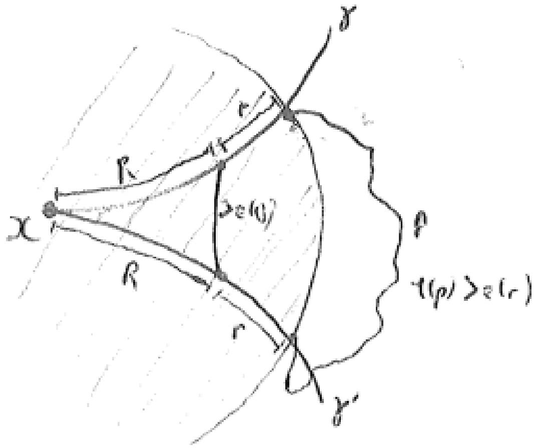
Divergence of geodesics

Write $B_x(R)$ for the closed ball $\{y : d(y,x) \leq R\}$, and write $B_x(R)^o$ for its interior.

Definition 3.14. A **divergence function** for a geodesic metric space (X,d) is an unbounded function $e : \mathbb{R}_{\geq 0} \to \mathbb{R}$ such that for all geodesics γ, γ' with $\gamma(0) = \gamma'(0) =: x$, and for all $R, r \geq 0$, if $d(\gamma(R), \gamma'(R)) > e(0)$ and $R + r \in \mathrm{dom}(\gamma) \cap \mathrm{dom}(\gamma')$, then for any rectifiable path p in $X \setminus B_x(R+r)^o$ from $\gamma(R+r)$ to $\gamma'(R+r)$, we have $\ell(p) > e(r)$.



We say **geodesics diverge exponentially** in X if there is a divergence function e with $\lim_{r\to\infty} e(r)a^{-r} = +\infty$ for some $a > 1$.

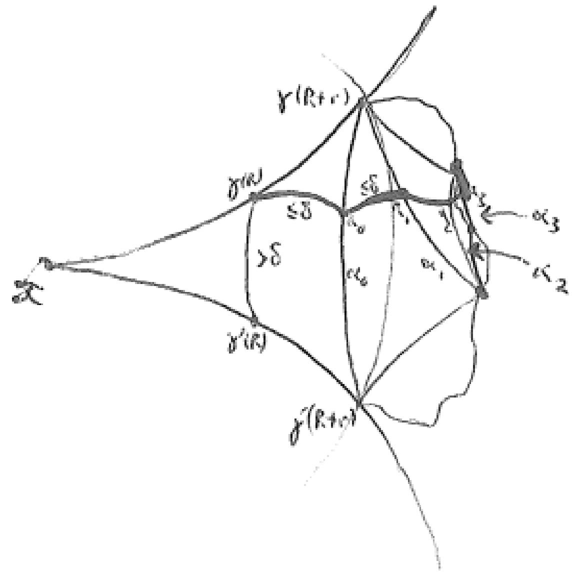
Theorem 3.15. *Suppose (X,d) is a hyperbolic geodesic metric space. Then geodesics in X diverge exponentially.*

Proof. Let δ be such that geodesic triangles in X are δ-thin.

Let γ, γ' be geodesics with $\gamma(0) = \gamma'(0) =: x$, let $R, r \geq 0$, suppose $R + r \in \mathrm{dom}(\gamma) \cap \mathrm{dom}(\gamma')$, and suppose $d(\gamma(R), \gamma'(R)) > \delta$. Let p be a rectifiable path in $X \setminus B_x(R+r)^o$ from $\gamma(R+r)$ to $\gamma'(R+r)$.

We conclude by finding a lower bound on $\ell(p)$ which is exponential in r and depends only on δ.

We will recursively define points a_i on geodesics α_i for $i \geq 0$.



First, pick a geodesic $\alpha_0 = [\gamma(R+r), \gamma'(R+r)]$. Since $d(\gamma(R), \gamma'(R)) > \delta$ and triangles are δ-thin, $d(\gamma(R), a_0) \leq \delta$ for some $a_0 \in \alpha_0$ (namely $a_0 = \alpha_0(r)$).

Given a geodesic α_i with endpoints b_1, b_2 on p and $a_i \in \alpha_i$, let $m \in p$ be the midpoint between b_1 and b_2 on p, and pick geodesics $[b_j, m]$. Then since triangles are δ-thin, for some $j \in \{1, 2\}$ there is $c \in [b_j, m]$ with $d(a_i, c) \leq \delta$. Let $a_{i+1} := c$ and $\alpha_{i+1} := [b_j, m]$.

Since we pick the midpoint at each stage, $\ell(\alpha_i) \leq 2^{-i}\ell(p)$ for all i.

Let $n := \lceil \log_2(\ell(p)) \rceil$. Then $d(a_n, p) \leq \ell(\alpha_n) \leq 2^{-n}\ell(p) \leq 1$.

So

$$\begin{aligned} R + r &\leq d(x, p) && \text{(since } p \subseteq X \setminus B_x(R+r)^o\text{)} \\ &\leq d(x, \gamma(R)) + d(\gamma(R), a_0) + \sum_{0 \leq i < n} d(a_i, a_{i+1}) + d(a_n, p) \\ &\leq R + (n+1)\delta + 1 \\ &\leq R + (\log_2(\ell(p)) + 2)\delta + 1, \end{aligned}$$

so $\ell(p) \geq 2^{\frac{r-1}{\delta} - 2}$, and

$$e(r) := \begin{cases} \delta & \text{if } r = 0 \\ 2^{\frac{r-1}{\delta} - 2} & r > 0 \end{cases}$$

is a divergence function as required. □

Say *geodesics diverge superlinearly* in a geodesic metric space X if there is a divergence function e with

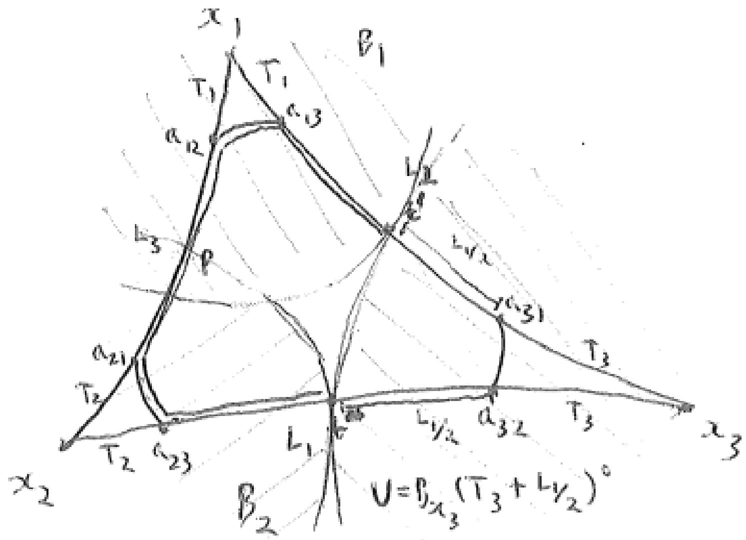
$$\lim_{r \to \infty} \frac{e(r)}{r} = +\infty.$$

Note that exponential divergence implies superlinear divergence.

Theorem 3.16. *If geodesics diverge superlinearly in a geodesic metric space X, then X is hyperbolic.*

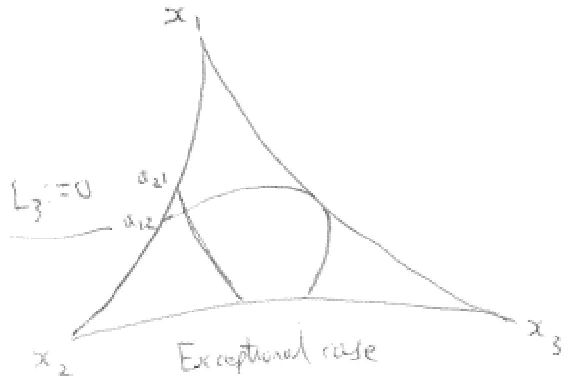
Proof. Let e be a divergence function with $\lim_{r\to\infty} \frac{e(r)}{r} = +\infty$. We may assume $e(0) > 0$.

Let $\Delta = [x_1, x_2, x_3]$ be a geodesic triangle.



Let $T_1 \in [0, \min(d(x_1, x_2), d(x_1, x_3))]$ be maximal such that $t \leq T_1 \Rightarrow d([x_1, x_2](t), [x_1, x_3](t)) \leq e(0)$, and let $a_{12} := [x_1, x_2](T_1)$ and $a_{13} := [x_1, x_3](T_1)$. For $i = 2, 3$, define T_i and a_{ij} for $j \neq i$ analogously.

For $\{i, j, k\} = \{1, 2, 3\}$, let $L_k := d(a_{ij}, a_{ji})$ if $[x_i, a_{ji}] \cap [x_j, a_{ij}] = \emptyset$, and let $L_k := 0$ otherwise.



Pick geodesics $[a_{ij}, a_{ik}]$.

We may assume $L_1 \geq L_2 \geq L_3$. So Δ is $(e(0) + \frac{L_1}{2})$-slim. We conclude by showing that L_1 is bounded by a constant depending only on e.

We may assume $L_1 > 2e(0)$. We have

$$\begin{aligned} T_3 + L_2 + e(0) + L_3 + T_2 &\geq d(x_3, a_{13}) + d(a_{13}, a_{12}) + d(a_{12}, x_2) \\ &\geq d(x_3, x_2) \\ &= T_3 + L_1 + T_2, \end{aligned}$$

so

$$2L_2 \geq L_1 - e(0). \tag{1}$$

In particular, $L_2 \geq \frac{1}{2}e(0) > 0$.

We may also assume that $L_3 = d(a_{12}, a_{21})$. Indeed, otherwise we may define $T_2' := d(x_2, a_{12})$, and define $a_{21}' := [x_2, x_1](T_2') = a_{12}$ and $a_{23}' := [x_2, x_3](T_2')$ and $L_1' := d(a_{23}', a_{32})$ correspondingly. Then $d(a_{12}', a_{21}) = 0 = L_3$. Now $T_2' \leq T_2$ and $L_1' \geq L_1$, so redefining $T_2 := T_2'$, $a_{21} := a_{21}'$, $a_{23} := a_{23}'$ and $L_1 := L_1'$ leaves our assumptions and goal intact; only the maximality property in the definition of T_2 is lost, and we will not use this property.

Let $t := [x_3, x_2](T_3 + \frac{L_1}{2})$, the midpoint of $[a_{32}, a_{23}]$. Let $t' := [x_3, x_1](T_3 + \frac{L_1}{2})$. Let p be the concatenation of the geodesics

$$[t, a_{23}], [a_{23}, a_{21}], [a_{21}, a_{12}], [a_{12}, a_{13}], [a_{13}, t'].$$

Let $U := B_{x_3}(T_3 + \frac{L_1}{2})^o$.

Claim. *p is a path in $X \setminus U$.*

Proof. Let $B_1 := B_{x_1}(T_1 + L_2 - \frac{L_1}{2})$ and $B_2 := B_{x_2}(T_2 + \frac{L_1}{2})$. Then $(B_1 \cup B_2) \subseteq X \setminus U$.

We have $[a_{23}, a_{21}] \subseteq B_2$ since we assumed $L_1 > 2e(0)$ and so $d(a_{23}, a_{21}) \leq e(0) < \frac{L_1}{2}$.

Now $L_2 - \frac{L_1}{2} \geq \frac{e(0)}{2}$ by (1). So since $d(a_{13}, a_{12}) \leq e(0)$ and $d(x_1, a_{13}) = T_1 = d(x_1, a_{12})$, we have $[a_{13}, a_{12}] \subseteq B_1$.

Finally, $L_3 \leq (L_2 - \frac{L_1}{2}) + \frac{L_1}{2}$, so $[a_{12}, a_{21}] \subseteq B_1 \cup B_2$.

So we conclude that $p \subseteq B_1 \cup B_2 \subseteq X \setminus U$. $\square$

Now $L_1, L_2 > 0$, and so by the definition of T_3 there is $0 < \epsilon < e(0) < \frac{L_1}{2}$ such that $d([x_3, x_1](T_3 + \epsilon), [x_3, x_2](T_3 + \epsilon)) > e(0)$, and so

$$e(\frac{L_1}{2} - \epsilon) \leq \ell(p) \leq \frac{L_1}{2} + e(0) + L_3 + e(0) + (L_2 - \frac{L_1}{2}) \leq 2L_1 + 2e(0).$$

It follows from superlinearity of e that L_1 is bounded, as required. $\square$

– Martin Bays 2019