Definable quotient spaces, functional transcendence, and algebraicity [1]

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[1] In honor of Boris Zilber on the occasion of his 75th birthday.

Origins: Zilber-Pink conjecture

The Zilber-Pink conjectures predict that for S a special variety and $X \subseteq S$ an irreducible algebraic subvariety which is not contained in any proper special subvariety, then

$$X \cap \bigcup_{S' \subseteq S \text{ special }, \dim S' + \dim X < dim S} X \cap S'$$

is not Zariski dense in X.

- R. Pink makes this precise using the formalism of mixed Shimura varieties.
- B. Zilber's original (precise) formulation takes the form of the Conjecture on Intersection with Tori in which S is a power of the multiplicative group. With iterations of the theory of analytic covers the meaning of "special subvariety" is extended.
- Key ingredients in the proofs of weak forms of Zilber-Pink are Ax-Schanuel-type theorems.

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Aims

- I will not talk about Zilber-Pink not its variants nor complementary likely intersection theorems.
- Instead, I will focus on the definable quotient space formalism which can serve as a general, precise way to study special varieties that come from homogeneous spaces.
- I will go into some detail about a technical algebraization lemma that plays a key role in strengthening Ax-Schanuel theorems and in some applications to Hodge theory.

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Forms of Ax-Schanuel

If $U \subseteq \mathbb{C}^k$ is a connected domain in complex k-space, and $f = (f_1, \ldots, f_n) : U \to \mathbb{C}^n$ is an analytic map from U to $\mathbb{C}^n$, then $\operatorname{tr.deg}_{\mathbb{C}} \mathbb{C}(f_1, \ldots, f_n, \exp(f_1), \ldots, \exp(f_n)) \geq n + \operatorname{rk}(df)$ unless there is some nonzero vector $(q_1, \ldots, q_n) \in \mathbb{Q}^n$ for which $\sum_{i=1}^n q_i f_i$ is constant.

Equivalently, the dimension of $\overline{(f, \exp(f))(U)}^{\text{Zar}}$ is at least $n + \operatorname{rk}(df)$ unless $\exp(f)(U)$ is contained in a translate of a proper algebraic subgroup of $(\mathbb{C}^\times)^n$.

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Definable quotient spaces

Fix an o-minimal expansion $\mathscr{R}$ of the ordered field of real numbers.

A definable quotient space is given by the data of

- a definable group G,
- a definable compact subgroup $M \leq G$ of G,
- a discrete subgroup $\Gamma \leq G$ of G, and
- $\mathscr{F} \subseteq D := G/M$ a definable fundamental set for the action of Γ on D

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We write $S_{\Gamma,G,M;\mathscr{F}}$ both for the quotient space $\Gamma\backslash D = \Gamma\backslash G/M$ regarded as a definable, real analytic space where the definable structure comes from $\mathscr{F}$, and for the data $(G, M, \Gamma, \mathscr{F})$ giving this space.

$\pi_\Gamma : D \to S_{\Gamma,G,M;\mathscr{F}}$ is the quotient map.

Morphisms of definable quotient spaces

A morphism of definable quotient spaces $f : S_{\Gamma_1, G_1, M_1; \mathscr{F}_1} \to S_{\Gamma_2, G_2, M_2; \mathscr{F}_2}$ is given by a definable map of groups $\varphi : G_1 \to G_2$ and an element $a \in G_2$ for which

- $\varphi(M_1) \leq M_2$,
- $\varphi(\Gamma_1) \leq a^{-1}\Gamma_2 a$, and
- there is a finite set $\Xi \subseteq \Gamma_2$ with $a\overline{\varphi}(\mathscr{F}_1) \subseteq \bigcup_{\xi \in \Xi} \xi \mathscr{F}_2$, where $\overline{\varphi} : D_1 \to D_2$ is the map induced on the quotient space.

Example:exponential strip

- $G = \mathbb{R}^2$ (regarded as $\mathbb{C}$)
- $M = 0$ (so $D = \mathbb{C}$)
- $\Gamma = \{0\} \times 2\pi\mathbb{Z}$
- $\mathscr{F} = \{(x, y) : 0 \leq y < 2\pi\}$

We might regard $\pi : D \to S_{\Gamma,G,M;\mathscr{F}}$ as a an $\mathbb{R}_{\text{alg}}$ version of the complex exponential function.

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Example: modular curves

$Y_0(N) = S_{\Gamma,G,M;\mathscr{F}}$ where

- $G = \mathrm{PSL}_2(\mathbb{R})$,
- $\Gamma = \Gamma(N) := \ker[\mathrm{PSL}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{Z}/n\mathbb{Z})]$,
- $M = \mathrm{SO}_2(\mathbb{R})$,
- $G/M \cong D = \mathfrak{h}$,
- When $N = 1$, we take the standard fundamental domain

$$\mathscr{F} = \{\tau \in \mathscr{F} : |\operatorname{Re}(\tau)| \leq 1 \ \& \ |\tau| \geq 1\}$$

More generally, it is possible to find a finite set $\Xi \subseteq \mathrm{PSL}_2(\mathbb{Z})$ of coset representatives of $\Gamma_0(N)$ so that $\mathscr{F}' := \bigcup_{\xi \in \Xi} \xi\mathscr{F}$ is a fundamental domain for the action of $\Gamma_0(N)$.

Definable complex quotient spaces

A definable complex quotient space is a definable quotient space $S_{\Gamma,G,M;\mathscr{F}}$ together with the data of

- a complex algebraic group G,
- an algebraic subgroup $\mathrm{B} \leq \mathrm{G}$, and
- a definable embedding $G \hookrightarrow \mathrm{G}(\mathbb{C})$

for which

- $M = \mathrm{B}(\mathbb{C}) \cap G$, and
- $D = G/M \subseteq (\mathrm{G}/\mathrm{B})(\mathbb{C}) =: \check{D}(\mathbb{C})$ is an open domain in the complex points of the algebraic variety $\check{D}$.

A morphism $f : S_{\Gamma_1,G_1,M_1;\mathscr{F}_1} \to S_{\Gamma_2,G_2,M_2;\mathscr{F}_2}$ of definable complex quotient spaces is a morphism of definable quotient spaces for which the definable map of groups is given by a map of algebraic groups $\varphi : \mathrm{G}_1 \to \mathrm{G}_2$ for which $\varphi(\mathrm{B}_1) \leq \mathrm{B}_2$.

Example: complex tori as definable complex quotient spaces

Let $\Lambda \leq \mathbb{C}^g$ be a lattice with basis $\lambda_1, \ldots, \lambda_{2g}$. We regard the complex torus $T = \mathbb{C}^d/\Lambda$ as a definable complex quotient space by taking

- $\mathsf{G} = \mathbb{G}_a^{2g}$,
- $\mathsf{B} := \ker[\psi_\Lambda : \mathsf{G} \to \mathbb{G}_a^g]$ where $\psi_\Lambda(z_1, \ldots, z_{2g}) := \sum_{i=1}^{2g} z_i \lambda_i$,
- $\Gamma := \mathbb{Z}^{2g}$, and
- $\mathscr{F} := [0, 1)^{2g}$.

Special subvarieties

If S is a definable complex quotient spaces and $\rho : S' \to S$ is a map of definable complex quotient spaces, then $\rho(S')$ is a special subvariety of S.

In practice, we restrict to some subcategory S of definable complex quotient spaces and require that $\rho : S' \to S$ be a S-morphism and say that $\rho(S')$ is S-special.

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Examples of subcategories of definable complex quotient spaces

- The category of complex tori with translations by torsion points is adapted to Mordell-Lang-type problems.
- In the initial work of Bakker, Klingler, and Tsimerman (on which our definition of definable quotient space is based), the category S consists of arithmetic quotients: the algebraic group G is required to be a semisimple $\mathbb{Q}$-algebraic group, $G = \mathrm{G}(\mathbb{R})^+$, M is maximal compact, Γ is an arithmetic group, the fundamental domains are constructed from Siegel sets, and the morphisms are defined over $\mathbb{Q}$ in the sense that the affine map of algebraic groups is defined over $\mathbb{Q}$.

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Weakly special varieties

Fix a subcategory S of definable complex quotient spaces. For a pair of maps

$$\begin{array}{ccccc} & & S_1 & & \\ & \stackrel{\zeta}{\swarrow} & & \stackrel{\xi}{\searrow} & \\ S & & & & S_2 \end{array}$$

in S and $b \in S_2$, the image of the fiber $\zeta(\xi^{-1}\{b\})$ is an S-weakly special subvariety of S.

The main example to keep in mind is the case that $S = S_1 = S' \times S_2$, $\xi : S_1 \to S_2$ is the projection to the second coordinate, and the weakly special varieties take the form $S' \times \{b\}$.

Ax-Schanuel condition

Let S be a category of definable complex quotient spaces. We say that a definable holomorphic map

$$f : X^{\mathrm{an}} \to S$$

from the analytification of an algebraic variety X to some $S \in \mathsf{S}$ satisfies the Ax-Schanuel condition if for every natural number $k \geq 0$ and pair of analytic maps $g : \Delta^k \to X^{\mathrm{an}}$ and $h : \Delta^k \to D$ (where Δ is the unit disc) for which the diagram

$$\begin{array}{ccc} \Delta^k & \xrightarrow{h} & D \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \pi} \\ X^{\mathrm{an}} & \xrightarrow{f} & S \end{array}$$

commutes either

$$\dim \overline{(g,h)(\Delta^k)}^{\mathrm{Zariski}} \geq \mathrm{rk}(dh) + \dim \check{D}$$

or there is a proper weakly special subvariety $S' \subsetneq S$ so that

$$f \circ g(\Delta^k) \subseteq S' .$$

Examples of Ax-Schanuel condition

- Ax's theorem says that inclusions of algebraic varieties into powers of the multiplicative group satisfy Ax-Schanuel when S is taken so that the special varieties are components of algebraic subgroups.
- Bakker and Tsimerman showed that period mappings for variations of Hodge structure satisfy Ax-Schanuel relative to the category of arithmetic quotients.
- More generally, by work of Bakker, Brunebarbe, Klingler, and Tsimerman (establishing definability) and Chiu and separately Gao and Klingler, period mappings for variation of mixed Hodge structure satisfy Ax-Schanuel.

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Uniform Ax-Schanuel

It has been proven by other people (e.g. V. Aslanyan, J. Kirby, J. Pila, B Zilber, and others) in various contexts that a uniform version of Ax-Schanuel follows from a differential algebraic interpretation of the Ax-Schanuel theorem.

We show that this holds at the level of definable maps to definable complex quotient spaces under a technical condition on the category S of definable complex quotient spaces.

What makes the abstraction nontrivial is that in general the complex quotient space S does not come with the structure of an algebraic variety so that it is not clear the class of (weakly) special subvarieties is visible to differential algebra.

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Special varieties as algebraic varieties

If $f : X^{\text{an}} \to S$ is a definable holomorphic map from a quasiprojective complex algebraic variety to a definable complex quotient space and $S' \subseteq S$ is a special subvariety of S, why should $f^{-1}S'$ be an algebraic variety?

Answer: Peterzil-Starchenko definable Chow

Why should a family of weakly special subvarieties of S pull back to a constructible family of algebraic subvarieties of X?

This is harder, but the result still holds using definable Chow and a GAGA theorem of Raynaud.

Why should the pullbacks of all weakly special subvarieties of S fall into countably many families of subvarieties of X?

In general, we do not know whether that would be true. To deduce this we use a condition we call well-parameterization of weakly special varieties.

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Well-parameterization of weakly special varieties

We say that a category S of definable complex quotient spaces well-parameterizes the weakly special varieties if for each $S \in \mathsf{S}$ there is a countable collection of pairs of S-maps

$$\begin{array}{ccc} & S_{1,i} & \\ {}^{\zeta_i}\swarrow & & \searrow^{\xi_i} \\ S & & S_{2,i} \end{array}$$

for $i \in \mathbb{N}$ so that for any weakly special subvariety $S' \subseteq S$ there is some $i \in \mathbb{N}$ and some $b \in S_{2,i}$ so that $S' = \zeta_i \xi_i^{-1}\{b\}$.

Examples with well-parameterization of the weakly special varieties

- In any category S in which there are only countably many objects and morphisms between the objects, then well-parameterization of the weakly special varieties is obvious. This holds, for example, for arithmetic quotients.
- If we take S to be the category of complex tori, then even though this category is uncountable, for any given complex torus S there is a countable set of maps of complex tori $\rho_i : T_i \to S$ so that for any given affine map of complex tori $\psi : T \to S$, there is some $i \in \mathbb{N}$, some $a \in A$, and a map of complex tori $\phi : T \to T_i$ so that $\psi(x) = a + \rho_i(\phi(x))$.

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Example:Hopf manifolds

Hopf manifolds are constructed as

$$M = \mathbb{Z} \backslash (\mathbb{C}^n \smallsetminus \{(0, \ldots, 0)\})$$

where $\mathbb{Z}$ acts via $n \cdot x = \Phi^n(x)$ for some given analytic contraction mapping Φ.

Consider the case where $n = 2$ and $\Phi(x, y) = (px, qy)$ with $0 < |p|, |q| < 1$ and p and q are multiplicatively independent.

For each $\alpha \in \mathbb{C}^\times$ the map

$$(x, y) \to (\alpha x, \alpha y)$$

descends to an automorphism of M. These automorphisms, regarded as special subvarieties of $M \times M$, do not come from a weakly special family.

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Algebraic differential equations for weakly special varieties

Once we know that the preimages of the weakly special varieties in X comprise countably many constructible families of algebraic subvarieties, the Ax-Schanuel condition may be re-expressed in terms of algebraic differential equations.

From the differential-algebraic formulation the compactness theorem or other effective finiteness theorems may be used to deduce uniform versions of Ax-Schanuel from its non-uniform version.

Likewise, the lemma on the uniformity of families of weakly special varieties together with Ax-Schanuel may be used to strengthen the Cattani-Deligne-Kaplan theorem on the algebraicity of Hodge loci to show that the parameters required to defined the positive dimensional components are algebraic.

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Concluding remarks and questions

- The work on definable quotient spaces is joint with S. Eterović. The application to the field of definition of the Hodge locus is part of an incomplete project with J. Pila.
- The basic theory of definable quotient spaces is still opaque and is part of the thesis project of J. Brown.
- The formalism of definable complex quotient spaces does not suffice to describe all of the structures B. Zilber considers in his theory of analytic covers. Can the formalism be generalized in a useful way to include all of these covers? What model theoretic consequences would there be to say of a cover structure that it cannot be described using definable complex quotient spaces?

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Concluding remarks and questions

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