the locally modular, nontrivial case, one can one can prove existence of an abelian group A, isogenous to X, that is pure with respect to the Abelian structure. (Every definable subset of A^n is a finite Boolean combination of cosets of definable subgroups.)

Zilber's trichotomy conjecture

We say *Zilber's conjecture holds* for a theory T if every minimal X is of trivial geometry, is locally modular, or is isogenous to an algebraically closed field (interpretable appropriately in T). An additional part of the conjecture is that the field is *pure*; it has no additional structure.

The finite model property

Theorem 1 (Zilber)**.** . *Let T be a theory categorical in every infinite power. There exist* finite *models M_n, such that any sentence of T holds in all but finitely many M_n.*

The proof required:

Theorem 2 (Zilber)**.** *Let T be a nontrivial strongly minimal structure; assume it is $\aleph_0$-categorical. Then D is isogenous (over a parameter) to a pure vector space over a finite field.*

Why does Theorem 1 require Theorem 2?
Within a definable, minimal D, take $A_n = \mathrm{acl}(a_1, \ldots, a_n)$.
Then $Th(A_n) \to T$ (ex.)
But there may be a type q over $A (= A_{n_0})$, that is *almost orthogonal* to D; it implies a complete type over D. So not realized in any A_n.
Zilber defines an *envelope* of A to be a maximal E containing A such that $D(acl(E)) = D(acl(A))$.

q will be realized in *some* envelope of A_0. If we knew that envelopes are unique up to isomorphism, then q is realized in every envelope; from this, easy to see $Th(E(A_n)) \to T$.
Trouble occurs if there exists another r, almost orthogonal to p and to q, but $p \in acl(A(q, r))$.
Can reduce to the case that $q \otimes p$ is realized in $A(q \otimes q)$. So we have a *plane* $P = acl(q^3) = acl(q \otimes p^2) = acl(q \otimes r^2)$. Let Q be the set of realizations of q in P.
p defines a parallelism class of lines in Q (in the Euclidean sense!).
So Boris needed to prove, precisely, the parallel postulate in the plane $acl(q^3) = acl(q, p^2) = acl(r, p^2)$.
This follows from Theorem 2.

Disproof and proof

How do intersections break up into components?
Coming to terms with the need for a topology. Closed definable sets.
Rabinobich's thesis.
H.-Zilber, Zariski geometries.

Differential fields.

Char. $p > 0$ equivalent to separably closed fields with $[K : K^p] = p$.
Axiomatizability of existentially closed differential fields (A. Robinson, Blum)
Likewise separably closed fields in char. $p > 0$ (with $[K : K^p] = p$.) (Ershov, Delon).

Minimal sets exist (stability). So does a reduction theory of finite-rank types to minimal sets, (Shelah, orthogonality and domination, semi-minimal types, socle,....;).

Zilber's conjecture holds in this form:
up to isogeny:
nontrivial minimal sets $\longleftrightarrow$ simple semi-abelian varieties.
(The map $\longleftarrow$ takes A to the differential Zariski closure of the kernel of the semi-abelian logarithm map).
locally modular $\longleftrightarrow$ simple abelian varieties.
For G_m, $x'/x = 0$ defines a pure field, the field of constants.

Starting from the Leibniz law alone.

Trichotomy for difference equations

w. Zoé Chatzidakis.
The model companion is not stable.
Two approaches: stability at the qf level, and simplicity.

up to isogeny:
nontrivial minimal sets $\longleftrightarrow$ simple dynamical semi-abelian varieties (all locally modular), and "dynamical pseudo-finite fields".
The conjecture survives the generalization from stability to simplicity.

O-minimal, C-minimal

O-minimal: Petezil-Starchenko 1998. (2/3)
2002 w. Pillay, linear groups.
(+1/3) 2009. (assuming complex analyticity).
Klinger, Bakker, Brunebarbe, Tsimerman 2022.

C-minimal: Delon-Maalouf- 2014. C-minimal, 1/3(between trivial and linear).

II. The theory GVF

arxiv.org/abs/2409.04570
Globally valued fields: foundations
Itaï Ben Yaacov, Pablo Destic, Ehud Hrushovski, Michał Szachniewicz

For later reference: Bilu, Chambert-Loir, Szpiro-Ullmo-Zhang, Zhang, Autissier, Gubler, Yamaki, Gao-Habbegger, Cantat-Gao-Habegger-Xie. Xie-Yuan, Chen-Moriwaki. (Many cases of Bogomolov conjecture and equidistribution).

Theorem 3 (Szachniewicz). *$\bar{Q}$ is existentially closed as a globally valued field.*

Michał Szachniewicz, *existential closedness of $\bar{Q}$ as a globally valued field via Arakelov geometry*, ArXiv 2306.06275
A similar theorem for the function-field type GVF $\overline{k(t)}$ is proved in notes by Ben Yaacov-H, [GVF2], Arxiv.

Let's look at some soft corollaries.

1. Effectiveness given finiteness

Let X be variety over $\mathbb{Q}$, and consider algebraic solutions.
If one bounds the degree but not the height, *even if one knows that the number of solutions of degree $\leq d$ is finite*, there is still no known algorithm to find them effectively.(Is this *provable)*?On the other hand if the height but not the degree is bounded:

Corollary 1. *Assume as known that X has only finitely many solutions in $\bar{Q}$ of height $< h_0$. Then these solutions can be computed effectively.*

Proof. If $\bar{Q} \leq L$ is any GVF extension, all solutions of X of height $< h_0$ must lie in $\bar{Q}$. Otherwise, one can recursively find infinitely many solutions in $\bar{Q}$: having found $a_1, \ldots, a_n$, find a new solution b in some GVF extension; by the e.c. theorem, a solution b' exists in $\bar{Q}$ with $b' \notin \{a_1, \ldots, a_n\}$ and $ht(b')$ as close as we like to $ht(b)$, hence $< h_0$.
In particular the number of solutions is bounded in any GVF. Search for algebraic solutions $s_1, \ldots, s_n$ and a formal proof from the GVF axioms that these are all the elements of X of height at most h_0. This is guaranteed to terminate, and in particular identify $s_1, \ldots, s_n$.

□

Corollary **??** also works with additional qf GVF constraints, e.g. canonical height bounds, local conditions.

Corollary 2. *Every GVF is a subfield (with induced height structure) of an ultrapower of* $\overline{k(t)}$ *for some* k*, or of* $\overline{\mathbb{Q}}$*.*