# Existential closedness beyond quasiminimality: $j$ and derivatives

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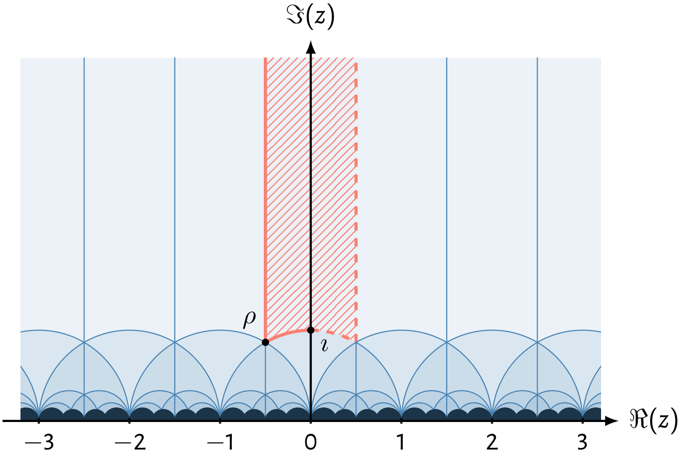
Supported by EPSRC EP/T018461/1

Geometry from the model theorist's point of view, Oxford 10–13 September 2024

From joint work with V. Aslanyan, S. Eterović

`https://arxiv.org/abs/2312.09974`

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# Quasiminimality

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**Definition.** Let Q be the quantifier with semantics 'there exist uncountably many'.

**Definition.** $M$ is quasiminimal if for every $\varphi(x)$ with parameters, $M \models \neg\, \mathrm{Q}\, x.\varphi(x)$ or $M \models \neg\, \mathrm{Q}\, x.\neg\varphi(x)$.

**Examples.** Ignoring countable structures, of course.

- (Strongly) minimal structures.
- $(\omega_1 \times \mathbb{Q}; <_{\mathrm{lex}})$, $(\mathbb{C}; \mathbb{Z}, +, \times)$; pseudoexponentiation $(\mathbb{B}; +, \times, \exp)$ (Zilber '05).
- Universal cover of $(\mathbb{C}^\times; \times)$ (Zilber '02–'06) and many follow ups (abelian and Shimura).
- Previous talks: raising to complex powers (Gallinaro-Kirby 2024), correspondences between elliptic curves, generic unary holomorphic function (Dmitrieva).

**Fact.** If $M$ is quasiminimal excellent,[1] then it is a model of an uncountably categorical $L_{\omega_1,\omega}(\mathrm{Q})$-sentence.

All of the above examples except one (which one?) are quasiminimal excellent.

**Conjecture** (Zilber '97–'05). $\mathbb{C}_{\exp} := (\mathbb{C}, +, \times, \exp)$ is quasiminimal excellent.

**Theorem** (Zilber '05+Bays-Kirby '18). If $\mathbb{C}_{\exp}$ is *exponentially-algebraically closed*, then $\mathbb{C}_{\exp}$ is q.m. excellent. Moreover, $\mathrm{cl}_{\mathrm{Q}}(A) := \{b : \varphi(b, A), \neg\, \mathrm{Q}\, x.\varphi(x, A) \text{ for some } \varphi\} = \mathrm{ecl}(A)$ (ecl on next slide).

[1] 'Closed substructures with closed embeddings generate an unbounded quasiminimal AEC' (see Vasey '18).



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# Exponential-algebraic closedness and existential closedness

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**Definition** (Macintyre '96?). $b \in \mathrm{ecl}(A)$ if there are an algebraic variety $V$ of dimension $n$ and a tuple $\bar{c}$ of length $n-1$ such that $(b\bar{c}, E(b\bar{c}))$ is a transversal intersection of $V$ with $\Gamma^n_{\exp}$.[2]

The above definition generalises to any abstract exponential field $K$ equipped a homomorphism $E : (K,+) \to (K^\times, \times)$: just replace 'transversal' with a suitable determinant being non-zero.

**Fact.** ecl is a closure operator (Macintyre) and a pregeometry (Wilkie for $\mathbb{R}$, Kirby '10).

**Theorem** (Ax '70, heavily rephrased). Let $V \subseteq \mathbb{C}^n \times (\mathbb{C}^\times)^n$ algebraic and of dimension $n$. If C is a *positive* dimensional component of $V \cap \Gamma^n_{\exp}$, then $C \subseteq (L + \bar{a}) \times (\mathbb{C}^\times)^n$ for some $\mathbb{Q}$-linear space $L$.

Such $C$ is an *unlikely* intersection: its dimension is bigger than it should. Compare with:

**EAC.** For all $V \subseteq K^n \times (K^\times)^n$ algebraic and of dimension $n$, if [conditions], then there is $(\bar{a}, \exp(\bar{a})) \in V$.

Thus exponential-algebraic closedness asks that *likely* intersections, i.e. the ones of dimension 0, exist.[3]

Equivalently, that $\mathbb{C}_{\exp}$ is existentially closed among fields with '$\dim_{\mathrm{ecl}}$-preserving embeddings over $\mathrm{ecl}(\varnothing)$'.[4]

The 'existential closedness' question can be formulated for other functions, regardless of quasiminimality.

[2] Here $\Gamma^n_{\exp}$ is the graph of $(x_1, \dots, x_n) \mapsto (e^{x_1}, \dots, e^{x_n})$.
[3] That is, $2n - \dim(V) - \dim(\Gamma_{\exp}) = 2n - n - n = 0$.
[4] This would actually be 'generic EAC'; EAC also says something about $\mathrm{ecl}(\varnothing)$. See Kirby '10, Bays–Kirby '18.



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# State of the art on existential closedness

- $p(z, e^z) = 0$ has infinitely many solutions unless $p \in \mathbb{C}[X] \cdot Y^{\mathbb{N}}$ (see Marker '06; this is $n = 1$).

Given $V \subseteq \mathbb{C}^n \times (\mathbb{C}^\times)^n$, $V \cap \Gamma^n_{\exp}$ is nonempty when:

- $V = L \times W$ 'free rotund' for $K$-affine $L \subseteq \mathbb{C}^n$, $W \subseteq (\mathbb{C}^\times)^n$ (Zilber '03-'12 for $K \subseteq \mathbb{R}$ 'generic'; Gallinaro '23).
- The projection of $V$ to $\mathbb{C}^n$ has dimension $n$ (Brownawell-Masser '17, D'Aquino-Fornasiero-Terzo '18).
- The projection of $V$ to $\mathbb{C}^n$ has dimension 1 and is 'free' (M-Masser '24). In particular, $n = 2$ is solved.
- $V = W_1 \times W_2$ with $W_1 \subseteq \mathbb{C}^n$ (Gallinaro).

Given $A$ semiabelian of dimension $g$, $V \subseteq \mathbb{C}^g \times A$ 'free rotund', $V \cap \Gamma^n_{\exp_A}$ is nonempty when:

- $A$ abelian, $V = L \times W$ for $K$-linear $L \subseteq \mathbb{C}^g$ (Gallinaro '24).
- $A$ (split semi-)abelian: the projection of $V$ to $\mathbb{C}^g$ has dimension $g$ (Aslanyan-Kirby-M '23).
- $A = E_1 \times E_2$, $V = \Delta \times W$ (where $\Delta$ is the diagonal; Dmitrieva).

Given $S$ Shimura variety with uniformizer $q : \Omega \subseteq \mathbb{C}^N \to S$, $V \subseteq \mathbb{C}^N \times S$ 'broad Hodge-generic':

- The projection of $V$ to $\mathbb{C}^N$ has dimension $N$ (Eterović-Herrero for $S = \mathbb{C}^N$, $q = j^N$; Eterović-Zhao).
- $V = L \times W$ with $L \subseteq \mathbb{C}^N$ 'totally geodesic' (Gallinaro for $S = \mathbb{C}^N$, $q = j^N$; Eterović-Zhao).

Also differential/blurred e.c. (Kirby, Aslanyan-Eterović-Kirby); $\Gamma$ function (Eterović-Padgett).



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# The $j$-function

The Klein $j$-invariant (or just $j$-function) is the unique holomorphic function $j : \mathbb{H} \to \mathbb{C}$ such that:

- $j(\tau) = j(\tau') \iff \tau' = \frac{a\tau+b}{c\tau+d}$ for some $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ (that is, $ad - bc = 1$);
- $j(\imath) = 1728$ and $j(z) \sim e^{-2\pi \imath z}$ for $\Im(z) \to +\infty$.

$j$ parametrizes elliptic curves up to isomorphism. It is differentially algebraic:

$$j''' = \frac{3}{2}\frac{(j'')^2}{j'} - \frac{j^2 - 1968j + 2654208}{2j^2(j-1728)^2}(j')^3.$$

Let $\mathbf{J} := (j, j', j'')$, $\mathbf{Y} := (Y_0, Y_1, Y_2)$.

**Theorem** (Aslanyan-Eterović-M). Let $p, q \in \mathbb{C}[X, \mathbf{Y}] \setminus \mathbb{C}$ be coprime. Then there is $\tau \in \mathbb{H}$ such that $p(\tau, \mathbf{J}(\tau)) = 0 \neq q(\tau, \mathbf{J}(\tau))$ unless $p \in \mathbb{C}[X] \cdot Y_0^{\mathbb{N}} \cdot (Y_0 - 1728)^{\mathbb{N}} \cdot Y_1^{\mathbb{N}}$.

We have $j'(\tau) = 0 \Leftrightarrow \tau \in \mathrm{SL}_2(\mathbb{Z}) \cdot \{\imath, \rho\} \Leftrightarrow j(\tau) = 0 \vee j(\tau) = 1728$.
Thus $p(\tau, \mathbf{J}(\tau)) = 0 \Leftrightarrow q(\tau, \mathbf{J}(\tau)) = 0$ for $p = Y_1, q = Y_0(Y_0 - 1728)$.

**Corollary.** $j''(\tau) = 0$ has solutions that are not in the $\mathrm{SL}_2(\mathbb{Z})$-orbit of $\rho$.



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# The Existential Closedness conjecture (for ZP enthusiasts)



**Theorem** (Pila–Tsimerman 2014, heavily rephrased). Let $V \subseteq \mathbb{C}^{4n}$ algebraic and of dimension $3n$. If $C$ is a positive dimensional component of $V \cap \Gamma_{\mathbf{J}}^n$,[5] then $C \subseteq \{z_N = a\}$ or $C \subseteq \{\tau_N = \gamma\tau_m\}$ for some $\gamma \in \mathrm{GL}_2^+(\mathbb{Q})$.

Just as before: unlikely intersections between $V$ and the graph of $\mathbf{J}$ come from $\mathrm{GL}_2^+(\mathbb{Q})$ or constant coordinates.

The Existential Closedness conjecture for $\mathbf{J}$ should assert that likely intersections exist:

**Existential closedness?** ($\mathbf{J}$). Let $V \subseteq \mathbb{C}^{4n}$ algebraic. If [conditions], then there is $(\bar{\tau}, \mathbf{J}(\bar{\tau})) \in V \cap \Gamma_{\mathbf{J}}^n$.

The omitted 'conditions' guarantee that the intersections remain likely even after transformations that preserve $\Gamma_{\mathbf{J}}$ (such as projecting to $(\tau_1, \mathbf{J}(\tau_1))$ which maps $\Gamma_{\mathbf{J}}^n$ to $\Gamma_{\mathbf{J}}^1$).

Our theorem is a very special case: here $V = \{p(x, y_0, y_1, y_2) = 0\} \subseteq \mathbb{C}^4$, and we prove that $V \cap \Gamma_{\mathbf{J}}^1$ is Zariski dense in $V$, unless $V$ is of a special form.

The new challenge is that $j', j''$ are *not* invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$:

$$j\left(\frac{a\tau+b}{c\tau+d}\right) = j(\tau), \quad j'\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 j'(\tau), \quad j''\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^4 j''(\tau) + 2c(c\tau+d)^3 j'(\tau).$$

[5] $\Gamma_{\mathbf{J}}^n = \{(\tau_1, \ldots, \tau_n, \mathbf{J}(\tau_1), \ldots, \mathbf{J}(\tau_n)) : \tau_i \in \mathbb{H}\}$.



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# An example: solving $j''(\tau) + j'(\tau) = 0$

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Let us sketch a strategy for the following system, where $q \in \mathbb{C}[X, \mathbf{Y}]$ is not divisible by $Y_1 + Y_2$:

A $j''(\tau) + j'(\tau) = 0$ (that is $p = Y_1 + Y_2$) and $q(\tau, \mathbf{J}(\tau)) \neq 0$.

We apply a suitable $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. Here $\gamma = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ (that is, $\tau \mapsto -\frac{1}{\tau}$) is enough.

$$j''\left(-\frac{1}{\tau}\right) + j'\left(-\frac{1}{\tau}\right) = \tau^4 \underbrace{j''(\tau)}_{p_4(\mathbf{J}(\tau))} + 2\tau^3 j'(\tau) + \tau^2 \underbrace{j'(\tau)}_{p_2(\mathbf{J}(\tau))}.$$

**Root finding** (pole version). Let $f_0, \ldots, f_\ell : \mathbb{H} \to \mathbb{C}$ be meromorphic, with $f_i(\tau + 1) = f_i(\tau)$, $f_\ell \not\equiv 0$, and

$$F(\tau) := \tau^\ell f_\ell(\tau) + \cdots + f_0(\tau).$$

If $\frac{f_k}{f_\ell}$ has a pole at $\tau \in \mathbb{H}$, then there are $\tau_m$ for large $m \in \mathbb{Z}$ such that $F(\tau_m + m) = 0$, and $\tau_m \to \tau$ for $|m| \to \infty$.

**Corollary.** Let $\tau$ with $j''(\tau) = 0$, $j'(\tau) \neq 0$. Then there is $\tau_m$ with $j''(\tau_m) = j''(\tau_m + m) = 0$ and $q(\tau_m + m, \mathbf{J}(\tau_m + m)) \neq 0$.

General strategy: apply a 'generic' $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, reduce to 'simpler' equation. Does it work?



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# Solving $j''(\tau) = 0$

We reduced $j'' + j' = 0 \neq q$ to $j'' = 0 \neq j'$. Let us try to solve:

B $j''(\tau) = 0$ (that is $p = Y_1$) and $q(\tau, \mathbf{J}(\tau)) \neq 0$ (where $Y_2$ does not divide $q$).

Unfortunately, $p = Y_2$ is '**J**-homogeneous': $p(Y_0, W^2Y_1, W^4Y_2) = W^4Y_2 = W^4p$.

**J**-homogeneous polynomials have the following funny transform. Fix $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ with $c \neq 0$:

$$p(\gamma\tau, \mathbf{J}(\gamma\tau)) = p(\tau, \mathbf{J}(\tau))c^N \underbrace{\left(\left(\tau + \frac{d}{c}\right)^N + \dots\right)}_{h\left(z+\frac{d}{c}, \mathbf{J}(\tau)\right)}; \text{ e.g. } j''(\gamma\tau) = j''(\tau)c^4\left(\left(\tau + \tfrac{d}{c}\right)^4 + 2c\left(\tau + \tfrac{d}{c}\right)^3 \frac{j'(\tau)}{j''(\tau)}\right)$$

The (A) strategy now fails: the leading coefficient is again $j''$! And yet, by contradiction (and very ineffectively):

- suppose we *cannot* apply the Root finding (even the 'cusp version', omitted in these slides);
- we deduce that the only zeroes are (conjugates of) $\rho$ and $i$ (with help from zero estimates);
- via the Open Mapping Theorem: $h(\tau + u, \mathbf{J}(\tau))$ does not vanish for $\tau \in \mathbb{H}$, $u \in \mathbb{R}$;
- get bound $|h(\tau + u, \mathbf{J}(\tau))| \succeq |z + u|^N$ for $\tau$ in the standard fundamental domain;
- but then $1/h$ extends holomorphically to $\mathbb{C}$ and vanishes on $\mathbb{R}$, contradiction! □

 General strategy: this works for every **J**-homogeneous $p$ containing $Y_2$.

# The other $\mathbf{J}$-homogeneous case: zero estimates



The actual general strategy is the following (with finer details not explained):

1. given $p(X, \mathbf{Y})$ *irreducible*, apply a 'generic' $\gamma \in \mathrm{SL}_2(\mathbb{Z})$; formally, let the generic transform of $p$ be

$$\Gamma(p)(Z, C, W, \mathbf{Y}) := p\left(Z, Y_0, W^2 Y_1, W^4 Y_2 + 2CW^3 Y_1\right) = p_N(Z, C, \mathbf{Y}) W^N + \cdots + p_{n_0}(Z, C, \mathbf{Y}) W^{n_0}$$

   so that $(c\tau + d)^{N+\ell} p(\gamma\tau, \mathbf{J}(\gamma\tau)) = \Gamma(p)(\gamma\tau, c, c\tau + d, \mathbf{J}(\tau))$; $p_N$ is always $\mathbf{J}$-homogeneous;
2. if $p_N$ contains $Y_2$: can solve $p_N(\tau, \mathbf{J}(\tau)) = 0 \neq p_k(\tau, \mathbf{J}(\tau))$ for (some) $p_k$; apply Root finding;
3. if $p_N$ does *not* contain $Y_2$: see below.

Fixed some $q \in \mathbb{C}[X, \mathbf{Y}]$, we have the following zero estimates for generic[6] $\gamma \in \mathrm{SL}_2(\mathbb{Z})$:

- for $j'(\tau_0) \neq 0$: $q(\tau, \mathbf{J}(\tau))$ vanishes at $\gamma\tau_0$ with multiplicity $s = \max\{s : (Y_0 - j(\tau_0))^s \text{ divides } q\}$;
- for $j'(\tau_0) = 0$, $q$ does not contain $Y_2$: $q(\tau, \mathbf{J}(\tau))$ vanishes at $\gamma\tau_0$ with multiplicity [explicit, but omitted];
- $q(\tau, \mathbf{J}(\tau))$ has 'exponential growth $e$' at the cusp of $\gamma\mathbb{F}$, where $e = \deg_T q(X, TY_0, TY_1, TY_2)$.

Take $\frac{p_{n_0}(\tau, c, \mathbf{J}(\tau))}{p_N(\tau, c, \mathbf{J}(\tau))}$: if it has no pole, the multiplicity of the numerator always beats the denominator; no exponential growth is similar. Summing up the zero estimates at a generic $\gamma$, we find $\frac{7}{6} \leq 1$ (!!). **Thanks!**

[6] Meaning outside of some proper Zariski closed subset.

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